

NOTES ON AN INEQUALITY

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Abstract: In this note we prove a generalized version of an inequality which was first introduced by A. Q. Ngo, *et al.* and later generalized and proved by W. J. Liu, *et al.* in the paper: "On an open problem concerning an integral inequality", *J. Inequal. Pure & Appl. Math.*, **8**(3) (2007), Art. 74.

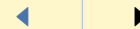
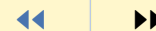
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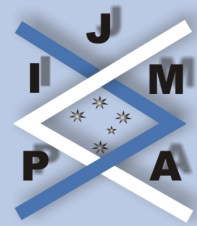
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1. Introduction

In [2] the following result was proved: If $f \geq 0$ is a continuous function on $[0, 1]$ such that

$$(1.1) \quad \int_x^1 f(t)dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1],$$

then

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx, \quad \forall \alpha > 0.$$

The following question was raised in [2]: If f satisfies the above assumptions, under what additional assumptions can one claim that:

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \quad \forall \alpha, \beta > 0?$$

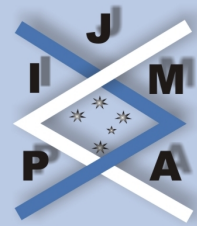
It was proved in [1] that if $f \geq 0$ is a continuous function on $[0, 1]$ satisfying

$$\int_x^b f^\alpha(t)dt \geq \int_x^b t^\alpha dt, \quad \alpha, b > 0, \quad \forall x \in [0, b],$$

then

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx, \quad \forall \beta > 0.$$

In this paper, we prove more general results, namely, Theorems 2.4 and 2.5 below.

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2. Results and Proofs

Let us recall the following result:

Lemma 2.1 (Young's inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y , we have:*

$$\alpha x + \beta y \geq x^\alpha y^\beta.$$

Throughout the paper, $[a, b]$ denotes a bounded interval and all functions are real-valued. Let us prove the following lemma:

Lemma 2.2. *Let $f \in L^1[a, b]$, $g \in C^1[a, b]$. Suppose $f \geq 0$, $g > 0$ is nondecreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a, b],$$

then $\forall \alpha > 0$ the following inequalities hold

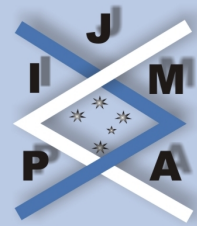
$$(2.1) \quad \int_a^b g^\alpha(x)f(x)dx \geq \int_a^b g^{\alpha+1}(x)dx,$$

$$(2.2) \quad \int_a^b f^{\alpha+1}(x)dx \geq \int_a^b f^\alpha(x)g(x)dx,$$

$$(2.3) \quad \int_a^b f^{\alpha+1}(x)dx \geq \int_a^b f(x)g^\alpha(x)dx.$$

Proof. First, let us prove (2.1). Let A, A^* denote

$$Af(x) := \int_a^x f(t)dt, \quad A^*f(x) := \int_x^b f(t)dt, \quad x \in [a, b], f \in L^1[a, b].$$



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Note that these are continuous functions. From the assumption one has

$$A^*f(x) \geq A^*g(x), \quad \forall x \in [a, b].$$

This means

$$(A^*f - A^*g)(x) \geq 0, \quad \forall x \in [a, b].$$

Then $\forall h \in L^1[a, b]$, $h \geq 0$, one obtains

$$(2.4) \quad \langle A^*f - A^*g, h \rangle := \int_a^b (A^*f - A^*g)(x)h(x)dx \geq 0.$$

Note that the left-hand side of (2.4) is finite since A^*f, A^*g are bounded and $h \in L^1[a, b]$. Thus, by Fubini's Theorem, one has

$$(2.5) \quad \langle f - g, Ah \rangle = \langle A^*f - A^*g, h \rangle \geq 0, \quad \forall h \geq 0, h \in L^1[a, b].$$

Denote $h(x) = \alpha g(x)^{\alpha-1} g'(x)$. One has

$$Ah(x) = \int_a^x h(t)dt = g^\alpha(x) - g^\alpha(a), \quad \forall x \in [a, b].$$

