Journal of Integer Sequences, Vol. 11 (2008),
Article 08.5.4

On the Partitions of a Number into Arithmetic Progressions

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Abstract

The paper investigates the enumeration of the set $\mathrm{AP}(n)$ of partitions of a positive integer n in which the nondecreasing sequence of parts form an arithmetic progression. We establish formulas for such partitions, and characterize a class of integers n with the property that the length of every member of $\mathrm{AP}(n)$ divides n. We prove that the number of such integers is small.

1 Introduction

A partition of a positive integer n is a nondecreasing sequence of positive integers whose sum is n. The summands are called parts of the partition. We will denote the partition $n_1, n_2, \ldots, n_k$ as the k-tuple $(n_1, n_2, \ldots, n_k)$.

We consider the problem of enumerating the set $\mathrm{AP}(n)$ of partitions of n in which the nondecreasing sequence of parts form an arithmetic progression (AP). Our investigation was in part motivated by sequence A049988 in Sloane's table [6], that is, the number of arithmetic progressions of positive, integers, nondecreasing with sum n. Cook and Sharpe [3] obtained necessary and sufficient conditions for a positive integer to possess a partition into arithmetic progressions with a prescribed common difference. Nyblom and Evans [5] undertook the enumeration problem and found the following representation for the number

$p_d(n)$ of partitions of n with common difference d.

$$p_d(n) = \begin{cases} \tau_1(n) - 2 - f(n), & \text{if } n = d\frac{m(m+1)}{2} \text{ for some } m > 1; \\ \tau_1(n) - 1 - f(n), & \text{otherwise,} \end{cases}$$

where $f(n) = |A_n|$ with $A_n = \{c : c|n, c \text{ odd}, c^2 < d(2n - c), 2n < dc(c-1)\}$, and $\tau_1(n)$ is the number of odd positive divisors of n. The authors also state a closed formula for $p_2(n)$.

In what follows we count partitions directly, and mostly according to the length k of an AP, an approach which reduces our domain to the natural subclass of k-partitions of n. This makes it possible to obtain simpler results with certain new consequences.

Let $\mathrm{AP}(n,k) = \{\pi \mid \pi \in \mathrm{AP}(n) \text{ and } \pi \text{ has } k \text{ parts }\}$, and let $\mathrm{ap}(n) = |\mathrm{AP}(n)|$, $\mathrm{ap}(n,k) = |\mathrm{AP}(n,k)|$ denote the cardinalities of the respective sets.

Note that $\mathrm{AP}(n,1) \neq \emptyset,\ n > 0$, and $\mathrm{AP}(n,2) \neq \emptyset,\ n > 1$, since $(n) \in \mathrm{AP}(n,1)$ and $(1, n-1) \in \mathrm{AP}(n,2)$ respectively.

The set $\mathrm{DP}(n)$ of partitions using a single integer (divisor of n) forms a distinguished subset of $\mathrm{AP}(n)$. The cardinality $\mathrm{dp}(n) = |\mathrm{DP}(n)|$ is given by the number $f(n,2)$ of ordered factorizations of n into two factors plus 2 (counting the partitions $(1,1,\ldots,1)$ and (n)): each factorization $n = rs,\ r,s > 0$, gives the s-tuple $(r,r,\ldots,r) \in \mathrm{DP}(n)$, and $n = sr$ gives the r-tuple $(s,s,\ldots,s) \in \mathrm{DP}(n)$. Since the first factor runs over the divisors of n, we have

$$\mathrm{dp}(n) = f(n,2) + 2 = \tau(n), \tag{1}$$

where $\tau(n)$ is the number of positive integral divisors of n.

