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ISOMETRIES ON LINEAR n-NORMED SPACES

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Abstract

The aim of this article is to generalize the Aleksandrov problem to the case of linear n-normed spaces.

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1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies

$$d_Y\big(f(x), f(y)\big) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all x, y in X with $d_X(x, y) = r$, we have $d_Y\big(f(x), f(y)\big) = r$. Then r is called a conservative (or preserved) distance for the mapping f. Aleksandrov [1] posed the following problem:

Remark 1. *Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.*

The Aleksandrov problem has been investigated in several papers (see [3] – [10]). Th.M. Rassias and P. Šemrl [9] proved the following theorem for mappings satisfying the strong distance one preserving property (SDOPP), i.e., for every $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

Theorem 1.1 ([9]). *Let X and Y be real normed linear spaces with dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping with Lipschitz constant $\kappa = 1$. Assume that f is a surjective mapping satisfying (SDOPP). Then f is an isometry.*

Definition 1.1 ([2]). *Let X be a real linear space with $\dim X \geq n$ and $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ a function. Then $(X, \|\cdot, \dots, \cdot\|)$ is called a linear n-normed space if*

(nN_1) $\qquad \|x_1, \dots, x_n\| = 0 \iff x_1, \dots, x_n$ *are linearly dependent*

(nN_2) $\|x_1, \dots, x_n\| = \|x_{j_1}, \dots, x_{j_n}\|$

for every permutation $(j_1, \dots j_n)$ *of* $(1, \dots, n)$

(nN_3) $$\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$$

(nN_4) $$\|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$$

for all $\alpha \in \mathbb{R}$ *and all* $x, y, x_1, \dots, x_n \in X$. *The function* $\|\cdot, \dots, \cdot\|$ *is called the* n*-norm on* X.

In [3], Chu *et al.* defined the notion of weak n-isometry and proved the Rassias and Šemrl's theorem in linear n-normed spaces.

Definition 1.2 ([3]). *We call* $f : X \to Y$ *a weak* n*-Lipschitz mapping if there is a* $\kappa \geq 0$ *such that*

$$\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| \leq \kappa \|x_1 - x_0, \dots, x_n - x_0\|$$

for all $x_0, x_1, \dots, x_n \in X$. *The smallest such* κ *is called the weak* n*-Lipschitz constant.*

Definition 1.3 ([3]). *Let* X *and* Y *be linear* n*-normed spaces and* $f : X \to Y$ *a mapping. We call* f *a weak* n*-isometry if*

$$\|x_1 - x_0, \dots, x_n - x_0\| = \|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\|$$

for all $x_0, x_1, \dots, x_n \in X$.

For a mapping $f : X \to Y$, consider the following condition which is called the *weak n-distance one preserving property*: For $x_0, x_1, \ldots, x_n \in X$ with $\|x_1 - y_1, \ldots, x_n - y_n\| = 1$, $\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| = 1$.

Theorem 1.2 ([3]). *Let $f : X \to Y$ be a weak n-Lipschitz mapping with weak n-Lipschitz constant $\kappa \leq 1$. Assume that if $x_0, x_1, \ldots, x_m$ are m-colinear then $f(x_0)$, $f(x_1)$, $\ldots$, $f(x_m)$ are m-colinear, $m = 2, n$, and that f satisfies the weak $n-$distance one preserving property. Then f is a weak n-isometry.*

In this paper, we introduce the concept of n-isometry which is suitable for representing the notion of n-distance preserving mappings in linear n-normed spaces. We prove also that the Rassias and Šemrl theorem holds under some conditions when X and Y are linear n-normed spaces.

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2. The Aleksandrov Problem in Linear n-normed Spaces

In this section, let X and Y be linear n-normed spaces with dimension greater than $n-1$.

Definition 2.1. *Let X and Y be linear n-normed spaces and $f : X \to Y$ a mapping. We call f an n-isometry if*

$$\|x_1 - y_1, \dots, x_n - y_n\| = \|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\|$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

For a mapping $f : X \to Y$, consider the following condition which is called the n*-distance one preserving property* : For $x_1, \dots, x_n, y_1, \dots, y_n \in X$ with $\|x_1 - y_1, \dots, x_n - y_n\| = 1$, $\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = 1$.

Lemma 2.1 ([3, Lemma 2.3]). *Let $x_1, x_2, \dots, x_n$ be elements of a linear n-normed space X and γ a real number. Then*

$$\|x_1, \dots, x_i, \dots, x_j, \dots, x_n\| = \|x_1, \dots, x_i, \dots, x_j + \gamma x_i, \dots, x_n\|.$$

for all $1 \le i \ne j \le n$.

