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Unisequences and Nearest Integer Continued Fraction Midpoint Criteria for Pell's Equation

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Abstract

The nearest integer continued fractions of Hurwitz, Minnegerode (NICF-H) and in Perron's book *Die Lehre von den Kettenbrüchen* (NICF-P) are closely related. Midpoint criteria for solving Pell's equation $x^2 - Dy^2 = \pm 1$ in terms of the NICF-H expansion of $\sqrt{D}$ were derived by H. C. Williams using singular continued fractions. We derive these criteria without the use of singular continued fractions. We use an algorithm for converting the regular continued fraction expansion of $\sqrt{D}$ to its NICF-P expansion.

1 Introduction

In Perron's book [7, p. 143], a *nearest integer* continued fraction (Kettenbruch nach nächsten Ganzen) expansion (NICF-P) of an irrational number ξ_0 is defined recursively by

$$\xi_n = q_n + \frac{\epsilon_{n+1}}{\xi_{n+1}}, -\tfrac{1}{2} < \xi_n - q_n < \tfrac{1}{2}, \tag{1}$$

where $\epsilon_{n+1} = \pm 1$, q_n is an integer (the nearest integer to ξ_n) and $\operatorname{sign}(\epsilon_{n+1}) = \operatorname{sign}(\xi_n - q_n)$. Then we have the expansion

$$\xi_0 = q_0 + \frac{\epsilon_1|}{|q_1} + \cdots + \frac{\epsilon_n|}{|q_n} + \cdots \tag{2}$$

where

$$q_n \geq 2,\ q_n + \epsilon_{n+1} \geq 2 \text{ for } n \geq 1. \tag{3}$$

(Satz 10, [7, p. 169]).

A. Hurwitz [1] and B. Minnegerode [6] defined a related nearest integer continued fraction (NICF-H) by

$$\xi_n' = q_n' - \frac{1}{\xi_{n+1}'}, -\tfrac{1}{2} < \xi_n' - q_n' < \tfrac{1}{2}, \tag{4}$$

where q_n' is an integer. Then

$$\xi_0' = q_0' - \frac{1|}{|q_1'} - \cdots - \frac{1|}{|q_n'} - \cdots \tag{5}$$

and we have $|q_n'| \geq 2$ for $n \geq 1$. Also if one of $q_1', q_2', \ldots$, say q_n', equals 2 (resp. -2), then $q_{n+1}' < 0$ (resp. $q_{n+1}' > 0$) (Hurwitz [1, p. 372]). Section 2 relates the two types of continued fraction.

In 1980, H. C. Williams gave six midpoint criteria for solving Pell's equation $x^2 - Dy^2 = \pm 1$ in terms of the NICF-H expansion of $\sqrt{D}$ (see Theorems 6 and 7, [9, pp. 12–13]). His proof made extensive use of the singular continued fraction expansion of $\sqrt{D}$. Theorem 6 of section 3 of our paper gives the corresponding criteria for the NICF-P expansion of $\sqrt{D}$. In an attempt to give a derivation of the latter criteria without the use of singular continued fractions, the author studied the conversion of the regular continued fraction (RCF) expansion of $\sqrt{D}$ to the NICF given by Lemma 9 of section 5, where the RCF is defined recursively by

$$\xi_n = a_n + \frac{1}{\xi_{n+1}},$$

with $a_n = \lfloor \xi_n \rfloor$, the integer part of ξ_n. Theorem 8 of section 4 shows that the central part of a least period determines which of the criteria hold. Finally, Theorem 18, section 7, describes the case where there are only odd-length *unisequences*, i.e., consecutive sequences of partial quotients equal to 1, in the RCF expansion of $\sqrt{D}$; in this case the NICF-P expansion of $\sqrt{D}$ exhibits the usual symmetry properties of the RCF expansion.