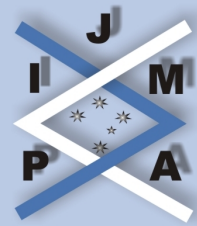
By the assumption,

$$(2.6) \quad \langle f - g, g^\alpha(a) \rangle = g^\alpha(a) \int_a^b (f(x) - g(x))dx \geq 0.$$

Since $h \geq 0$, from (2.5) and (2.6) one gets

$$(2.7) \quad \langle f - g, g^\alpha \rangle = \langle f - g, Ah \rangle + \langle f - g, g^\alpha(a) \rangle \geq 0, \quad \forall \alpha \geq 0.$$

Hence, (2.1) is obtained.



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Since

$$(f(x) - g(x))(f^\alpha(x) - g^\alpha(x)) \geq 0, \quad \forall x \in [a, b], \quad \forall \alpha \geq 0,$$

one gets

$$(2.8) \quad \langle f - g, f^\alpha - g^\alpha \rangle \geq 0, \quad \forall \alpha \geq 0.$$

Inequalities (2.7) and (2.8) imply

$$\langle f - g, f^\alpha \rangle = \langle f - g, f^\alpha - g^\alpha \rangle + \langle f - g, g^\alpha \rangle \geq 0, \quad \forall \alpha > 0.$$

Thus, (2.2) holds.

By Lemma 2.1,

$$\frac{1}{\alpha + 1} f^{\alpha+1}(x) + \frac{\alpha}{\alpha + 1} g^{\alpha+1}(x) \geq g^\alpha(x) f(x), \quad \forall x \in [a, b].$$

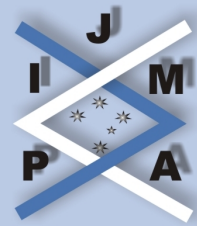
Thus,

$$(2.9) \quad \frac{1}{\alpha + 1} \int_a^b f^{\alpha+1}(x) dx + \frac{\alpha}{\alpha + 1} \int_a^b g^{\alpha+1}(x) dx \\ \geq \int_a^b g^\alpha(x) f(x) dx, \quad \forall \alpha > 0.$$

From (2.1) and (2.9) one obtains

$$\int_a^b f^{\alpha+1}(x) dx \geq \int_a^b g^\alpha(x) f(x) dx, \quad \forall \alpha \geq 0.$$

The proof is complete. \square

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In particular, one has the following result

Corollary 2.3. Suppose $f \in L^1[a, b]$, $g \in C^1[a, b]$, $f, g \geq 0$, g is nondecreasing. If

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a, b]$$

then the following inequality holds

$$(2.10) \quad \int_a^b f^\beta(x)dx \geq \int_a^b g^\beta(x)dx, \quad \forall \beta \geq 1.$$

Proof. Denote $f_\epsilon := f + \epsilon$, $g_\epsilon := g + \epsilon$ where $\epsilon > 0$. It is clear that $g_\epsilon > 0$ and

$$\int_x^b f_\epsilon(t)dt \geq \int_x^b g_\epsilon(t)dt, \quad \forall x \in [a, b].$$

By (2.1) and (2.3) in Lemma 2.2 one has

$$(2.11) \quad \int_a^b f_\epsilon^\beta(x)dx \geq \int_a^b g_\epsilon^\beta(x)dx, \quad \forall \beta \geq 1.$$

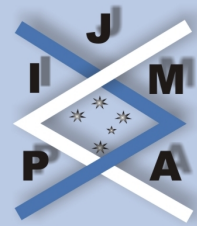
Inequality (2.10) is obtained from (2.11) by letting $\epsilon \rightarrow 0$. □

Theorem 2.4. Suppose $f \in L^1[a, b]$, $g \in C^1[a, b]$, $f, g \geq 0$, g is nondecreasing. If

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a, b],$$

then $\forall \alpha, \beta \geq 0$, $\alpha + \beta \geq 1$, the following inequality holds

$$(2.12) \quad \int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^\alpha(x)g^\beta(x)dx.$$