Definition 2.2 ([3]). *The points $x_0, x_1, \dots, x_n$ of X are said to be n-colinear if for every i, $\{x_j - x_i \mid 0 \le j \ne i \le n\}$ is linearly dependent.*

Remark 2. *The points x_0, x_1 and x_2 are 2-colinear if and only if $x_2 - x_0 = t(x_1 - x_0)$ for some real number t.*

Theorem 2.2. *Let $f : X \to Y$ be a weak n-Lipschitz mapping with weak n-Lipschitz constant $\kappa \leq 1$. Assume that if $x_0, x_1, \dots, x_m$ are m-colinear then $f(x_0)$, $f(x_1)$, $\dots$, $f(x_m)$ are m-colinear, $m = 2, n$, and that f satisfies the weak $n-$distance one preserving property. Then f satisfies*

$$\|x_1 - y_1, \dots, x_n - y_n\| = \|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\|$$

for all $x_1, \dots, x_n, y_1, \dots y_n \in X$ with x_1, y_1, y_j 2-colinear for $j = 2, 3, \dots, n$.

Proof. By Theorem 1.2, f is a weak n-isometry. Hence

$$\|x_1 - y, \dots, x_n - y\| = \|f(x_1) - f(y), \dots, f(x_n) - f(y)\| \tag{2.1}$$

for all $x_1, \dots, x_n, y \in X$.

If $x_1, y_1, y_2 \in X$ are 2-colinear then there exists a $t \in \mathbb{R}$ such that $y_1 - y_2 = t(y_1 - x_1)$. By Lemma 2.1,

$$\begin{aligned}\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &= \|x_1 - y_1, (x_2 - y_1) + (y_1 - y_2), \dots, x_n - y_n\| \\ &= \|x_1 - y_1, (x_2 - y_1) + (-t)(x_1 - y_1), \dots, x_n - y_n\| \\ &= \|x_1 - y_1, x_2 - y_1, \dots, x_n - y_n\|\end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$ with x_1, y_1, y_2 2-colinear. By the same method as above, one can obtain that if x_1, y_1, y_j are 2-colinear for $j = 3, \dots, n$ then

$$\begin{aligned}\|x_1 - y_1, x_2 - y_2, x_3 - y_3, \dots, x_n - y_n\| \\ &= \|x_1 - y_1, x_2 - y_1, x_3 - y_3, \dots, x_n - y_n\|\end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&= \|x_1 - y_1, x_2 - y_1, x_3 - y_1, \dots, x_n - y_n\| \\
&= \cdots \\
&= \|x_1 - y_1, x_2 - y_1, x_3 - y_1, \dots, x_n - y_1\|
\end{aligned}
$$

for all $x_1, \dots, x_n, y_1, \dots y_n \in X$ with x_1, y_1, y_j 2-colinear for $j = 2, 3, \dots, n$.

By the assumption, if $x_1, y_1, y_2 \in X$ are 2-colinear then $f(x_1), f(y_1), f(y_2) \in Y$ are 2-colinear. So there exists a $t \in \mathbb{R}$ such that $f(y_1) - f(y_2) = t(f(y_1) - f(x_1))$. By Lemma 2.1,

$$
\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\
&= \|f(x_1) - f(y_1), (f(x_2) - f(y_1)) + (f(y_1) - f(y_2)), \dots, f(x_n) - f(y_n)\| \\
&= \|f(x_1) - f(y_1), (f(x_2) - f(y_1)) + (-t)(f(x_1) - f(y_1)), \dots, f(x_n) - f(y_n)\| \\
&= \|f(x_1) - f(y_1), f(x_2) - f(y_1), \dots, f(x_n) - f(y_n)\|
\end{aligned}
$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$ with x_1, y_1, y_2 2-colinear. If x_1, y_1, y_j are 2-colinear for $j = 3, \dots, n$ then $f(x_1), f(y_1), f(y_j)$ are 2-colinear for $j = 3, \dots, n$. By the same method as above, one can obtain that if $f(x_1), f(y_1), f(y_j)$ are 2-colinear for $j = 3, \dots, n$, then

$$
\begin{aligned}
(2.3)\quad &\|f(x_1) - f(y_1), f(x_2) - f(y_2), f(x_3) - f(y_3), \dots, f(x_n) - f(y_n)\| \\
&= \|f(x_1) - f(y_1), f(x_2) - f(y_1), f(x_3) - f(y_3), \dots, f(x_n) - f(y_n)\| \\
&= \|f(x_1) - f(y_1), f(x_2) - f(y_1), f(x_3) - f(y_1), \dots, f(x_n) - f(y_n)\| \\
&= \dots \\
&= \|f(x_1) - f(y_1), f(x_2) - f(y_1), f(x_3) - f(y_1), \dots, f(x_n) - f(y_1)\|
\end{aligned}
$$

for all $x_1, \dots, x_n, y_1, \dots y_n \in X$ with x_1, y_1, y_j 2-colinear for $j = 2, 3, \dots, n$.

