Journal of Integer Sequences, Vol. 12 (2009),
Article 09.5.5

The Number of Crossings in a Regular Drawing of the Complete Bipartite Graph

Stéphane Legendre
Team of Mathematical Eco-Evolution
Ecole Normale Supérieure
75005 Paris
France
legendre@ens.fr

Abstract

The regular drawing of the complete bipartite graph $K_{n,n}$ produces a striking pattern comprising simple and multiple crossings. We compute the number $c(n)$ of crossings and give an asymptotic estimate for this sequence.

1 Introduction

A regular drawing of the complete bipartite graph $K_{n,n}$ is obtained in the following way. Draw vertically n uniformly spaced nodes on the left, draw vertically n uniformly spaced nodes on the right, and join by straight lines the left and right nodes in all possible manners. A striking pattern appears, as in Figure 1.

This combinatorial pattern is one of many devised by ancient scholars — Ramon Lull (circa 1235–1316), Giordano Bruno (1548–1600) and Wilhem Leibniz (1646–1716) among others — who aimed at explaining phenomena in terms of extensive combinations of primordial entities. Athanasius Kircher (1601–1680) uses the $K_{n,m}$ pattern in several instances in his book *Ars Magna Sciendi* [1]. His drawing of $K_{18,18}$ is reproduced in the novel *Foucault's Pendulum* by Umberto Eco [2]. As the protagonists of the novel delve into esoteric matters, the $K_{n,n}$ pattern suggests to them that the Map would be reconstructed, provided that a device could compute all combinations.

Here we look more prosaically for a formula giving the number $c(n)$ of crossings in a regular drawing of $K_{n,n}$. Figure 1 displays the first values of $c(n)$.

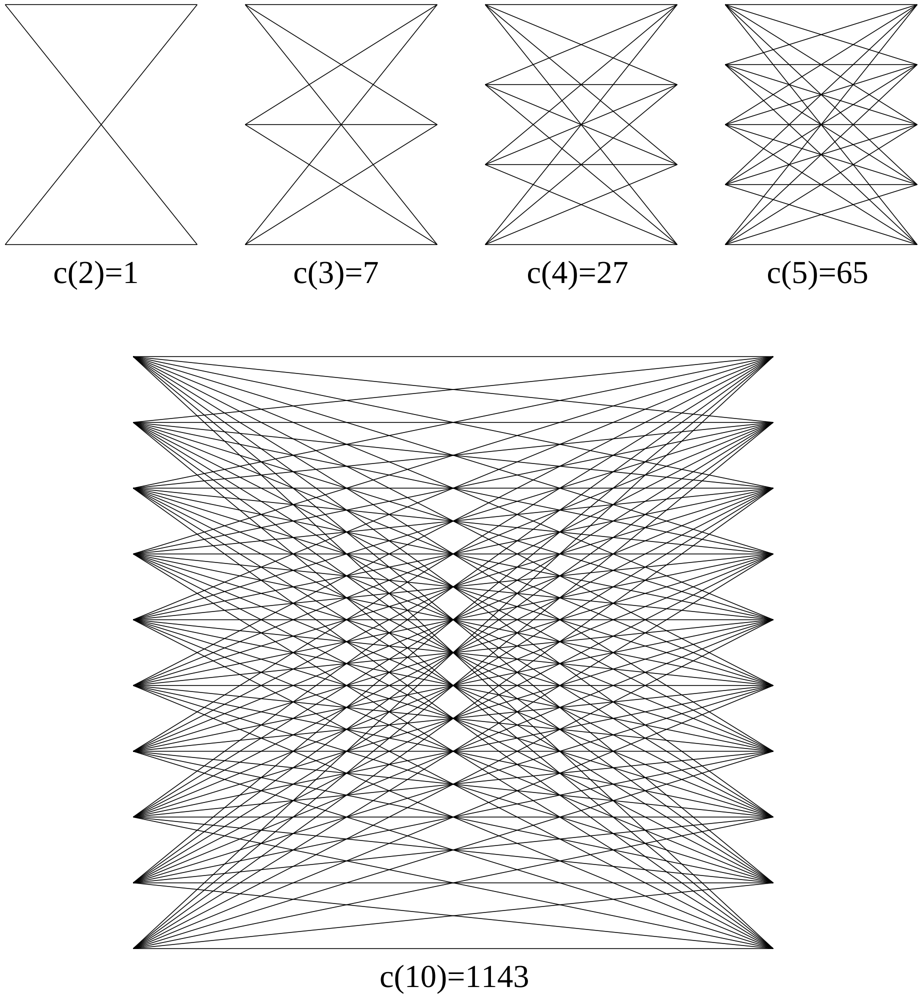

Figure 1: Regular drawing of $K_{n,n}$ for $n = 2, 3, 4, 5$ and 10, with the corresponding number $c(n)$ of crossings.