2 Connections between the NICF-H and NICF-P expansions of an irrational number.

Lemma 1. *Let $q_n', \xi_n', A_n'/B_n'$ denote the n-th partial denominator, complete quotient and convergent of the* NICF-H *expansion of an irrational number ξ_0 and $q_n, \epsilon_n, \xi_n, A_n/B_n$ denote* *the n-th partial denominator, partial numerator, complete quotient and convergent of the* NICF-P *expansion of ξ_0, where*

$$A_{-1} = 1 = A'_{-1}, \quad B_{-1} = 0 = B'_{-1},$$
$$A_{-2} = 0 = A'_{-2}, \quad B_{-2} = 1 = -B'_{-2}.$$

and for $n \geq -1$,

$$A_{n+1} = q_{n+1}A_n + \epsilon_{n+1}A_{n-1}, \quad B_{n+1} = q_{n+1}B_n + \epsilon_{n+1}B_{n-1}$$
$$A'_{n+1} = q'_{n+1}A'_n - A'_{n-1}, \quad B'_{n+1} = q'_{n+1}B'_n - B'_{n-1},$$

where $\epsilon_0 = 1$. Then

$$q'_n = t_n q_n, \quad \xi'_n = t_n \xi_n, \quad n \geq 0, \tag{6}$$

where $t_0 = 1$ and $t_n = (-1)^n \epsilon_1 \cdots \epsilon_n$, if $n \geq 1$.

$$A'_n = s_n A_n, \quad B'_n = s_n B_n, \quad n \geq -2, \tag{7}$$

where $s_{-2} = -1, s_{-1} = 1$ and $s_{n+1} = -s_{n-1}\epsilon_{n+1}$ for $n \geq -1$.

Remark 2. It follows that $s_0 = 1$ and

$$s_{2i} = (-1)^i \epsilon_{2i}\epsilon_{2i-2} \cdots \epsilon_2, \quad \text{if } i \geq 1, \tag{8}$$
$$s_{2i+1} = (-1)^{i+1} \epsilon_{2i+1}\epsilon_{2i-1} \cdots \epsilon_1, \quad \text{if } i \geq 0, \tag{9}$$
$$s_{n+1}s_n = t_{n+1}, \quad \text{if } n \geq -1. \tag{10}$$

Proof. We prove (6) by induction on $n \geq 0$. These are true when $n = 0$. So we assume that $n \geq 0$ and (6) hold. Then

$$\xi'_{n+1} = \frac{1}{q'_n - \xi'_n}, \quad \xi_{n+1} = \frac{\epsilon_{n+1}}{q_n - \xi_n}, \tag{11}$$
$$q'_n = [\xi'_n], \quad q_n = [\xi_n], \tag{12}$$

where $[x]$ denotes the nearest integer to x. Then

$$\begin{aligned} \xi'_{n+1} &= \frac{1}{t_n q_n - t_n \xi_n} \\ &= \frac{t_n}{q_n - \xi_n} = t_n(-\epsilon_{n+1}\xi_{n+1}) \\ &= t_{n+1}\xi_{n+1}. \end{aligned}$$

Next,

$$q'_{n+1} = [\xi'_{n+1}] = [t_{n+1}\xi_{n+1}] = t_{n+1}[\xi_{n+1}] = t_{n+1}q_{n+1}.$$

Finally, we prove (7) by induction on $n \geq -2$. These hold for $n = -2$ and -1. So we assume $n \geq -1$ and

$$A'_{n-1} = s_{n-1}A_{n-1}, \quad B'_{n-1} = s_{n-1}B_{n-1}, \quad A'_n = s_nA_n, \quad B'_n = s_nB_n.$$

Then

$$\begin{aligned} A'_{n+1} &= q'_{n+1}A'_n - A'_{n-1} \\ &= (t_{n+1}q_{n+1})(s_nA_n) - s_{n-1}A_{n-1} \\ &= q_{n+1}s_{n+1}A_n - (-s_{n+1}\epsilon_{n+1})A_{n-1} \\ &= s_{n+1}(q_{n+1}A_n + \epsilon_{n+1}A_{n-1}) \\ &= s_{n+1}A_{n+1}. \end{aligned}$$