By (2.1), (2.2) and (2.3),

$$\|x_1 - y_1, \ldots, x_n - y_n\| = \|f(x_1) - f(y_1), \ldots, f(x_n) - f(y_n)\|$$

for all $x_1, \ldots, x_n, y_1, \ldots y_n \in X$ with x_1, y_1, y_j 2-colinear for $j = 2, 3, \ldots, n$. □

Now we introduce the concept of n-Lipschitz mapping and prove that the n-Lipschitz mapping satisfying the n-distance one preserving property is an n-isometry under some conditions.

Definition 2.3. *We call $f : X \to Y$ an n-Lipschitz mapping if there is a $\kappa \geq 0$ such that*

$$\|f(x_1) - f(y_1), \ldots, f(x_n) - f(y_n)\| \leq \kappa \|x_1 - y_1, \ldots, x_n - y_n\|$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. The smallest such κ is called the n-Lipschitz constant.

Lemma 2.3 ([3, Lemma 2.4]). *For $x_1, x_1^{'} \in X$, if x_1 and $x_1^{'}$ are linearly dependent with the same direction, that is, $x_1^{'} = \alpha x_1$ for some $\alpha > 0$, then*

$$\left\| x_1 + x_1^{'}, x_2, \ldots, x_n \right\| = \|x_1, x_2, \ldots, x_n\| + \|x_1^{'}, x_2, \ldots, x_n\|$$

for all $x_2, \ldots, x_n \in X$.

Lemma 2.4. *Assume that if x_0, x_1 and x_2 are 2-colinear then $f(x_0), f(x_1)$ and $f(x_2)$ are 2-colinear, and that f satisfies the n-distance one preserving property. Then f preserves the n-distance k for each $k \in \mathbb{N}$.*

Proof. Suppose that there exist $x_0, x_1 \in X$ with $x_0 \neq x_1$ such that $f(x_0) = f(x_1)$. Since $\dim X \geq n$, there are $x_2, \dots, x_n \in X$ such that $x_1 - x_0, x_2 - x_0, \dots, x_n - x_0$ are linearly independent. Since $\|x_1 - x_0, x_2 - x_0, \dots, x_n - x_0\| \neq 0$, we can set

$$z_2 := x_0 + \frac{x_2 - x_0}{\|x_1 - x_0, x_2 - x_0, \dots, x_n - x_0\|}.$$

Then we have

$$\begin{aligned} &\|x_1 - x_0, z_2 - x_0, x_3 - x_0, \dots, x_n - x_0\| \\ &= \left\| x_1 - x_0, \frac{x_2 - x_0}{\|x_1 - x_0, x_2 - x_0, \dots, x_n - x_0\|}, x_3 - x_0, \dots, x_n - x_0 \right\| \\ &= 1. \end{aligned}$$

Since f preserves the n-distance 1,

$$\|f(x_1) - f(x_0), f(z_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = 1.$$

But it follows from $f(x_0) = f(x_1)$ that

$$\|f(x_1) - f(x_0), f(z_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = 0,$$

which is a contradiction. Hence f is injective.

Let $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $k \in \mathbb{N}$ and

$$\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = k.$$

We put

$$z_i = y_1 + \frac{i}{k}(x_1 - y_1), \quad i = 0, 1, \dots, k.$$

Then

$$\begin{aligned}
\|z_{i+1} - z_i, x_2 - y_2, \dots, x_n - y_n\| \\
&= \left\| y_1 + \frac{i+1}{k}(x_1 - y_1) - \left(y_1 + \frac{i}{k}(x_1 - y_1) \right), x_2 - y_2, \dots, x_n - y_n \right\| \\
&= \left\| \frac{1}{k}(x_1 - y_1), x_2 - y_2, \dots, x_n - y_n \right\| \\
&= \frac{1}{k}\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{k}{k} = 1
\end{aligned}$$

for all $i = 0, 1, \dots, k-1$. Since f satisfies the n-distance one preserving property,

$$\|f(z_{i+1}) - f(z_i), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| = 1 \tag{2.4}$$

for all $i = 0, 1, \dots, k-1$. Since z_0, z_1 and z_2 are 2-colinear, $f(z_0), f(z_1)$ and $f(z_2)$ are also 2-colinear. Thus there is a real number t_0 such that

$$f(z_2) - f(z_1) = t_0(f(z_1) - f(z_0)).$$

By (2.4),

$$\begin{aligned}
\|f(z_1) - f(z_0), &f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\
&= \|f(z_2) - f(z_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\
&= \|t_0(f(z_1) - f(z_0)), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\
&= |t_0| \|f(z_1) - f(z_0), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\|.
\end{aligned}$$