If p is an odd prime then each $\pi \in \mathrm{AP}(p)$ can have at most two distinct parts since the sum of a k-term AP with a positive common difference is composite if $k \geq 3$. Since $\mathrm{dp}(n) = 2$ for prime p, observe that

$$|\mathrm{AP}(p) \setminus \mathrm{DP}(p)| = \left| \left\{ (i, p-i) \mid 1 \leq i \leq \frac{p-1}{2} \right\} \right| = \frac{p-1}{2}.$$

Consequently $\mathrm{ap}(p) = 2 + (p-1)/2 = (p+3)/2$, from which we obtain the following result on the enumeration of $\mathrm{AP}(n)$ for prime n.

Proposition 1. $\mathrm{ap}(2) = 2$, *and if p is an odd prime, then* $\mathrm{ap}(p) = \dfrac{p+3}{2}$.

Proposition 1 has the following extension. Let $\mathrm{AP}_t(n)$ denote the subset of $\mathrm{AP}(n)$ containing partitions with at most t distinct parts and $\mathrm{ap}_t(n) = |\mathrm{AP}_t(n)|$. Then $\mathrm{ap}_2(n) = \tau(n) + \frac{n}{2} - 1$ if n is even, and $\mathrm{ap}_2(n) = \tau(n) + \frac{(n-1)}{2}$ if n is odd. Hence we have the following result.

Proposition 2. *If n is a positive integer, then* $\mathrm{ap}_2(n) = \tau(n) + \left\lfloor \dfrac{n-1}{2} \right\rfloor$, *where $\lfloor x \rfloor$ is the greatest integer $\leq x$.*

The next-step result follows from the summation of the parts of a partition in $\mathrm{AP}(n)$.

$$a + (a+d) + \cdots + (a + (k-1)d) = ka + \binom{k}{2} d = n, \quad d \geq 0, \tag{2}$$

for some integers a, d, k, $a \geq 1$, $d \geq 0$, $1 \leq k \leq n$.

Then $k = 3$ implies $3a + 3d = n$. Hence $\mathrm{ap}(n, 3) > 0$ if and only if n is a multiple of 3. If $3|n$, the solution set of $a + d = n/3$ is clearly $\{(a, d) = (r, (n - 3r)/3) \mid 1 \leq r \leq n/3\}$. Thus $\mathrm{ap}_3(n) = \mathrm{ap}_2(n) + n/3$.

Proposition 3. *If n is a positive integer such that $3|n$, then* $\mathrm{ap}_3(n) = \tau(n) + \left\lfloor \dfrac{(5n-3)}{6} \right\rfloor$.

The formula for $\mathrm{ap}_t(n)$, $t \geq 4$, is uneconomical, $(2, 5, 8, 11) \in \mathrm{AP}(26)$ but $4 \nmid 26$. Let $\mathrm{Div}(n)$ denote the set of divisors of n. Since $k \in \mathrm{Div}(n)$ implies $\mathrm{ap}(n, k) > 0$, we define the set $\mathrm{APDiv}(n) = \{k \mid \mathrm{ap}(n, k) > 0\}$. Then $\mathrm{Div}(n) \subseteq \mathrm{APDiv}(n)$ in general.

In Section 2, we derive the general formula for $\mathrm{ap}(n, k)$. This will make it possible, in Section 3, to obtain more inclusive formulas in the spirit of those stated above, bearing in mind $\mathrm{Div}(n) \subseteq \mathrm{APDiv}(n)$. In particular, we characterize the class of numbers n for which $\mathrm{Div}(n) = \mathrm{APDiv}(n)$, and show that such numbers are few.

2 The formula for ap(n, k)

We present the main theorem of this section.

Theorem 4. *1. Let n be a positive integer and $k > 0$ an even number such that ap$(n, k) >$ 0. Then*

$$\mathrm{ap}(n, k) = \left\lfloor \frac{n + k(k-2)}{k(k-1)} \right\rfloor, \qquad \text{if } k|n \text{ and} \tag{3}$$

$$\mathrm{ap}(n, k) = \left\lfloor \frac{2n + k(k-3)}{2k(k-1)} \right\rfloor, \qquad \text{if } k \nmid n. \tag{4}$$

2. Let n be a positve integer and k an odd number such that $\mathrm{ap}(n, k) > 0$, $k > 1$. *Then*

$$\mathrm{ap}(n, k) = \left\lfloor \frac{2n + k(k-3)}{k(k-1)} \right\rfloor. \tag{5}$$

Proof. The proof makes use of the following result [2].