Similarly $B'_{n+1} = s_{n+1}B_{n+1}$. ☐

Corollary 3. *Suppose $\xi_n, \ldots, \xi_{n+k-1}$ is a least period of* NICF-P *complete quotients for a quadratic irrational ξ_0.*

(a) If $\epsilon_{n+1} \cdots \epsilon_{n+k} = (-1)^k$, then

$$\xi'_n, \ldots, \xi'_{n+k-1}$$

is a least period of NICF-H *complete quotients for ξ_0.*

(b) If $\epsilon_{n+1} \cdots \epsilon_{n+k} = (-1)^{k+1}$, then

$$\xi'_n, \ldots, \xi'_{n+k-1}, -\xi'_n, \ldots, -\xi'_{n+k-1} \tag{13}$$

is a least period of NICF-H *complete quotients for ξ_0. Moreover*

$$\xi_0 = q'_0 - \frac{1|}{|q'_1} - \cdots - \frac{1|}{|\underset{*}{q'_n}} - \cdots - \frac{1|}{|q'_{n+k-1}} - \frac{1|}{|-q'_n} - \cdots - \frac{1|}{|\underset{*}{-q'_{n+k-1}}},$$

where the asterisks correspond to the least period (13).

Proof. Suppose $\xi_n, \ldots, \xi_{n+k-1}$ is a least period of NICF-P complete quotients for ξ_0. Then $\xi_n = \xi_{n+k}$. Hence from (6),

$$\begin{aligned} t_n\xi'_n &= t_{n+k}\xi'_{n+k} \\ (-1)^n\epsilon_1 \cdots \epsilon_n\xi'_n &= (-1)^{n+k}\epsilon_1 \cdots \epsilon_{n+k}\xi'_{n+k} \\ \xi'_n &= (-1)^k\epsilon_{n+1} \cdots \epsilon_{n+k}\xi'_{n+k}. \end{aligned} \tag{14}$$

(a) Suppose $\epsilon_{n+1} \cdots \epsilon_{n+k} = (-1)^k$. Then (14) gives

$$\xi'_n = \xi'_{n+k}.$$

Then because $\xi'_n, \ldots, \xi'_{n+k-1}$ are distinct, they form a least period of complete quotients for the NICF-H expansion of ξ_0.

(b) Suppose $\epsilon_{n+1}\cdots\epsilon_{n+k} = (-1)^{k+1}$. Then (14) gives

$$\xi'_n = -\xi'_{n+k}.$$

Similarly

$$\xi'_{n+1} = -\xi'_{n+k+1}, \ldots, \xi'_{n+k-1} = -\xi'_{n+2k-1}.$$

Also $\xi'_n = -\xi'_{n+k} = -(-\xi'_{n+2k}) = \xi'_{n+2k}$. Hence

$$\xi'_n, \ldots, \xi'_{n+k-1}, \xi'_{n+k}, \ldots, \xi'_{n+2k-1} \tag{15}$$

form a period of complete quotients for the NICF-H expansion of ξ_0. However sequence (15) is identical with

$$\xi'_n, \ldots, \xi'_{n+k-1}, -\xi'_n, \ldots, -\xi'_{n+k-1},$$

whose members are distinct. Hence (15) form a least period of complete quotients for the NICF-H expansion of ξ_0. □

Corollary 4. *Let k and p be the period-lengths of the* NICF-P *and* RCF *expansions of a quadratic irrational ξ_0 not equivalent to $(1+\sqrt{5})/2$. Then*

(a) if p is even, the period-length of the NICF-H *expansion of ξ_0 is equal to k;*

(b) if p is odd, the period-length of the NICF-H *expansion of ξ_0 is equal to $2k$ and the NICF-H expansion has the form*

$$\xi_0 = q'_0 - \frac{1}{\left\lfloor q'_1 \right.} - \cdots - \frac{1}{\left\lfloor \underset{*}{q'_n} \right.} - \cdots - \frac{1}{\left\lfloor q'_{n+k-1} \right.} - \frac{1}{\left\lfloor -q'_n \right.} - \cdots - \frac{1}{\left\lfloor \underset{*}{-q'_{n+k-1}} \right.}.$$