So we have $t_0 = \pm 1$. If $t_0 = -1$, $f(z_2) - f(z_1) = -f(z_1) + f(z_0)$, that is,

$$f(z_2) = f(z_0).$$

Since f is injective, $z_2 = z_0$, which is a contradiction. Thus $t_0 = 1$. Hence

$$f(z_2) - f(z_1) = f(z_1) - f(z_0).$$

Similarly, one can obtain that

$$f(z_{i+1}) - f(z_i) = f(z_i) - f(z_{i-1})$$

for all $i = 2, 3, \ldots, k-1$. Thus

$$f(z_{i+1}) - f(z_i) = f(z_1) - f(z_0)$$

for all $i = 1, 2, \ldots, k-1$. Hence

$$\begin{aligned} f(x_1) - f(y_1) &= f(z_k) - f(z_0) \\ &= f(z_k) - f(z_{k-1}) + f(z_{k-1}) - f(z_{k-2}) + \cdots + f(z_1) - f(z_0) \\ &= k(f(z_1) - f(z_0)). \end{aligned}$$

Therefore,

$$\begin{aligned} \|f(x_1) - f(y_1), &f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| \\ &= \|k(f(z_1) - f(z_0)), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| \\ &= k\|(f(z_1) - f(z_0)), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| \\ &= k, \end{aligned}$$

which completes the proof. $\square$

Theorem 2.5. *Let $f : X \to Y$ be an n-Lipschitz mapping with n-Lipschitz constant $\kappa = 1$. Assume that if x_0, x_1, x_2 are 2-colinear then $f(x_0), f(x_1), f(x_2)$ are 2-colinear, and that if $x_1 - y_1, \dots, x_n - y_n$ are linearly dependent then $f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)$ are linearly dependent. If f satisfies the n-distance one preserving property, then f is an n-isometry.*

Proof. By Lemma 2.4, f preserves the n-distance k for each $k \in \mathbb{N}$. For $x_1, \dots, x_n, y_1, \dots, y_n \in X$, there are two cases depending upon whether $\|x_1 - y_1, \dots, x_n - y_n\| = 0$ or not. In the case $\|x_1 - y_1, \dots, x_n - y_n\| = 0$, $x_1 - y_1, \dots, x_n - y_n$ are linearly dependent. By the assumption, $f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)$ are linearly dependent. Hence

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = 0.$$

In the case $\|x_1 - y_1, \dots, x_n - y_n\| > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\|x_1 - y_1, \dots, x_n - y_n\| < n_0.$$

Assume that

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| < \|x_1 - y_1, \dots, x_n - y_n\|.$$

Set

$$w = y_1 + \frac{n_0}{\|x_1 - y_1, \dots, x_n - y_n\|}(x_1 - y_1).$$

Then we obtain that

$$\|w - y_1, x_2 - y_2, \dots, x_n - y_n\|$$

$$= \left\| y_1 + \frac{n_0}{\|x_1 - y_1, \dots, x_n - y_n\|}(x_1 - y_1) - y_1, x_2 - y_2, \dots, x_n - y_n \right\|$$
$$= \frac{n_0}{\|x_1 - y_1, \dots, x_n - y_n\|}\|x_1 - y_1, \dots, x_n - y_n\| = n_0.$$

By Lemma 2.4,

$$\|f(w) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| = n_0.$$

By the definition of w,

$$w - x_1 = \left(\frac{n_0}{\|x_1 - y_1, \dots, x_n - y_n\|} - 1\right)(x_1 - y_1).$$

Since

$$\frac{n_0}{\|x_1 - y_1, \dots, x_n - y_n\|} > 1,$$

$w - x_1$ and $x_1 - y_1$ have the same direction. By Lemma 2.3,

$$\begin{aligned}&\|w - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\quad = \|w - x_1, x_2 - y_2, \dots, x_n - y_n\| + \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|.\end{aligned}$$

So we have

$$\begin{aligned}\|f(w) - f(x_1), &f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\leq \|w - x_1, x_2 - y_2, \dots, x_n - y_n\| \\ &= n_0 - \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|.\end{aligned}$$

By the assumption,

$$\begin{aligned} n_0 &= \|f(w) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\le \|f(w) - f(x_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\qquad\qquad + \|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &< n_0 - \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| + \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &= n_0, \end{aligned}$$

which is a contradiction. Hence f is an n-isometry. □

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