Proof. Lemma 2.1 shows that

$$\frac{\alpha}{\alpha + \beta} f(x)^{\alpha + \beta} + \frac{\beta}{\alpha + \beta} g(x)^{\alpha + \beta} \geq f^\alpha(x) g^\beta(x), \quad \forall x \in [a, b], \forall \alpha, \beta > 0.$$

Therefore, $\forall \alpha, \beta > 0$ one has

$$(2.13) \quad \frac{\alpha}{\alpha + \beta} \int_a^b f(x)^{\alpha + \beta} dx + \frac{\beta}{\alpha + \beta} \int_a^b g(x)^{\alpha + \beta} dx \geq \int_a^b f^\alpha(x) g^\beta(x) dx.$$

Corollary 2.3 implies

$$(2.14) \quad \int_a^b f(x)^{\alpha + \beta} dx \geq \int_a^b g(x)^{\alpha + \beta} dx, \quad \forall \alpha, \beta \geq 0, \alpha + \beta \geq 1.$$

Inequality (2.12) is obtained from (2.13) and (2.14). \square

Remark 1. Theorem 2.4 is not true if we drop the assumption $\alpha + \beta \geq 1$. Indeed, take $g \equiv 1$, $[a, b] = [0, 1]$, and define

$$f(x) = c(1 - x)^{c-1}, \quad 0 \leq x \leq 1,$$

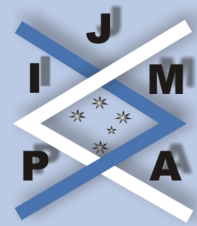
where $c \in (0, 1)$. One has

$$(1 - x)^c = \int_x^1 f(t) dt \geq \int_x^1 g(t) dt = (1 - x), \quad \forall x \in [0, 1], c \in (0, 1),$$

but

$$\frac{2\sqrt{c}}{c + 1} = \int_0^1 \sqrt{f(t)} dt < \int_0^1 \sqrt{g(t)} dt = 1, \quad \forall c \in (0, 1).$$

Assuming that the condition $g \in C^1[a, b]$ can be dropped and replaced by $g \in L^1[a, b]$, we have the following result:

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Theorem 2.5. Suppose $f, g \in L^1[a, b]$, $f, g \geq 0$, g is nondecreasing. If

$$(2.15) \quad \int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a, b],$$

then

$$(2.16) \quad \int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^\alpha(x)g^\beta(x)dx, \quad \forall \alpha, \beta \geq 0, \alpha + \beta \geq 1.$$

Proof. Since $C^1[a, b]$ is dense in L^1 , there exists a sequence $(g_n)_{n=1}^\infty \in C^1[a, b]$ such that g_n is nondecreasing, $g_n \nearrow g$ a.e. Since $g_n \nearrow g$ a.e.,

$$(2.17) \quad \int_x^b g(t)dt \geq \int_x^b g_n(t)dt, \quad \forall x \in [a, b], \forall n.$$

Inequalities (2.15), (2.17) and Theorem 2.4 imply

$$(2.18) \quad \int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^\alpha(x)g_n^\beta(x)dx, \quad \forall n, \forall \alpha, \beta \geq 0, \alpha + \beta \geq 1.$$

Since $f^\alpha g_n^\beta \nearrow f^\alpha g^\beta$ a.e., $f^\alpha g_n^\beta \geq 0$ is measurable satisfying (2.18), by the Monotone convergence theorem (see [3, 4]) $\|f^\alpha g_n^\beta \rightarrow f^\alpha g^\beta\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^\alpha(x)g^\beta(x)dx, \quad \forall \alpha, \beta \geq 0, \alpha + \beta \geq 1.$$

The proof is complete. □

Remark 2. One may wish to extend Theorem 2.5 to the case where $[a, b]$ is unbounded. Note that the case $b = \infty$ is not meaningful. It is because if $g \neq 0$ a.e.,

then both sides of (2.15) are infinite. If $b < \infty$ and $a = -\infty$ and inequality (2.15) holds for $a = \infty$, then it holds as well for all finite $a < 0$. Hence, inequality (2.16) holds for all $a < 0$. Thus, by letting $a \rightarrow -\infty$ in Theorem 2.5, one gets the result of Theorem 2.5 in the case $a = -\infty$.



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