Proof. Let $\xi_n, \ldots, \xi_{n+k-1}$ be a least period of NICF-P complete quotients. Suppose that r of the partial numerators $\epsilon_{n+1}, \ldots, \epsilon_{n+k}$ of the NICF-P expansion of ξ_0 are equal to -1. Now by Theorem 4 of Matthews and Robertson [5], $p = k + r$ and hence

$$\epsilon_{n+1}\cdots\epsilon_{n+k} = (-1)^r = (-1)^{k+p}.$$

Then according as p is even or odd, $\epsilon_{n+1}\cdots\epsilon_{n+k} = (-1)^k$ or $(-1)^{k+1}$ and Corollary 3 applies. □

Remark 5. This result was obtained by Hurwitz and Minnegerode for the special case $\xi_0 = \sqrt{D}$.

We give some examples.

(1) $\xi_0 = (12 + \sqrt{1792})/16$, (Tables 1 and 2). Here $k = 4, r = 2$, $\xi_1, \xi_2, \xi_3, \xi_4$ form a period of NICF-P complete quotients, $\epsilon_2\epsilon_3\epsilon_4\epsilon_5 = (-1)(1)(-1)(1) = 1 = (-1)^k$ and the NICF-P and NICF-H expansions have the same period-length. Also $p = 6$.

(2) $\xi_0 = (5 + \sqrt{13})/4$, (Table 3 and 4). Here $k = 3, r = 2$ and ξ_0, ξ_1, ξ_2 form a period of NICF-P complete quotients, $\epsilon_1\epsilon_2\epsilon_3 = (1)(-1)(-1) = 1 = (-1)^{k+1}$ and the NICF-H period-length is twice the NICF-P period-length. Also $p = 5$.

Table 1: NICF-P expansion for $(12+\sqrt{1792})/16$

i	ξ_i	ϵ_i	b_i	A_i/B_i
0	$\frac{12+\sqrt{1792}}{16}$	1	3	3/1
1	$\frac{36+\sqrt{1792}}{31}$	1	3	10/3
2	$\frac{57+\sqrt{1792}}{47}$	-1	2	17/5
3	$\frac{37+\sqrt{1792}}{9}$	1	9	163/48
4	$\frac{44+\sqrt{1792}}{16}$	-1	5	798/235
5	$\frac{36+\sqrt{1792}}{31}$	1	3	2557/753

Table 2: NICF-H expansion for $(12+\sqrt{1792})/16$

i	ξ_i	ϵ_i	b_i	A_i/B_i
0	$\frac{12+\sqrt{1792}}{16}$	-1	3	3/1
1	$\frac{36+\sqrt{1792}}{-31}$	-1	-3	$-10/-3$
2	$\frac{57+\sqrt{1792}}{-47}$	-1	-2	17/5
3	$\frac{37+\sqrt{1792}}{9}$	-1	9	163/48
4	$\frac{44+\sqrt{1792}}{16}$	-1	5	798/235
5	$\frac{36+\sqrt{1792}}{-31}$	-1	-3	$-2557/-753$

3 NICF-P midpoint criteria for Pell's equation

Theorem 6. *Let k and p be the respective period-lengths of NICF-P and RCF expansions of $\sqrt{D}$. Then precisely one of the following must hold for the NICF-P expansion of $\sqrt{D}$:*

1) $P_\rho = P_{\rho+1}$, $k = 2\rho, p = 2h$. *Then*

$$A_{k-1} = B_{\rho-1}A_\rho + \epsilon_\rho A_{\rho-1}B_{\rho-2},$$
$$B_{k-1} = B_{\rho-1}(B_\rho + \epsilon_\rho B_{\rho-2}).$$