The linear Diophantine equation $ax + by = c$ has a solution if and only if $g|c$, where $g = \gcd(a, b)$. If (x_0, y_0) is any particular solution of this equation, then all other solutions are given by,

$$x = x_0 + \left(\frac{b}{g}\right) t, \; y = y_0 - \left(\frac{a}{g}\right) t$$

where t is an arbitrary integer. ∎

In view of equation (2), the enumerative function $\mathrm{ap}(n, k)$ is given by

$$\mathrm{ap}(n, k) = \sum_{\substack{ka + \ell d = n \\ a \geq 1,\, d \geq 0}} 1,$$

where $\ell = k\dfrac{(k-1)}{2}$. However, $ka + \ell d = n$ has a solution if and only if $\gcd(k,\ell)|n$. For the case k odd, it follows that $\gcd(k,\ell) = k$ leading to a solution if and only if $k|n$, that is, n is a multiple of k. In which case one particular solution is $d_0 = 0$ and $a_0 = \frac{n}{k}$, an integer. And hence, the other solutions are given by

$$a = \frac{n}{k} + \frac{\ell}{\gcd(k,\ell)}t \geq 1 \text{ and } d = -\frac{k}{\gcd(k,\ell)}t \geq 0,$$

where t is an integer. Thus,

$$\begin{aligned}
\operatorname{ap}(n,k) &= \sum_{\substack{a=\frac{n}{k}+\frac{\ell}{\gcd(k,\ell)}t\geq 1 \\ d=-\frac{k}{\gcd(k,\ell)}t\geq 0}} 1 \\
&= \sum_{(1-\frac{n}{k})\frac{\gcd(k,\ell)}{\ell}\leq t\leq 0} 1 \;=\; \left\lfloor \frac{2(n-k)}{k(k-1)} \right\rfloor + 1,
\end{aligned}$$

a result equivalent to formula (5).

Next we consider the case when k is even and note that

$$\gcd(k,\ell) = \gcd\left(2\frac{k}{2}, \frac{k}{2}(k-1)\right) = \frac{k}{2},$$

that is, $ka + \ell d = n$ has a solution if and only if $\frac{k}{2} \mid n$, which implies $n = s\frac{k}{2}$, for some positive integer s. Clearly, for s even, the previous argument goes through save for a minor adjustment, leading to the result,

$$\operatorname{ap}(n,k) = \left\lfloor \frac{(n-k)}{k(k-1)} \right\rfloor + 1,$$

where n is a multiple of k, a result equivalent to formula (3).

On the other hand, if s is odd, then one valid solution is

$$d_0 = 1 \text{ which implies } a_0 = \frac{(s\frac{k}{2} - \ell)}{k} = \frac{\frac{k}{2}(s-(k-1))}{k}, \text{ which is an integer when } s \text{ is odd .}$$

The condition $a \geq 1$ implies the condition $n \geq \frac{k}{2}(k+1)$. Therefore,

$$\begin{aligned}
\operatorname{ap}(n,k) &= \sum_{\substack{a=\frac{n-\ell}{k}+\frac{\ell}{\gcd(k,\ell)}t\geq 1 \\ d=1-\frac{k}{\gcd(k,\ell)}t\geq 0}} 1 \\
&= \sum_{\frac{\gcd(k,\ell)}{\ell}\left(1-\frac{(n-\ell)}{k}\right)\leq t\leq \frac{\gcd(k,\ell)}{k}} 1.
\end{aligned}$$

Since $\gcd(k,\ell) = k/2$, this gives $t \leq \frac{1}{2}$ and hence,

$$\operatorname{ap}(n,k) = \left\lfloor \frac{(2n - k(k+1))}{2k(k-1)} \right\rfloor + 1,$$

where $n = s\frac{k}{2} \geq \frac{k}{2}(k+1)$, a result equivalent to formula (4). This exhausts all possible outcomes.

