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A REVERSE HARDY-HILBERT-TYPE INTEGRAL INEQUALITY

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ABSTRACT. By estimating a weight function, a reverse Hardy-Hilbert-type integral inequality with a best constant factor is obtained. As an application, some equivalent forms and some particular results have been established.

Key words and phrases: Hardy-Hilbert-type integral inequality; weight function; β function; Hölder's inequality.

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1. INTRODUCTION

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) \geq 0$, $g(x) \geq 0$, and $0 < \int_0^\infty f^p(x)dx < \infty$, $0 < \int_0^\infty g^q(x)dx < \infty$. Then the Hardy-Hilbert's integral inequality is as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty g^q(x)dx\right]^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible (see [1]). For (1.1), Yang et al. [2], [3], [4], [8] and [9] gave some strengthened versions and extensions. In particular, in [7], Yang obtained

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \times \left[\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_\alpha^\infty (x-\alpha)^{1-\lambda} g^q(x)dx\right]^{\frac{1}{q}}, \tag{1.2}$$

where $\alpha \in \mathbb{R}$, $\lambda > 2 - \min\{p, q\}$,

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Recently, in [5], Xi gave a reverse Hardy-Hilbert-type inequality

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} > \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

where $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $1.5 \le \lambda < 3$ and $a_n \ge 0, b_n > 0$, such that $0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}a_n^p}{2n+3-\lambda} < \infty$, $0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda} < \infty$.

In this paper, by estimating the weight function, a reverse Hardy-Hilbert-type integral inequality with a best constant factor is obtained. As an application, some equivalent forms and some particular results have been established.

2. SOME LEMMAS

Lemma 2.1.

$$B(p,q) = B(q,p) = \int_0^{\infty}\frac{1}{(1+u)^{p+q}}u^{p-1}du \quad (p,q>0), \tag{2.1}$$

where $B(p,q)$ is the β function.

Proof. See [6]. □

Lemma 2.2. *Let $p<0$ or $0<p<1$, and $\frac{1}{p}+\frac{1}{q}=1$, $f(x),\ g(x) \ge 0$, $f \in L^p(E),\ g \in L^q(E)$, $x \in \mathbb{R}^k$ and the set E is Borel measurable in $\mathbb{R}^k$, where k is a positive integer. Then*

$$\int_E f(x)g(x)dx \ge \left(\int_E f^p(x)dx\right)^{\frac{1}{p}}\left(\int_E g^q(x)dx\right)^{\frac{1}{q}}, \tag{2.2}$$

where the equality holds if and only if there exist non-negative real numbers a and b, such that they are not all zero and $af^p(x) = bg^q(x)$, a. e. in E.

Proof. See [1, Section 6.9, Theorem 189]. □

Lemma 2.3. *Let $p<0$ or $0<p<1$, and $\frac{1}{p}+\frac{1}{q}=1$, $2-\max\{p,q\} < \lambda < 2-\min\{p,q\},\ y > \alpha$, The weight function $\omega_\lambda(y,p)$ is defined by*

$$\omega_\lambda(y,p) = \int_\alpha^{\infty}\frac{1}{(x+y-2\alpha)^{\lambda}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}dx, \quad y \in (0,\ \infty). \tag{2.3}$$

Then we have $\frac{p+\lambda-2}{p} > 0,\ \frac{q+\lambda-2}{q} > 0$, and

$$\omega_\lambda(y,p) = (y-\alpha)^{1-\lambda}B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right). \tag{2.4}$$

Proof. If $p<0$, then $0<q<1$. Since $1<\lambda<2-p$, $\frac{p+\lambda-2}{p}>0$ and $\frac{q+\lambda-2}{q}>0$. If $0<p<1$, we have $q<0$, $1<\lambda<2-q$, $\frac{p+\lambda-2}{p}>0,\ \frac{q+\lambda-2}{q}>0$.