2) $P_{\rho+1} = P_\rho + Q_\rho$, $k = 2\rho, p = 2h$. *Then*

$$A_{k-1} = B_{\rho-1}A_\rho + A_{\rho-1}B_{\rho-2} - A_{\rho-1}B_{\rho-1},$$
$$B_{k-1} = B_{\rho-1}(B_\rho + B_{\rho-2} - B_{\rho-1}).$$

3) $Q_\rho = Q_{\rho+1}$ *and*

$$\begin{array}{ll}\text{(a)} & \epsilon_{\rho+1} = -1, k = 2\rho+1, p = 2h, \ \textit{or} \\ \text{(b)} & \epsilon_{\rho+1} = 1, k = 2\rho+1, p = 2h-1.\end{array}$$

Then

$$A_{k-1} = A_\rho B_\rho + \epsilon_{\rho+1}A_{\rho-1}B_{\rho-1},$$
$$B_{k-1} = B_\rho^2 + \epsilon_{\rho+1}B_{\rho-1}^2.$$

Table 3: NICF-P expansion for $(5+\sqrt{13})/4$

i	ξ_i	ϵ_i	b_i	A_i/B_i
0	$\frac{5+\sqrt{13}}{4}$	1	2	$2/1$
1	$\frac{3+\sqrt{13}}{1}$	1	7	$15/7$
2	$\frac{4+\sqrt{13}}{3}$	-1	3	$43/20$
3	$\frac{5+\sqrt{13}}{4}$	-1	2	$71/33$

Table 4: NICF-H expansion for $(5+\sqrt{13})/4$

i	ξ_i	ϵ_i	b_i	A_i/B_i
0	$\frac{5+\sqrt{13}}{4}$	1	2	$2/1$
1	$\frac{3+\sqrt{13}}{-1}$	-1	-7	$-15/-7$
2	$\frac{4+\sqrt{13}}{-3}$	-1	-3	$43/20$
3	$\frac{5+\sqrt{13}}{-4}$	-1	-2	$-71/-33$
4	$\frac{3+\sqrt{13}}{1}$	-1	7	$-540/-251$
5	$\frac{4+\sqrt{13}}{3}$	-1	3	$-1549/-720$
6	$\frac{5+\sqrt{13}}{4}$	-1	2	$-2558/-1189$

4) $P_{\rho+1} = Q_\rho + \frac{1}{2}Q_{\rho+1}, \epsilon_{\rho+1} = -1, k = 2\rho+1, p = 2h-1$. *Then*

$$\begin{aligned} A_{k-1} &= A_\rho B_\rho + 2A_{\rho-1}B_{\rho-1} - (A_\rho B_{\rho-1} + B_\rho A_{\rho-1}),\\ B_{k-1} &= B_\rho^2 + 2B_{\rho-1}^2 - 2B_\rho B_{\rho-1}.\end{aligned}$$

5) $P_\rho = Q_\rho + \frac{1}{2}Q_{\rho-1}, \epsilon_\rho = -1, k = 2\rho, p = 2h-1$. *Then*

$$\begin{aligned} A_{k-1} &= 2A_{\rho-1}B_{\rho-1} + A_{\rho-2}B_{\rho-2} - (A_{\rho-1}B_{\rho-2} + B_{\rho-1}A_{\rho-2})\\ B_{k-1} &= 2B_{\rho-1}^2 + B_{\rho-2}^2 - 2B_{\rho-1}B_{\rho-2}.\end{aligned}$$

Proof. We only exhibit the calculations for criterion 1) of Theorem 6. This corresponds to criterion 1) of Theorem 6, Williams [9, p. 12], which states that $P'_\rho = P'_{\rho+1}$, $k = 2\rho$, $p = 2h$ and

$$|A'_{k-1}| = |B'_{\rho-1}A'_\rho - A'_{\rho-1}B'_{\rho-2}|, \tag{16}$$
$$|B'_{k-1}| = |B'_{\rho-1}(B'_\rho - B'_{\rho-2})|. \tag{17}$$