3 The formula for $\mathrm{ap}(n)$

Using the results of Section 2, we can write the sum of $\mathrm{ap}(n,k)$ over all divisors k of n, denoted $\mathrm{divap}(n)$. Thus,

Theorem 5.

$$\mathrm{divap}(n) = 1 + \sum_{k|n,\ k>1 \text{ is even}} \left\lfloor \frac{n+k(k-2)}{k(k-1)} \right\rfloor + \sum_{k|n,\ k>1 \text{ is odd}} \left\lfloor \frac{2n+k(k-3)}{k(k-1)} \right\rfloor .$$

Alternatively,

$$\mathrm{divap}(n) = \tau(n) + n - \sigma(n) + \sum_{\substack{k|n \\ k>1,\ k \text{ even}}} \left\lfloor \frac{n-1}{k-1} \right\rfloor + \sum_{\substack{k|n \\ k>1,\ k \text{ odd}}} \left\lfloor \frac{n+k(n-2)}{k(k-1)} \right\rfloor .$$

Proof.

$$\begin{aligned}
\mathrm{divap}(n) &= \sum_{k|n} \mathrm{AP}(n,k) = \sum_{k|n,\ k \text{ odd}} \mathrm{AP}(n,k) + \sum_{k|n,\ k \text{ even}} \mathrm{AP}(n,k) \\
&= 1 + \sum_{\substack{k|n \\ k>1,\ k \text{ odd}}} \left(\left\lfloor \frac{2(n-k)}{k(k-1)} \right\rfloor + 1 \right) + \sum_{\substack{k|n \\ k>1,\ k \text{ even}}} \left(\left\lfloor \frac{n-k}{k(k-1)} \right\rfloor + 1 \right) \\
&= 1 + \sum_{\substack{k|n \\ k>1,\ k \text{ odd}}} 1 + \sum_{\substack{k|n \\ k>1,\ k \text{ even}}} 1 + \sum_{\substack{k|n \\ k>1,\ k \text{ odd}}} \left\lfloor \frac{2(n-k)}{k(k-1)} \right\rfloor + \sum_{\substack{k|n \\ k>1,\ k \text{ even}}} \left\lfloor \frac{n-k}{k(k-1)} \right\rfloor \\
&= \tau(n) - \left(\sum_{\substack{k|n \\ k<n,\ k \text{ odd}}} \frac{n}{k} + \sum_{\substack{k|n \\ k<n,\ k \text{ even}}} \frac{n}{k} \right) \\
&\quad + \sum_{\substack{k|n \\ k>1,\ k \text{ odd}}} \left\lfloor \frac{2(n-1)}{k-1} - \frac{n}{k} \right\rfloor + \sum_{\substack{k|n \\ k>1,\ k \text{ even}}} \left\lfloor \frac{n-1}{k-1} \right\rfloor \\
&= \tau(n) - \big(\sigma(n) - n\big) + \sum_{\substack{k|n \\ k>1,\ k \text{ even}}} \left\lfloor \frac{n-1}{k-1} \right\rfloor + \sum_{\substack{k|n \\ k>1,\ k \text{ odd}}} \left\lfloor \frac{n+k(n-2)}{k(k-1)} \right\rfloor .
\end{aligned}$$

□

Note that since $n = \frac{k}{2}\big(2a+(k-1)d\big)$ by (2), it follows that if k is odd and $\mathrm{ap}(n,k) > 0$, then $k|n$, and if $k \nmid n$ and $\mathrm{ap}(n,k) > 0$, then k is even and $n \equiv 0 \pmod{\frac{k}{2}}$.