Setting $t = \frac{x-\alpha}{y-\alpha}$, by $\frac{p+\lambda-2}{p} + \frac{q+\lambda-2}{q} = \lambda$, we have

$$\begin{aligned}\int_\alpha^\infty \frac{1}{(x+y-2\alpha)^\lambda}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}} dx &= (y-\alpha)^{1-\lambda}\int_0^\infty \frac{1}{(1+t)^\lambda} t^{-\frac{2-\lambda}{p}} dt \\ &= (y-\alpha)^{1-\lambda}\int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{p+\lambda-2}{p}-1} dt \\ &= (y-\alpha)^{1-\lambda} B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right).\end{aligned}$$

The lemma is thus proved. □

Lemma 2.4. *Let* $p < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - q < \lambda < 2 - p$, $0 < \varepsilon < \frac{q+\lambda-2}{2}$, *then we have*

$$\begin{aligned}I &= \int_{\alpha+1}^\infty \int_{\alpha+1}^\infty \frac{1}{(x+y-2\alpha)^\lambda}(x-\alpha)^{\frac{\lambda-2-\varepsilon}{p}}(y-\alpha)^{\frac{\lambda-2-\varepsilon}{q}} dxdy \\ &= \frac{1}{\varepsilon} B\left(\frac{q+\lambda-2}{q} - \frac{\varepsilon}{q}, \frac{p+\lambda-2}{p} + \frac{\varepsilon}{q}\right) - O_\varepsilon(1).\end{aligned} \tag{2.5}$$

Proof. Setting $x - \alpha = s, y - \alpha = t$, then

$$\begin{aligned}I &= \int_{\alpha+1}^\infty \int_{\alpha+1}^\infty \frac{1}{(x+y-2\alpha)^\lambda}(x-\alpha)^{\frac{\lambda-2-\varepsilon}{p}}(y-\alpha)^{\frac{\lambda-2-\varepsilon}{q}} dxdy \\ &= \int_1^\infty \int_1^\infty \frac{1}{(s+t)^\lambda} s^{\frac{\lambda-2-\varepsilon}{p}} t^{\frac{\lambda-2-\varepsilon}{q}} dsdt \\ &= \int_1^\infty s^{\frac{\lambda-2-\varepsilon}{p}} \left[\int_1^\infty \frac{1}{(s+t)^\lambda} t^{\frac{\lambda-2-\varepsilon}{q}} dt\right] ds.\end{aligned}$$

Let $u = \frac{t}{s}$, we have

$$\begin{aligned}I &= \int_1^\infty s^{-1-\varepsilon}\left[\int_{\frac{1}{s}}^\infty \frac{1}{(1+u)^\lambda} u^{\frac{q+\lambda-2-\varepsilon}{q}-1} du\right] ds \\ &= \int_1^\infty s^{-1-\varepsilon}\left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{q+\lambda-2-\varepsilon}{q}-1} du\right] ds \\ &\qquad - \int_1^\infty s^{-1-\varepsilon}\left[\int_0^{\frac{1}{s}} \frac{1}{(1+u)^\lambda} u^{\frac{q+\lambda-2-\varepsilon}{q}-1} du\right] ds \\ &= \frac{1}{\varepsilon} B\left(\frac{q+\lambda-2}{q} - \frac{\varepsilon}{q}, \frac{p+\lambda-2}{p} + \frac{\varepsilon}{q}\right) \\ &\qquad - \int_1^\infty s^{-1-\varepsilon}\left[\int_0^{\frac{1}{s}} \frac{1}{(1+u)^\lambda} u^{\frac{q+\lambda-2-\varepsilon}{q}-1} du\right] ds.\end{aligned}$$

By $1 < 2 - q < \lambda,\ \ 0 < \varepsilon < \frac{q+\lambda-2}{2}$. Hence

$$\begin{aligned}0 &< \int_1^\infty s^{-1-\varepsilon}\left[\int_0^{\frac{1}{s}} \frac{1}{(1+u)^\lambda} u^{\frac{q+\lambda-2-\varepsilon}{q}-1}du\right]ds \\ &< \int_1^\infty s^{-1-\varepsilon}\left[\int_0^{\frac{1}{s}} u^{\frac{q+\lambda-2-\varepsilon}{q}-1}du\right]ds \\ &< \int_1^\infty s^{-1}\left[\int_0^{\frac{1}{s}} u^{\frac{q+\lambda-2}{q}-1}du\right]ds \\ &= \left(\frac{q}{q+\lambda-2}\right)^2.\end{aligned}$$

The lemma is proved. □

3. MAIN RESULTS

Theorem 3.1. *Let $p < 0$ or $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $2-\max\{p,q\} < \lambda < 2-\min\{p,q\}$, $f(x), g(x) \geq 0$, and $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx < \infty$, $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} g^q(x)dx < \infty$, $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned}\text{(3.1)}\quad \int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy &> B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \\ &\times \left[\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_\alpha^\infty (x-\alpha)^{1-\lambda} g^q(x)dx\right]^{\frac{1}{q}},\end{aligned}$$

where the constant factor $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible.

