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# A Combinatorial Interpretation for an Identity of Barrucand

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**Abstract**

The binomial coefficient identity, $\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$, appeared as Problem 75-4 in Siam Review in 1975. The published solution equated constant terms in a suitable polynomial identity. Here we give a combinatorial interpretation in terms of card deals.

## 1 Introduction

Pierre Barrucand [1] proposed the identity

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k},$$

in the Problems and Solutions section of Siam Review in 1975 and it was considered sufficiently interesting to be included in the problem compilation [2]. The published solution [3] equated constant terms in the identity

$$\big(1+(1+x)(1+y/x)(1+1/y)\big)^n=\left(1+\frac{1+x}{y}\right)^n\left(1+y\left(1+\frac{1}{x}\right)\right)^n.$$

The problem was also solved by G. E. Andrews, M. E. H. Ismail, and O. G. Ruehr using hypergeometric functions, by C. L. Mallows using probability, and by the proposer using differential equations. The sequence generated by each side of the identity, $(1,3,15,93,639,\ldots)_{n\geq 0}$, is A002893 in The On-Line Encyclopedia of Integer Sequences.

Here we show that the identity counts certain derangement-type card deals in two different ways.

## 2 The card deals

To construct these card deals, start with a deck of $3n$ cards, $n$ each colored red, green and blue in denominations 1 through $n$. Next choose a subset $S$ of the denominations and partition all the cards of these denominations into a list of three equal size sets such that the first set contains no red cards, the second no green cards, and the third no blue cards. Or, more picturesquely, deal all cards of the chosen denominations into three equal size hands to players designated red, green and blue in such a way that no player receives a card of her own color. Let $\mathcal{T}_n$ denote the set of all triples (deals) obtained in this way. For example, $\mathcal{T}_2$ is shown on the next page, with deals classified by the set $S$ of denominations.

The left side of Barrucand's identity counts these deals by size of the denomination set $S$: the number of deals in $\mathcal{T}_n$ with $|S| = k$ is $\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3$. The right side counts them by number of distinct denominations occurring in the red player's hand: the number of deals in $\mathcal{T}_n$ with $k$ distinct denominations in red's hand is $\binom{n}{k}^2\binom{2k}{k}$.

We now proceed to verify these assertions. Since there are $\binom{n}{k}$ ways to choose a subset $S$ of size $k$ from the denominations, the first assertion will obviously follow from

**Proposition 1.** *The number of ways to deal all $3n$ cards so that no player receives a card of her own color is $\sum_{j=0}^{n}\binom{n}{j}^3$* [A000172].

*Proof.* Let us count these deals by number $j$ of green cards in red's hand. If there are $j$ green cards in red's hand, then the balance of red's hand must consist of $n - j$ blue cards, red cards not being allowed. The remaining $n-j$ green cards must be in blue's hand and the remaining $j$ blue cards in green's hand. This forces $j$ red cards in blue's hand and $n - j$ red cards in green's hand. Thus the deal is determined by a choice of $j$ green cards and a choice of $n - j$ blue cards to make up red's hand, and a choice of $j$ red cards for blue's hand—$\binom{n}{j}^3$ choices in all. $\square$

As for the second assertion, let $D$ denote the set of denominations appearing in the red player's hand. Since the number of deals depends on $D$ only through its size and since there are $\binom{n}{k}$ ways to choose a set $D$ of size $k$, it suffices to establish the following count.

**Proposition 2.** *The number of deals in $\mathcal{T}_n$ for which the denominations appearing in the red player's hand are $1, 2, \ldots, k$ is $\binom{n}{k}\binom{2k}{k}$* [A026375].

*Proof.* Partition the set of denominations $D = \{1, 2, \ldots, k\}$ occurring in red's hand into three blocks: $A$, those appearing on both blue and green cards (in red's hand); $B$, those appearing on blue cards only; $C$, those appearing on green cards only. Set $|A| = a,\ |B| = b,\ |C| = c$. Thus $a + b + c = k$ and $2a + b + c$ is the size of each hand. This implies that the number of denominations not in $\{1, 2, \ldots, k\}$ but involved in the deal is $a$; call this set $E$. The green cards with denominations in $B \cup E$ must occur in blue's hand. This accounts for $|B \cup E| = a + b$ cards in blue's hand and so the rest of her hand must consist of $a + c$ red cards.