Hence the set-difference $\mathrm{APDiv}(n) \setminus \mathrm{Div}(n)$ is given by

$$\mathrm{Ek}(n) = \mathrm{APDiv}(n) \setminus \mathrm{Div}(n) = \left\{ k = 2v \mid k \nmid n, v|n, n \geq \binom{k+1}{2} \right\}.$$

That is,

$$\mathrm{Ek}(n) = \left\{ 2v \in 2^{\alpha+1}\mathrm{Div}(m) \mid n \geq v(2v+1) \right\}, \tag{6}$$

where m is the unique odd number satisfying $n = 2^{\alpha}m, \alpha \geq 0$, and $rS = \{rs \mid s \in S\}$.

The importance of $\mathrm{Ek}(n)$ lies in the following statement.

$$\mathrm{ap}(n) = \mathrm{divap}(n) \ \text{ if and only if } \ \mathrm{Ek}(n) = \emptyset. \tag{7}$$

In the case $\mathrm{Ek}(n) \neq \emptyset$ we have

$$\mathrm{ap}(n) = \mathrm{divap}(n) + \mathrm{extap}(n), \ \text{where,} \quad \mathrm{extap}(n) = \sum_{k \in \mathrm{Ek}(n)} \left\lfloor \frac{2n + k(k-3)}{2k(k-1)} \right\rfloor. \tag{8}$$

Since $1 \in \mathrm{Div}(m)$, we obtain

$$\mathrm{Ek}(n) = \emptyset \ \text{ if and only if } \ m < 2^{\alpha+1} + 1. \tag{9}$$

In particular $\mathrm{Ek}(2^{\alpha}) = \emptyset$, $\alpha \geq 0$. Hence the next result follows from Theorem 5 and (7).

Proposition 6. *If α is a nonnegative integer, then*

$$\mathrm{ap}(2^{\alpha}) = 1 + \sum_{j=1}^{\alpha} \left\lfloor \frac{2^{\alpha-j} + 2^{j} - 2}{2^{j} - 1} \right\rfloor = 1 + \alpha + \sum_{j=1}^{\alpha} \left\lfloor \frac{2^{\alpha-j} - 1}{2^{j} - 1} \right\rfloor.$$

Proposition 6 is a special case of the next result.

Theorem 7. *The following assertions are equivalent for any even integer n.*

(i) $\mathrm{ap}(n) = \mathrm{divap}(n)$.

(ii) n can be expressed in the form $n = 2^{j}(r + 2^{j-1})$, $r = 0, 1, \ldots, 3 \cdot 2^{j-1}$, where j is a positive integer.

Proof. Let $n = 2^{j}(r + 2^{j-1}) = 2^{\alpha}m$, $\alpha \geq j$, where m is odd.
$m = 2^{j-\alpha}(r + 2^{j-1}) \leq 2^{j-\alpha}(3 \cdot 2^{j-1} + 2^{j-1}) = 2^{2j-\alpha+1} < 2^{\alpha+1} + 1$. So $\mathrm{Ek}(n) = \emptyset$ by (9). Hence $(ii) \Rightarrow (i)$.

Conversely, notice that $3 \cdot 2^{j-1} < r = 3 \cdot 2^{j-1} + 1$ gives $n = 2^{j}(2^{j+1} + 1) \equiv 2^{j}m$, which implies $\mathrm{Ek}(n) \neq \emptyset$ or $\mathrm{ap}(n) > \mathrm{apdiv}(n)$, a contradiction of (i). $\square$

Remarks

(i) The special even numbers of Theorem 7 form an AP from 2^{2j-1} to 2^{2j+1} for each j (with common difference 2^{j}), say $R(j)$. For example, $R(1) = (2, 4, 6, 8)$ and $R(2) = (8, 12, 16, 20, 24, 28, 32)$.

(ii) Theorem 7 implies Proposition 6 since 2^{2j-1}, $2^{2j} \in R(j)$, when $r = 0,\ 2^{j-1}$.