Then from equations (7),

$$\begin{aligned} B'_{\rho-1}A'_\rho - A'_{\rho-1}B'_{\rho-2} &= s_{\rho-1}s_\rho B_{\rho-1}A_\rho - s_{\rho-1}s_{\rho-2}A_{\rho-1}B_{\rho-2}\\ &= t_\rho B_{\rho-1}A_\rho - t_{\rho-1}A_{\rho-1}B_{\rho-2}\\ &= -t_{\rho-1}\epsilon_\rho B_{\rho-1}A_\rho - t_{\rho-1}A_{\rho-1}B_{\rho-2}\\ &= -t_{\rho-1}\epsilon_\rho(B_{\rho-1}A_\rho + \epsilon_\rho A_{\rho-1}B_{\rho-2}).\end{aligned}$$

Hence

$$|B'_{\rho-1}A'_\rho - A'_{\rho-1}B'_{\rho-2}| = B_{\rho-1}A_\rho + \epsilon_\rho A_{\rho-1}B_{\rho-2} \tag{18}$$

as $B_{\rho-1}A_\rho \geq A_{\rho-1}B_{\rho-2}$. Hence (16) and (18) give the first result of criterion 1) above. Next,

$$\begin{aligned}
B'_{\rho-1}(B'_\rho - B'_{\rho-2}) &= s_{\rho-1}B_{\rho-1}(s_\rho B_\rho - s_{\rho-2}B_{\rho-2}) \\
&= B_{\rho-1}(t_\rho B_\rho - t_{\rho-1}B_{\rho-2}) \\
&= B_{\rho-1}(-t_{\rho-1}\epsilon_\rho B_\rho - t_{\rho-1}B_{\rho-2}) \\
&= -t_{\rho-1}\epsilon_\rho B_{\rho-1}(B_\rho + \epsilon_\rho B_{\rho-2}).
\end{aligned}$$

Hence

$$|B'_{\rho-1}(B'_\rho - B'_{\rho-2})| = B_{\rho-1}(B_\rho + \epsilon_\rho B_{\rho-2}) \tag{19}$$

and (17) and (19) give the second result of criterion 1) above. □

Remark 7. John Robertson in an email to the author, dated November 26, 2007, noted the following errors in Williams [9, pp. 12–13]:

(i) Criterion 3), Theorem 6, page 12, should be

$$\begin{aligned}
|A'_{\pi-1}| &= A'_\rho B'_\rho - A'_{\rho-1}B'_{\rho-1}, \\
|B'_{\pi-1}| &= {B'}_\rho^2 - {B'}_{\rho-1}^2.
\end{aligned}$$

(ii) Criterion 6), Theorem 7, page 13, should be

$$\begin{aligned}
|A'_{\pi-1}| &= 2A'_{\rho-1}B'_{\rho-1} + A'_{\rho-2}B'_{\rho-2} - |A'_{\rho-1}B'_{\rho-2} + B'_{\rho-1}A'_{\rho-2}|, \\
|B'_{\pi-1}| &= 2{B'}_{\rho-1}^2 + {B'}_{\rho-2}^2 - 2|B'_{\rho-2}B'_{\rho-1}|.
\end{aligned}$$

4 Midpoint criteria in terms of unisequences

The RCF of $\sqrt{D}$, with period-length p, has the form

$$\sqrt{D} = \begin{cases} [a_0, \overline{a_1, \ldots, a_{h-1}, a_{h-1} \ldots, a_1, 2a_0}] & \text{if } p = 2h-1; \\ [a_0, \overline{a_1, \ldots, a_{h-1}, a_h, a_{h-1}, \ldots, a_1, 2a_0}] & \text{if } p = 2h. \end{cases} \tag{20}$$