Proof. By (2.2), we have

$$\begin{aligned}\text{(3.2)}\quad \int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy &= \int_\alpha^\infty\int_\alpha^\infty \left[\frac{f(x)}{(x+y-2\alpha)^{\frac{\lambda}{p}}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{pq}}\right] \\ &\quad\times \left[\frac{g(y)}{(x+y-2\alpha)^{\frac{\lambda}{q}}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{pq}}\right]dxdy \\ &\geq \int_\alpha^\infty\int_\alpha^\infty \left[\frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dx\right]^{\frac{1}{p}} \\ &\quad\times \int_\alpha^\infty\int_\alpha^\infty \left[\frac{g^q(y)}{(x+y-2\alpha)^\lambda}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}dy\right]^{\frac{1}{q}}.\end{aligned}$$

If (3.2) takes the form of equality, then by (2.2), there exist non-negative numbers a and b such that they are not all zero and

$$a\frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}} = b\frac{g^q(y)}{(x+y-2\alpha)^\lambda}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}, \quad \text{a.e. in } (\alpha,\infty)\times(\alpha,\infty).$$

It follows that

$$a(x-\alpha)^{2-\lambda}f^p(x) = b(y-\alpha)^{2-\lambda}g^q(y) = c,$$

a. e. in $(0,\ \infty)\times(0,\ \infty)$. Without loss of generality, suppose $a\neq 0$. One has

$$(x-\alpha)^{1-\lambda}f^p(x)=\frac{c}{a}(x-\alpha)^{-1},$$

a. e. in $(0,\ \infty)\times(0,\ \infty)$, which contradicts $0<\int_\alpha^\infty(x-\alpha)^{1-\lambda}f^P(x)dx<\infty$. Therefore, by (2.3), we have

$$\int_\alpha^\infty\int_\alpha^\infty\frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy>\left[\int_\alpha^\infty\omega_\lambda(x,q)(x-\alpha)^{1-\lambda}f^p(x)dx\right]^{\frac{1}{p}}\times\left[\int_\alpha^\infty\omega_\lambda(y,p)(y-\alpha)^{1-\lambda}g^q(y)dy\right]^{\frac{1}{q}}.$$

By (2.4), we obtain (3.1).

Without loss of generality, suppose $p<0$, which implies that $0<q<1$. For $0<\varepsilon<\frac{q+\lambda-2}{2}$ set

$$f_\varepsilon(x)=0,\ \ x\in(\alpha,\alpha+1);\qquad f_\varepsilon(x)=(x-\alpha)^{\frac{\lambda-2-\varepsilon}{p}},\ \ x\in[\alpha+1,\infty),$$
$$g_\varepsilon(x)=0,\ \ x\in(\alpha,\alpha+1);\qquad g_\varepsilon(x)=(x-\alpha)^{\frac{\lambda-2-\varepsilon}{q}},\ \ x\in[\alpha+1,\infty).$$

If the constant factor $B\left(\frac{p+\lambda-2}{p},\ \frac{q+\lambda-2}{q}\right)$ in (3.1) is not the best possible, then, there exists a positive constant $K>B\left(\frac{p+\lambda-2}{p},\ \frac{q+\lambda-2}{q}\right)$, such that (3.1) is still valid if $B\left(\frac{p+\lambda-2}{p},\ \frac{q+\lambda-2}{q}\right)$ is replaced by K. By (2.5), we have

$$\begin{aligned}&B\left(\frac{q+\lambda-2}{q}-\frac{\varepsilon}{q},\frac{p+\lambda-2}{p}+\frac{\varepsilon}{q}\right)-\varepsilon O_\varepsilon(1)\\&\quad=\varepsilon\int_\alpha^\infty\int_\alpha^\infty\frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y-2\alpha)^\lambda}dxdy\\&\quad>\varepsilon K\left\{\int_{\alpha+1}^\infty(x-\alpha)^{1-\lambda}f_\varepsilon^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_{\alpha+1}^\infty(y-\alpha)^{1-\lambda}g_\varepsilon^q(y)dy\right\}^{\frac{1}{q}}=K.\end{aligned}$$

Letting $\varepsilon\longrightarrow 0^+$, we obtain $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\geq K$. Hence the constant factor $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ in (3.1) is the best possible when $2-q<\lambda<2-p, 0<\varepsilon<\frac{q+\lambda-2}{2}$.