(iii) For fixed j, there are integers $n = 2^j(r+2^{j-1})$ with $r \notin \{0, 1, \ldots, 3 \cdot 2^{j-1}\}$ which satisfy Theorem 7(i). However, this is not a violation of the theorem since $n \in R(j)$ for some (legal) j and r. For example if $j = 1$, then $n = 12$ corresponds to $r = 5 > 3$. So $12 \notin R(1)$ even though $\mathrm{ap}(12) = \mathrm{apdiv}(12)$. But note that $12 \in R(2)$. This phenomenon is explained by removing the restriction on r and observing that $i < j \Rightarrow R_\infty(i) \supset R_\infty(j)$, where $R_\infty(j) = \left\{2^j(r + 2^{j-1}) \mid r \geq 0\right\}$.

But if n is odd, (9) reduces to $n < 3$, owing to the fact that $\mathrm{ap}(n, 2) > 0$ for each $n > 1$, including prime n. So, for odd numbers, we can skip the $1 \in \mathrm{Div}(m)$ and use the least prime factor of n, to obtain the adjusted version of (9).

Let $n > 1$ be an odd positive integer, and let p denote the least prime divisor of n.

$$\mathrm{Ek}(n) = \{2\} \text{ if and only if } \frac{n}{p} < 2p + 1. \tag{10}$$

The next theorem characterizes odd numbers n satisfying (10), i.e., $\mathrm{ap}(n) = \mathrm{divap}(n) + \mathrm{ap}(n, 2)$.

Theorem 8. *The following assertions are equivalent for any odd integer $n > 1$.*

(i) $\mathrm{ap}(n) = \mathrm{divap}(n) + \frac{(n-1)}{2}$.

(ii) n is prime, or $n = p_1p_2$, where p_1, p_2 are primes such that $p_2 < 2p_1 + 1$.

Proof. Clearly (i) is true if n is prime, since there are $\frac{(n-1)}{2}$ partitions of n into two parts (see Proposition 1). The proof follows from (10) and the observation that $n = p_1p_2 \cdots p_r$, $r > 2$, implies $\frac{n}{p_1} \geq 2p_1 + 1$, which implies that $\mathrm{Ek}(n)$ properly contains $\{2\}$. □

Corollary 9. *If p is an odd prime, then*

$$\mathrm{ap}(p^2) = \frac{p^2 + 9}{2}.$$

Corollary 9 is also a corollary of the next theorem.

Theorem 10. *If p is an odd prime and α is a positive integer, then*

$$\mathrm{ap}(p^\alpha) = 1 + \sum_{i=1}^{\alpha} \left\lfloor \frac{2p^{\alpha-i} + p^i - 3}{p^i - 1} \right\rfloor + \sum_{i=0}^{\left\lfloor \frac{\alpha-1}{2} \right\rfloor} \left\lfloor \frac{p^{\alpha-i} + 2p^i - 3}{2(2p^i - 1)} \right\rfloor.$$

Proof. $\mathrm{Div}(p^\alpha) = \{1, p, \ldots, p^\alpha\}$. So for each i, $0 \leq i \leq \alpha$, we have $p^{\alpha-i} \geq 2p^i + 1$ if and only if $\alpha > 2i$ if and only if $0 \leq i \leq \lfloor(\alpha-1)/2\rfloor$. Thus $\mathrm{Ek}(p^\alpha) = \left\{2p^i \mid 0 \leq i \leq \lfloor(\alpha-1)/2\rfloor\right\}$. Substituting for p^i in (7) and (8), and simplifying the following summations gives the theorem.

$$\mathrm{ap}(p^\alpha) = 1 + \sum_{i=1}^{\alpha} \mathrm{divap}(p^\alpha, p^i) + \sum_{i=0}^{\left\lfloor \frac{\alpha-1}{2} \right\rfloor} \mathrm{extap}(p^\alpha, 2p^i).$$