| denomination set $S$ | # | avoid red | avoid green | avoid blue |
|---|---|---|---|---|
| $\{1,2\}$ | 1 | $\textcolor{blue}{\boxed{1}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{red}{\boxed{2}}$ | $\textcolor{green}{\boxed{1}}$ $\textcolor{green}{\boxed{2}}$ |
| | 2 | $\textcolor{blue}{\boxed{1}}$ $\textcolor{green}{\boxed{1}}$ | $\textcolor{red}{\boxed{2}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{green}{\boxed{2}}$ |
| | 3 | $\textcolor{blue}{\boxed{1}}$ $\textcolor{green}{\boxed{1}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{2}}$ $\textcolor{green}{\boxed{2}}$ |
| | 4 | $\textcolor{blue}{\boxed{1}}$ $\textcolor{green}{\boxed{2}}$ | $\textcolor{red}{\boxed{2}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{green}{\boxed{1}}$ |
| | 5 | $\textcolor{blue}{\boxed{1}}$ $\textcolor{green}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{green}{\boxed{1}}$ $\textcolor{red}{\boxed{2}}$ |
| | 6 | $\textcolor{green}{\boxed{1}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{blue}{\boxed{1}}$ $\textcolor{red}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{green}{\boxed{2}}$ |
| | 7 | $\textcolor{green}{\boxed{1}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{blue}{\boxed{1}}$ | $\textcolor{red}{\boxed{2}}$ $\textcolor{green}{\boxed{2}}$ |
| | 8 | $\textcolor{blue}{\boxed{2}}$ $\textcolor{green}{\boxed{2}}$ | $\textcolor{blue}{\boxed{1}}$ $\textcolor{red}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{green}{\boxed{1}}$ |
| | 9 | $\textcolor{blue}{\boxed{2}}$ $\textcolor{green}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{blue}{\boxed{1}}$ | $\textcolor{green}{\boxed{1}}$ $\textcolor{red}{\boxed{2}}$ |
| | 10 | $\textcolor{green}{\boxed{1}}$ $\textcolor{green}{\boxed{2}}$ | $\textcolor{blue}{\boxed{1}}$ $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{1}}$ $\textcolor{red}{\boxed{2}}$ |
| $\{1\}$ | 11 | $\textcolor{blue}{\boxed{1}}$ | $\textcolor{red}{\boxed{1}}$ | $\textcolor{green}{\boxed{1}}$ |
| | 12 | $\textcolor{green}{\boxed{1}}$ | $\textcolor{blue}{\boxed{1}}$ | $\textcolor{red}{\boxed{1}}$ |
| $\{2\}$ | 13 | $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{2}}$ | $\textcolor{green}{\boxed{2}}$ |
| | 14 | $\textcolor{green}{\boxed{2}}$ | $\textcolor{blue}{\boxed{2}}$ | $\textcolor{red}{\boxed{2}}$ |
| $\emptyset$ | 15 | $\emptyset$ | $\emptyset$ | $\emptyset$ |

The 15 deals in $\mathcal{T}_2$

Thus the deal is determined by a choice of the sets $A$ and $B$ ($C$ is then determined), the set $E$, and a choice of $a+c$ red cards (from the $k+a$ available) for blue's hand. These choices are counted by the sum over nonnegative $a$ and $b$ of the product $\binom{k}{a}$ [choose $A$] $\times\binom{k-a}{b}$ [choose $B$] $\times\binom{n-k}{a}$ [choose $E$] $\times\binom{k+a}{a+c}$ [choose red cards for blue's hand]. This sum can be written

$$\sum_{a\geq 0}\binom{k}{a}\binom{n-k}{n-k-a}\sum_{b\geq 0}\binom{k-a}{b}\binom{k+a}{k-b}.$$

An application of the Vandermonde convolution (see [4, Id. 132, p. 66] for a combinatorial proof) to the inner sum yields $\binom{2k}{k}$, independent of $a$, and then another application evaluates the entire sum as $\binom{n}{n-k}\binom{2k}{k}=\binom{n}{k}\binom{2k}{k}$. $\square$

# 3 Acknowledgment

I thank Zerinvary Lajos for pointing out that the counting sequence of Proposition 1 is a special case of the Dinner-Diner matching numbers A059066.

# References

[1] P. Barrucand, A combinatorial identity, Problem 75-4, *SIAM Review*, **17** (1975), 168.

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(Concerned with sequences A000172, A002893, A026375, and A059066.)

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