2 Simple and multiple crossings

Let us assume that nodes on the left have integer coordinates $(0, i)$, $1 \le i \le n$, and nodes on the right have integer coordinates $(1, j)$, $1 \le j \le n$.

We define an (a, b)-*crossing* as the intersection point of two lines, each joining a left node to a right, where a is the distance between left nodes, $1 \le a \le n-1$, and b is the distance between right nodes, $1 \le b \le n-1$.

Let $\langle i, a | j, b \rangle$ denote the (a, b)-crossing whose nodes have ordinates i, $i + a$ on the left, $1 \le i < i + a \le n$, and $j - b$, j on the right, $1 \le j - b < j \le n$, as in Figure 2. When a, b, i, j are understood from context, we shall abbreviate $\langle i, a | j, b \rangle$ as $\mathcal{C}$.

A crossing has *multiplicity* m if it is the intersection of $m + 1$ lines. It is *simple* if $m = 1$ (two lines intersect), *multiple* if $m \ge 2$.

Proposition 1. *For given a and b, the (a, b)-crossings $\langle i, a | j, b \rangle$ have the same abscissa*

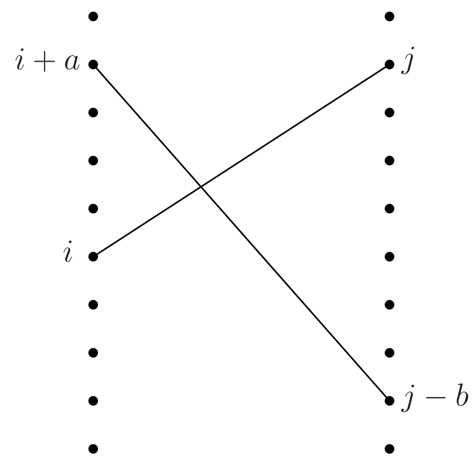

Figure 2: An (a, b)-crossing.

$x = \frac{a}{a+b}$ when $a \leq b$, and $x = \frac{b}{a+b}$ when $a > b$, and ordinates $y = \frac{aj+bi}{a+b}$ with $1 \leq i \leq n-a$ and $b \leq j \leq n$.

Proof. By the theorem of Thales, $\frac{x}{a} = \frac{1-x}{b}$ gives the abscissa x, and $\frac{(i+a)-(j-b)}{1} = \frac{y-(j-b)}{\frac{b}{a+b}}$ gives the ordinate y. ☐

Corollary 2. *The (a, b)-crossing $\mathcal{C}$ with $\gcd(a, b) = 1$ and the (a', b')-crossing $\mathcal{C}'$ have the same abscissa if and only if there exists $d \geq 1$ such that $a' = da$ and $b' = db$.*

Proof. If $\frac{a}{a+b} = \frac{a'}{a'+b'}$ then $ab' = ba'$. As $\gcd(a, b) = 1$, a divides a' and b divides b', so that there exist $d \geq 1$ and $e \geq 1$ such that $a' = da$, $b' = eb$. As $ab' = ba'$, we get $d = e$. The converse is obvious. ☐

Corollary 3. *When disregarding superimposition, the number of (a, b)-crossings is $(n-a)(n-b)$.*

Proof. From Proposition 1, i can take $n-a$ values and j can take $n-b$ values. ☐

Proposition 4. *Let $\mathcal{C} = \langle i, a|j, b\rangle$ be an (a, b)-crossing with $\gcd(a, b) = 1$, and $\mathcal{C}' = \langle i', da|j', db\rangle$ a (da, db)-crossing with $d \geq 1$. Then $\mathcal{C}$ and $\mathcal{C}'$ are superimposed if and only if there exists $k \in \mathbb{Z}$ such that $i' = i + ka$ and $j' = j - kb$.*

Proof. Crossings $\mathcal{C}$ and $\mathcal{C}'$ have the same abscissa by Corollary 2. By Proposition 1, they have the same ordinate if and only if $aj + bi = aj' + bi'$, or $a(j - j') = b(i' - i)$. As $\gcd(a, b) = 1$, the latter condition is equivalent to the existence of $k \in \mathbb{Z}$ such that $i' - i = ka$ and $j' - j = -kb$. ☐