□

Given an odd prime p let α and c be nonnegative integers, $0 \leq c \leq \alpha$. We claim that

$$\left\lfloor \frac{2p^{\alpha-c}+p^c-3}{p^c-1} \right\rfloor = 1 + \frac{2p^r(p^{(q-1)c}-1)}{p^c-1}, \quad \alpha = qc+r,\, 0 \leq r < c. \tag{11}$$

Denote the left side of (11) by $u(\alpha, c)$, and write $h(\alpha, c) = \frac{(2p^{\alpha-c}-2p^c)}{(p^c-1)}$, so that $\lfloor h(\alpha, c)+3 \rfloor = u(\alpha, c)$. Then clearly, $u(2c, c) = 1$ and $\alpha < 2c$ implies $\alpha - c < c$ which implies $\left\lfloor \frac{h(\alpha,c)}{2} \right\rfloor = -1$ which implies $u(\alpha, c) = 1$. But if $2c < \alpha < 3c$, we have $\left\lfloor \frac{h(\alpha,c)}{2} \right\rfloor = p^{\alpha-2c} + \left\lfloor \frac{h(\alpha,2c)}{2} \right\rfloor$ which gives $\left\lfloor \frac{h(\alpha,c)}{2} \right\rfloor = p^{\alpha-2c} - 1$. Iterating the procedure we obtain an expression of the form $\left\lfloor \frac{h(\alpha,c)}{2} \right\rfloor = p^{\alpha-2c} + p^{\alpha-3c} + \cdots + p^{\alpha-qc} + \left\lfloor \frac{h(\alpha,qc)}{2} \right\rfloor$, where $q = \left\lfloor \frac{\alpha}{c} \right\rfloor$, giving $\left\lfloor \frac{h(\alpha,qc)}{2} \right\rfloor = -1$. If we compute $\frac{\alpha}{c}$ and reverse the order of summation, we obtain $\left\lfloor \frac{h(\alpha,c)}{2} \right\rfloor = p^r + p^{c+r} + \cdots + p^{(q-2)c+r} - 1$, where $\alpha = qc + r$. Summing the finite geometric series gives the result.

Hence we obtain from Theorem 10,

$$\operatorname{divap}(p^\alpha) = 1 + \alpha + \sum_{\substack{c=1 \\ \alpha=qc+r,\, 0\leq r<c}}^{\alpha} \frac{2p^r(p^{(q-1)c}-1)}{p^c-1}. \tag{12}$$

This results in a sequence of polynomials in p over the positive integers for $\operatorname{apdiv}(p^\alpha)$, $\alpha = 0, 1, 2, \ldots$, and consequently yielding simplified forms of $\operatorname{ap}(p^\alpha)$. The degree of $\operatorname{apdiv}(p^\alpha)$ is clearly $\max(\alpha - 2c \mid c > 0) = \alpha - 2$, $\alpha > 1$.

The polynomials $\operatorname{apdiv}(p^\alpha)$ are given below for $\alpha = 2, 3, \ldots, 9$.

$$\operatorname{divap}(p^2) = 5$$

$$\begin{aligned}
\operatorname{divap}(p^3) &= 2p + 6 \\
\operatorname{divap}(p^4) &= 2p^2 + 2p + 9 \\
\operatorname{divap}(p^5) &= 2p^3 + 2p^2 + 4p + 8 \\
\operatorname{divap}(p^6) &= 2p^4 + 2p^3 + 4p^2 + 2p + 13 \\
\operatorname{divap}(p^7) &= 2p^5 + 2p^4 + 4p^3 + 2p^2 + 6p + 10 \\
\operatorname{divap}(p^8) &= 2p^6 + 2p^5 + 4p^4 + 2p^3 + 6p^2 + 2p + 15 \\
\operatorname{divap}(p^9) &= 2p^7 + 2p^6 + 4p^5 + 2p^4 + 6p^3 + 2p^2 + 6p + 14.
\end{aligned}$$