The theorem is proved. □

In (3.1), when $\alpha=-\frac{1}{2}$, we have:

Corollary 3.2. *Let* $p<0$, *or* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\max\{p,q\}<\lambda<2-\min\{p,q\}$, $f(x),g(x)\geq 0$, $0<\int_{-\frac{1}{2}}^\infty\left(x+\frac{1}{2}\right)^{1-\lambda}f^p(x)dx<\infty$, $0<\int_{-\frac{1}{2}}^\infty\left(x+\frac{1}{2}\right)^{1-\lambda}g^q(x)dx<\infty$. *Then*

$$\int_{-\frac{1}{2}}^\infty\int_{-\frac{1}{2}}^\infty\frac{f(x)g(y)}{(x+y+1)^\lambda}dxdy>B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\times\left[\int_{-\frac{1}{2}}^\infty\left(x+\frac{1}{2}\right)^{1-\lambda}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\frac{1}{2}}^\infty\left(x+\frac{1}{2}\right)^{1-\lambda}g^q(x)dx\right]^{\frac{1}{q}},\tag{3.3}$$

where the constant factor $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *is the best possible.*

In (3.1), when $\lambda=2$, we have:

Corollary 3.3. *Let* $p < 0$, *or* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x), g(x) \geq 0$, $0 < \int_\alpha^\infty (x-\alpha)^{-1} f^p(x)dx < \infty$, $0 < \int_\alpha^\infty (x-\alpha)^{-1} g^q(x)dx < \infty$, $\alpha \in \mathbb{R}$. *Then*

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^2} dxdy > \left[\int_\alpha^\infty \frac{f^p(x)}{x-\alpha} dx\right]^{\frac{1}{p}} \left[\int_\alpha^\infty \frac{g^q(x)}{x-\alpha} dx\right]^{\frac{1}{q}}. \tag{3.4}$$

In (3.1), when $\alpha = -\frac{1}{2}$, $\lambda = 2$, we have:

Corollary 3.4. *Let* $p < 0$, *or* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x), g(x) \geq 0$, $0 < \int_{-\frac{1}{2}}^\infty (x+\frac{1}{2})^{-1} f^p(x)dx < \infty$, $0 < \int_{-\frac{1}{2}}^\infty \left(x+\frac{1}{2}\right)^{-1} g^q(x)dx < \infty$. *Then*

$$\int_{-\frac{1}{2}}^\infty \int_{-\frac{1}{2}}^\infty \frac{f(x)g(y)}{(x+y+1)^2} dxdy > \left[\int_{-\frac{1}{2}}^\infty \frac{f^p(x)}{x+\frac{1}{2}} dx\right]^{\frac{1}{p}} \left[\int_{-\frac{1}{2}}^\infty \frac{g^q(x)}{x+\frac{1}{2}} dx\right]^{\frac{1}{q}}. \tag{3.5}$$

4. AN EQUIVALENT FORM

Theorem 4.1. *Let* $p < 0$, $\frac{1}{p}+\frac{1}{q} = 1$, $2-q < \lambda < 2-p$, $f(x) \geq 0$, *and* $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx < \infty$, $\alpha \in \mathbb{R}$. *Then*

$$\int_\alpha^\infty (y-\alpha)^{(p-1)(\lambda-1)} \left[\int_\alpha^\infty \frac{f(x)}{(x+y-2\alpha)^\lambda} dx\right]^p dy < \left[B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)\right]^p \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx. \tag{4.1}$$

where the constant factor $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ *is the best possible. Inequalities (4.1) and (3.1) are equivalent as* $p < 0$.