Corollary 5. *If an (a, b)-crossings is simple then $\gcd(a, b) = 1$. Every multiple crossing contains an (a, b)-crossing such that $\gcd(a, b) = 1$.*

Proof. Assume that $\mathcal{C} = \langle i, a|j, b\rangle$ is simple and that there exist $d \geq 2$ and a', b' such that $a = da'$, $b = db'$. Then the (a', b')-crossing $\mathcal{C}' = \langle i', a'|j', b'\rangle$ with $i' = i$ and $j' = j$ is superimposed on $\mathcal{C}$ by Proposition 4, contradicting simplicity. Hence $\gcd(a, b) = 1$. Let $\mathcal{C}' = \langle i, a'|j, b'\rangle$ be an (a', b')-crossing contained in a multiple crossing. We set $d = \gcd(a', b')$, $a = a'/d$, and $b = b'/d$. Then the crossing $\mathcal{C} = \langle i, a|j, b\rangle$ has the desired properties by Proposition 4. □

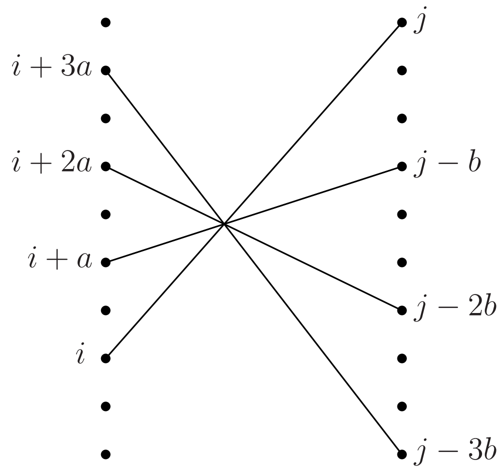

Figure 3: A multiple crossing.

Proposition 6. *Suppose $\mathcal{C}$ is an (a, b)-crossing with $\gcd(a, b) = 1$, superimposed to a multiple crossing $\mathcal{M}$ of multiplicity $m \geq 2$. Then $1 \leq ma \leq n-1$ and $1 \leq mb \leq n-1$. In particular, $2a \leq n-1$ and $2b \leq n-1$. Moreover, $\mathcal{M}$ is the superimposition of m (a, b)-crossings, $m-1$ $(2a, 2b)$-crossings, $m-2$ $(3a, 3b)$-crossings, ..., and a single (ma, mb)-crossing.*

Figure 3 displays a crossing of multiplicity $m = 3$. It is the superimposition of three $(2, 3)$-crossings, two $(4, 6)$-crossings and one $(6, 9)$-crossing.

Proof. Among the (a, b)-crossings contained in $\mathcal{M}$, we select $\langle i, a|j, b\rangle$ with $i + a \leq n$ and $1 \leq j-b$, such that i is minimum and j is maximum. For $k = 1, \ldots, m$, the m (a, b)-crossings $\langle i+(k-1)a, a|j-(k-1)b, b\rangle$ are superimposed to $\mathcal{M}$ by Proposition 4. For $k = m$, $i+ma \leq n$ gives $ma \leq n - i \leq n - 1$, while $1 \leq j - mb$ and $j \leq n$ give $mb \leq n - 1$. Moreover, for $k = 1, \ldots, m-1$, the $m-1$ $(2a, 2b)$-crossings $\langle i+(k-1)a, 2a|j-(k-1)b, 2b\rangle$ are superimposed to $\mathcal{M}$, ..., and the (ma, mb)-crossing $\langle i, ma|j, mb\rangle$ is superimposed to $\mathcal{M}$. □

Corollary 7. *The multiplicity of a (u, v)-crossing that is not superimposed to another (u, v)-crossing, is the greatest common divisor of u and v.*

Proof. This is a consequence of Corollary 5 and Proposition 6. □

Proposition 8. *The number of crossings of abscissa $\frac{a}{a+b}$ (or $\frac{b}{a+b}$) with $\gcd(a, b) = 1$ is*

$$\begin{array}{ll} (n-a)(n-b) - (n-2a)(n-2b), & \textit{if } 2a \leq n-1 \textit{ and } 2b \leq n-1; \\ (n-a)(n-b), & \textit{otherwise.} \end{array}$$

Proof. The crossings of abscissa $\frac{a}{a+b}$ are simple or multiple. If they are all simple, their number is $(n-a)(n-b)$ by Corollary 3, and this occurs when $2a > n-1$ or $2b > n-1$ by Proposition 6. If some crossings are simple and others are multiple, then $2a \le n-1$ and $2b \le n-1$ by Proposition 6. Each simple crossing is counted one time in the first term, and 0 times in the second term. Indeed, the second term is the number of $(2a, 2b)$-crossings (disregarding superimposition), and they are not simple by Corollary 5. By Proposition 6, each multiple crossing of multiplicity m contributes m to the first term and $m-1$ to the second term, hence contributes $m-(m-1) = 1$ to the tally. □

3 Number of crossings

We now state our main result.