On the other hand, given an odd prime p, the set of numbers $n = pp_2$, $p_2 \geq p$ (p_2 a prime) which satisfy Theorem 8 is nicely bounded: $p^2 \leq n < 2p^2$. So let $S(p)$ denote the set of all numbers n between p^2 and $2p^2$ inclusive which satisfy Theorem 8. We deduce that

$$|S(p)| = \left|\left\{p_2 \text{ prime} \mid p \leq p_2 < 2p\right\}\right| + \left|\left\{p_2 \text{ prime} \mid p^2 \leq p_2 < 2p^2\right\}\right|. \tag{13}$$

In terms of the sequence $a(n) =$ number of primes between n and $2n$ inclusive [4, A035250], a concise expression is $|S(p)| = a(p) + a(p^2)$.

We examine the size of the set of numbers which satisfy the "closure" relation $\mathrm{ap}(n) = \mathrm{divap}(n)$. By Theorem 7 all such numbers (> 1) are even. For a fixed positive integer j define $\mathrm{AE}(j) = \{2c \mid 2^{2j-2} \le c \le 2^{2j}\}$. Recalling the sets $R(j)$ defined in the remarks following Theorem 7, we have

$$|R(j)| = 3 \cdot 2^{j-1} + 1, \quad \text{and} \quad |\mathrm{AE}(j) - R(j)| = 3 \cdot 2^{j-1}(2^{j-1} - 1).$$

Note that $\mathrm{AE}(j) \setminus R(j)$ is the set of even numbers n within the range of elements of $R(j)$ which satisfy $\mathrm{ap}(n) > \mathrm{divap}(n)$. It follows that, for sufficiently large j,

$$\frac{|R(j)|}{|\mathrm{AE(j)}|} \le \frac{1}{2^{j-1}} \longrightarrow 0, \quad \text{and} \quad \frac{|\mathrm{AE(j)} \setminus \mathrm{R(j)}|}{|\mathrm{AE(j)}|} \le \frac{2^{j-1} - 1}{2^{j-1}} \longrightarrow 1.$$

We conclude that practically all even numbers satisfy the strict inequality $\mathrm{ap}(n) > \mathrm{divap}(n)$. Thus more readily so for all positive integers.

4 Conclusion

We close with some remarks on the set $\mathrm{Ek}(n)$. It follows from (6) that if $k \in \mathrm{Ek}(n)$, then $n = M(\frac{k}{2})$ for some odd integer M. Writing $n = 2^\alpha m$ as previously, and $k = 2^{\alpha+1} m_i$ (m_i odd), we have $n = M\frac{k}{2} = M(2^\alpha m_i) = M m_i \cdot 2^\alpha$, where $M m_i = m$, $m_i \ge 1$. Thus each element of $\mathrm{Ek}(n)$ corresponds to a decomposition of m into two factors. This gives $|\mathrm{Ek}(n)| \le \dfrac{\tau(m)}{2}$.
That is,

$$|\mathrm{Ek}(n)| \le \left\lfloor \frac{\tau(m)}{2} \right\rfloor, \quad \text{where} \quad m = \frac{n}{2^\alpha} \text{ is odd, } \alpha \ge 0.$$

The case of prime powers (see Theorem 10) shows that this upper bound is sharp.
The determination of the exact value of $|\mathrm{Ek}(n)|$ remains an open problem.

Acknowledgment

We wish to thank Jeffrey Shallit for bringing the Nyblom-Evans paper to our attention.

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2000 *Mathematics Subject Classification*: Primary 11P81; Secondary 05A15, 05A17.

Keywords: partition, arithmetic progression, divisor function, Diophantine equation.

(Concerned with sequences A004119, A035250, and A049988.)

Received October 8 2007; revised version received October 10 2007; December 5 2008. Published in *Journal of Integer Sequences*, December 13 2008.

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