Proof. Let

$$g(y) = (y-\alpha)^{(p-1)(\lambda-1)} \left[\int_\alpha^\infty \frac{f(x)}{(x+y-2\alpha)^\lambda} dx\right]^{p-1}, \qquad y \in (\alpha, \infty).$$

Then by (3.1) and $p < 0$, we have

$$\begin{aligned} 0 &< \int_\alpha^\infty (y-\alpha)^{1-\lambda} g^q(y)dy \\ &= \int_\alpha^\infty (y-\alpha)^{(p-1)(\lambda-1)} \left[\int_\alpha^\infty \frac{f(x)}{(x+y-2\alpha)^\lambda} dx\right]^p dy \\ &= \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy \\ &\geq B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left[\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx\right]^{\frac{1}{p}} \\ &\qquad \times \left[\int_\alpha^\infty (y-\alpha)^{1-\lambda} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{4.2}$$

Since $p < 0$, we obtain that

$$\int_\alpha^\infty (y-\alpha)^{1-\lambda} g^q(y)dy \leq \left[B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)\right]^p \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^p(x)dx < \infty. \tag{4.3}$$

By (4.3) and (4.2), we obtain (4.1).

Whereas, assume that (4.1) is true, by (2.2), we have

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}}dxdy$$
$$=\int_{\alpha}^{\infty}\left[(y-\alpha)^{\frac{\lambda-1}{q}}\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right]\left[(y-\alpha)^{\frac{1-\lambda}{q}}g(y)\right]dy$$
$$\geq\left\{\int_{\alpha}^{\infty}(y-\alpha)^{(p-1)(\lambda-1)}\left[\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right]^{p}dy\right\}^{\frac{1}{p}}\left[\int_{\alpha}^{\infty}(y-\alpha)^{1-\lambda}g^{q}(y)dy\right]^{\frac{1}{q}}.$$

Since $p<0$, by (4.1), then (3.1) is proved. Hence inequalities (4.1) and (3.1) are equivalent as $p<0$. □

Corollary 4.2. *Let* $p<0, \frac{1}{p}+\frac{1}{q}=1, 2-q<\lambda<2-p, f(x)\geq 0, 0<\int_{-\frac{1}{2}}^{\infty}\left(x+\frac{1}{2}\right)^{1-\lambda}f^{p}(x)dx<\infty$. *Then*

$$\int_{-\frac{1}{2}}^{\infty}\left(y+\frac{1}{2}\right)^{(p-1)(\lambda-1)}\left[\int_{-\frac{1}{2}}^{\infty}\frac{f(x)}{(x+y+1)^{\lambda}}dx\right]^{p}dy < \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^{p}\int_{-\frac{1}{2}}^{\infty}\left(x+\frac{1}{2}\right)^{1-\lambda}f^{p}(x)dx, \tag{4.4}$$

where the constant factor $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *is the best possible.*

Corollary 4.3. *Let* $p<0, \frac{1}{p}+\frac{1}{q}=1, f(x)\geq 0, 0<\int_{\alpha}^{\infty}(x-\alpha)^{-1}f^{p}(x)dx<\infty, \alpha\in\mathbb{R}$. *Then*

$$\int_{\alpha}^{\infty}(y-\alpha)^{(p-1)}\left[\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{2}}dx\right]^{p}dy<\int_{\alpha}^{\infty}\frac{f^{p}(x)}{x-\alpha}dx. \tag{4.5}$$

Corollary 4.4. *Let* $p<0, \frac{1}{p}+\frac{1}{q}=1, f(x)\geq 0, 0<\int_{-\frac{1}{2}}^{\infty}\left(x+\frac{1}{2}\right)^{-1}f^{p}(x)dx<\infty$. *Then*

$$\int_{-\frac{1}{2}}^{\infty}\left(y+\frac{1}{2}\right)^{(p-1)}\left[\int_{-\frac{1}{2}}^{\infty}\frac{f(x)}{(x+y+1)^{2}}dx\right]^{p}dy<\int_{-\frac{1}{2}}^{\infty}\frac{f^{p}(x)}{x+\frac{1}{2}}dx. \tag{4.6}$$

Theorem 4.5. *Let* $0<p<1, \frac{1}{p}+\frac{1}{q}=1, 2-p<\lambda<2-q, f(x)\geq 0,$ *and* $0<\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f^{p}(x)dx<\infty, \alpha\in\mathbb{R}$. *Then*

$$\int_{\alpha}^{\infty}(y-\alpha)^{(p-1)(\lambda-1)}\left[\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right]^{p}dy > \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^{p}\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f^{p}(x)dx, \tag{4.7}$$

where the constant factor $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *is the best possible. Inequalities (4.7) and (3.1) are equivalent as* $0<p<1$.