Proposition 9. *The number $c(n)$ of crossings in a regular drawing of the complete bipartite graph $K_{n,n}$ is*

$$c(n) = \sum_{\substack{1\le a,b\le n-1\\ \gcd(a,b)=1}} (n-a)(n-b) - \sum_{\substack{1\le 2a,2b\le n-1\\ \gcd(a,b)=1}} (n-2a)(n-2b).$$

Proof. As each crossing in the drawing contains an (a, b)-crossing with $\gcd(a, b) = 1$ by Corollary 5, we count the number of crossings using Proposition 8. Summing over all (a, b) such that $\gcd(a, b) = 1$, within the bounds of validity, gives the result. □

An alternative expression for $c(n)$ has been proposed by Philippe Paclet [3]:

Proposition 10. *Let $f(i, j)$ be the number of irreducible fractions p/q with $1 \le p \le i$ and $1 \le q \le j$, and $f'(i, j)$ the number of rationals admitting at least one reducible form p/q with $1 \le p \le i$ and $1 \le q \le j$. Then*

$$c(n) = \sum_{1\le i,j\le n-1} (f(i,j) - f'(i,j)).$$

Proof. We denote

$$s(n) = \sum_{\substack{1\le a,b\le n-1\\ \gcd(a,b)=1}} (n-a)(n-b), \quad s'(n) = \sum_{\substack{1\le 2a,2b\le n-1\\ \gcd(a,b)=1}} (n-2a)(n-2b),$$

so that $c(n) = s(n) - s'(n)$. By the definition of f,

$$\sum_{1\le i,j\le n-1} f(i,j) = \sum_{1\le i,j\le n-1} \sum_{\substack{1\le a\le i\\ 1\le b\le j\\ \gcd(a,b)=1}} 1 = \sum_{\substack{1\le a,b\le n-1\\ \gcd(a,b)=1}} \sum_{\substack{a\le i\le n-1\\ b\le j\le n-1}} 1 = \sum_{\substack{1\le a,b\le n-1\\ \gcd(a,b)=1}} (n-a)(n-b) = s(n).$$

Similarly,

$$\sum_{1\le i,j\le n-1} f'(i,j) = \sum_{\substack{1\le c,d\le n-1\\ \gcd(c,d)\ne 1}} \sum_{\substack{c\le i\le n-1\\ d\le j\le n-1}} 1.$$

The set $\{(c,d); 1 \le c,d \le n-1, \gcd(c,d) \ne 1\}$ is identical to the set $\{(a,b); 1 \le 2a, 2b \le n-1, \gcd(a,b)=1\}$. Indeed, $\gcd(c,d) \ne 1$ is equivalent to the existence of $m \ge 2$ such that $c = ma$, $d = mb$, with $\gcd(a,b) = 1$, $2a \le ma \le n-1$, $2b \le mb \le n-1$. We obtain

$$\sum_{1 \le i,j \le n-1} f'(i,j) = \sum_{\substack{1 \le 2a, 2b \le n-1 \\ \gcd(a,b)=1}} (n-2a)(n-2b) = s'(n).$$

$\square$

Sequence $s(n)$ is A115004 in Sloane [4]. Values of $c(n)$ are given in Table 1.

n	$s(n)$	$s'(n)$	$c(n)$	$d(n)$
1	-	-	0	0
2	1	0	1	1
3	8	1	7	9
4	31	4	27	36
5	80	15	65	100
6	179	32	147	225
7	332	71	261	441
8	585	124	461	784
9	948	211	737	1296
10	1463	320	1143	2025
11	2136	499	1637	3025
12	3065	716	2349	4356
13	4216	999	3217	6084
14	5729	1328	4401	8281
15	7568	1799	5769	11025
16	9797	2340	7457	14400
17	12456	3023	9433	18496
18	15737	3792	11945	23409
...	...	...	...	...
50	948514	235680	712835	1600625
...	...	...	...	...
100	15189547	3794060	11395487	24502500

Table 1: The number of crossings $c(n) = s(n) - s'(n)$ in the regular $K_{n,n}$ pattern. The sequence of triangular numbers squared, $d(n)$, enumerates the crossings when disregarding multiplicity.