Proof. Since $0 < p < 1,$ by (2.2) and (2.4),we have

$$\left[\int_\alpha^\infty \frac{f(x)}{(x+y-2\alpha)^\lambda}dx\right]^p = \left\{\int_\alpha^\infty \left[\frac{f(x)}{(x+y-2\alpha)^{\frac{\lambda}{p}}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{pq}}\right]\right.$$
$$\left.\times\left[\frac{1}{(x+y-2\alpha)^{\frac{\lambda}{q}}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{pq}}\right]dx\right\}^p$$
$$\geq \int_\alpha^\infty \frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dx$$
$$\times\left[\int_\alpha^\infty \frac{1}{(x+y-2\alpha)^\lambda}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}dx\right]^{p-1}$$
$$= \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^{p-1}(y-\alpha)^{(p-1)(1-\lambda)}$$
$$\times\int_\alpha^\infty \frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dx. \tag{4.8}$$

If (4.8) takes the form of equality, then by (2.2), there exist non-negative numbers a and b, such that they are not all zero and

$$a\frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}} = b\frac{1}{(x+y-2\alpha)^{\frac{\lambda}{p}}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{p}}, \quad \text{a.e. } (\alpha,\infty).$$

It follows that

$$a(x-\alpha)^{2-\lambda}f^p(x) = b(y-\alpha)^{2-\lambda}, \quad \text{a.e. in } (\alpha,\infty).$$

Obviously $a \neq 0$, (otherwise $a = b = 0$), one has

$$(x-\alpha)^{1-\lambda}f^p(x) = \frac{b}{a}(y-\alpha)^{2-\lambda}x^{-1}, \quad \text{a.e. } (\alpha,\infty)$$

which contradicts $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda}f^P(x)dx < \infty$. Hence

$$\int_\alpha^\infty (y-\alpha)^{(p-1)(\lambda-1)}\left[\int_\alpha^\infty \frac{f(x)}{(x+y-2\alpha)^\lambda}dx\right]^p dy$$
$$> \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^{p-1}\int_\alpha^\infty\left[\int_\alpha^\infty \frac{f^p(x)}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dx\right]dy$$
$$= \left[B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\right]^{p-1}$$
$$\times\int_\alpha^\infty\left[\int_\alpha^\infty \frac{1}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{q}}dy\right]f^p(x)dx.$$

By (2.3) and (2.4), we obtain (4.7).

Obviously, inequalities (4.7) and (3.1) are equivalent as $0 < p < 1$ and the constant factor $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ is the best possible. □

Corollary 4.6. *Let* $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, 2-p < \lambda < 2-q, 0 < \int_{-\frac{1}{2}}^{\infty} \left(x + \frac{1}{2}\right)^{1-\lambda} f^p(x)dx < \infty$, $f(x) \geq 0$. *Then*

$$\int_{-\frac{1}{2}}^{\infty} \left(y + \frac{1}{2}\right)^{(p-1)(\lambda-1)} \left[\int_{-\frac{1}{2}}^{\infty} \frac{f(x)}{(x+y+1)^{\lambda}}dx\right]^p dy > \left[B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)\right]^p \int_{-\frac{1}{2}}^{\infty} \left(x + \frac{1}{2}\right)^{1-\lambda} f^p(x)dx,$$

where the constant factor $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ *is the best possible.*

Corollary 4.7. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) \geq 0$, $0 < \int_{\alpha}^{\infty} (x-\alpha)^{-1} f^p(x)dx < \infty$, $\alpha \in \mathbb{R}$. *Then*

$$\int_{\alpha}^{\infty} (y-\alpha)^{(p-1)} \left[\int_{\alpha}^{\infty} \frac{f(x)}{(x+y-2\alpha)^2}dx\right]^p dy > \int_{\alpha}^{\infty} \frac{f^p(x)}{x-\alpha}dx. \tag{4.9}$$

Corollary 4.8. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) \geq 0$, $0 < \int_{-\frac{1}{2}}^{\infty} \left(x + \frac{1}{2}\right)^{-1} f^p(x)dx < \infty$. *Then*

$$\int_{-\frac{1}{2}}^{\infty} \left(y + \frac{1}{2}\right)^{(p-1)} \left[\int_{-\frac{1}{2}}^{\infty} \frac{f(x)}{(x+y+1)^2}dx\right]^p dy > \int_{-\frac{1}{2}}^{\infty} \frac{f^p(x)}{x+\frac{1}{2}}dx. \tag{4.10}$$

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