4 Asymptotics

When disregarding multiplicity, the number of crossings in the $K_{n,n}$ pattern is

$$d(n) = \binom{n}{2}\binom{n}{2} = \frac{n^2(n-1)^2}{4},$$

the square of the nth triangular number (A000537).

Proposition 11. *For large n, $c(n) \sim \frac{9}{2\pi^2} d(n)$.*

Proof. We write

$$\sum_{1 \le a,b \le n-1} (n-a)(n-b) = n^2 \sum 1 - n \sum (a+b) + \sum ab$$

with no condition of relative primality under the sums. We have

$$\sum (n-a)(n-b) = \sum ab = \frac{n^2(n-1)^2}{4} = d(n).$$

Hence $n^2 \sum 1 - n \sum (a+b) = 0$. We now sum with the condition $\gcd(a,b) = 1$. As the probability that two positive integers are relatively prime is $\frac{6}{\pi^2}$, for large n:

$$\sum_{\substack{1 \le a,b \le n-1 \\ \gcd(a,b)=1}} ab \sim \frac{6}{\pi^2} d(n), \text{ and } n^2 \sum_{\substack{1 \le a,b \le n-1 \\ \gcd(a,b)=1}} 1 - n \sum_{\substack{1 \le a,b \le n-1 \\ \gcd(a,b)=1}} (a+b) \sim 0.$$

Hence

$$s(n) \sim \frac{6}{\pi^2} d(n).$$

Let $m = \lfloor \frac{n+1}{2} \rfloor$, then

$$\sum_{1 \le a,b \le m-1} (n-2a)(n-2b) = n^2(m-1)^2 - 2mn(m-1)^2 + 4\frac{m^2(m-1)^2}{4}.$$

We sum with the condition $\gcd(a,b) = 1$. For large n, $m \sim \frac{n}{2}$. As in the previous computation, the first two terms ~ 0, and the last term $s'(n) \sim \frac{6}{\pi^2} 4d(\frac{n}{2}) \sim \frac{6}{\pi^2} 4\frac{d(n)}{16}$. We obtain

$$s'(n) \sim \frac{1}{4} s(n),$$

and $c(n) = s(n) - s'(n) \sim \frac{9}{2\pi^2} d(n)$. □

The equivalence $c(n) \sim \frac{9}{8\pi^2} n^4$, deduced from Proposition 11, appears to give a better estimate of $c(n)$. However, the approximation $\pi \approx \frac{3}{2} \frac{n^2}{\sqrt{2c(n)}}$ is not close; e.g., for $n = 100$, the approximation is 3.1420.

5 Concluding remarks

The regular drawing of $K_{n,n}$ can be considered an analogic device to compute the greatest common divisor of two positive integers a and b. In the drawing, take a units on the left, and b units on the right. Consider the corresponding (a,b)-crossing ($i = 0$, $j = b$). Then the multiplicity m of this crossing is the greatest common divisor of a and b (Corollary 7).

It can be noted that the abscissas of the (a,b)-crossings are Farey fractions. This suggests that number theoretical properties of the complete bipartite graph pattern deserve further exploration.

6 Acknowledgements

The referee is gratefully acknowledged for constructive suggestions that improved the manuscript.

References

[1] Athanasius Kircher, *Ars Magna Sciendi, XII Libros Digesta, qua Nova et Universali Methodo per Artificiosum Combinationum Contextum de Omni Re Proposita Plurimis et Prope Infinitis Rationibus Disputari, Omniumque Summaria Quaedam Cognitio Comparari Potest*, Apud Joannem Janssonium a Waesberge and Viduam Elizei Weyerstraet, Amsterdam, 1669.

[2] Umberto Ecco, *Foucault's Pendulum*, Harcourt Brace Jovanovich, San Diego, 1989, p. 473.

[3] Philippe Paclet, personnal communication, February 20, 2004.

[4] Neil Sloane, The On-Line Encyclopedia of Integer Sequences, `http://www.research.att.com/~njas/sequences`.

2000 *Mathematics Subject Classification*: Primary 05C62; Secondary 11A05.
Keywords: complete bipartite graph, greatest common divisor.

(Concerned with sequences A000537, A115004, and A159065.)

Received April 20 2009; revised version received July 10 2009. Published in *Journal of Integer Sequences*, July 10 2009.

Return to Journal of Integer Sequences home page.