We have Euler's midpoint formulae for solving Pell's equation $x^2 - Dy^2 = \pm 1$ using the regular continued fraction (see Dickson [3, p. 358]):

$$\begin{aligned}
Q_{h-1} &= Q_h, \\
A_{2h-2} &= A_{h-1}B_{h-1} + A_{h-2}B_{h-2}, \\
B_{2h-2} &= B_{h-1}^2 + B_{h-2}^2,
\end{aligned}$$

if $p = 2h - 1$;

$$\begin{aligned}
P_h &= P_{h+1}, \\
A_{2h-1} &= A_h B_{h-1} + A_{h-1}B_{h-2}, \\
B_{2h-1} &= B_{h-1}(B_h + B_{h-2}),
\end{aligned}$$

if $p = 2h$. We also need the following symmetry properties from Perron [7, p. 81]:

$$a_t = a_{p-t}, \quad t = 1, 2, \ldots, p-1, \tag{21}$$
$$P_{t+1} = P_{p-t}, \quad t = 0, 1, \ldots, p-1, \tag{22}$$
$$Q_t = Q_{p-t}, \quad t = 0, 1, \ldots, p. \tag{23}$$

Theorem 8. *Using the notation of* (20), *in relation to Theorem* 6, *we have*

(1) If $p = 2h - 1, h > 1$ *and* $a_{h-1} > 1$, *or* $p = 1$, *we get* criterion 3).

(2) If $p = 2h, h > 1$ *and* $a_{h-1} > 1, a_h > 1$, *or* $p = 2$ *and* $a_1 > 1$, *we get* criterion 1).

(3) Suppose $p = 2h$ *and* $a_{h-1} = 1, a_h > 1$, *so that* a_h *is enclosed by two* M*-unisequences. Then*

  (a) if $M \geq 2$ *is even, we get* criterion 2).

  (b) if M *is odd, we get* criterion 1).

(4) Suppose the centre of a period contains an M*-unisequence,* $M \geq 1$.

  (a) If M *is odd, then* $p = 2h$ *and we get* criterion 1) *if* $M = 4t + 3$, criterion 3) *if* $M = 4t + 1$.

  (b) If M *is even, then* $p = 2h - 1$ *and we get* criterion 4) *if* $M = 4t$, criterion 5) *if* $M = 4t + 2$.

Before we can prove Theorem 8, we need some results on the RCF to NICF-P conversion.

5 The RCF to NICF-P conversion and its properties

Lemma 9. *Let* $\xi_0 = \frac{P_0+\sqrt{D}}{Q_0}$ *have NICF-P and RCF expansions:*

$$\xi_0 = a'_0 + \frac{\epsilon_1|}{|a'_1} + \cdots = a_0 + \frac{1|}{|a_1} + \cdots,$$

with complete quotients ξ'_m, ξ_m, *respectively. Define* $f(m)$ *recursively for* $m \geq 0$ *by* $f(0) = 0$ *and*

$$f(m+1) = \begin{cases} f(m) + 1, & \text{if } \epsilon_{m+1} = 1; \\ f(m) + 2, & \text{if } \epsilon_{m+1} = -1. \end{cases} \tag{24}$$

Then for $m \geq 0$,

$$\epsilon_{m+1} = \begin{cases} 1, & \text{if } a_{f(m)+1} > 1; \\ -1, & \text{if } a_{f(m)+1} = 1, \end{cases} \tag{25}$$

$$\xi'_m = \begin{cases} \xi_{f(m)}, & \text{if } \epsilon_m = 1; \\ \xi_{f(m)} + 1, & \text{if } \epsilon_m = -1, \end{cases} \tag{26}$$

$$a'_m = \begin{cases} a_{f(m)}, & \textit{if } \epsilon_m = 1 \textit{ and } \epsilon_{m+1} = 1; \\ a_{f(m)} + 1, & \textit{if } \epsilon_m \epsilon_{m+1} = -1; \\ a_{f(m)} + 2, & \textit{if } \epsilon_m = -1 \textit{ and } \epsilon_{m+1} = -1. \end{cases} \tag{27}$$

Proof. See Theorem 2, Matthews and Robertson [5]. □

Remark 10. By virtue of (24) and (26), we say that the ξ'_m are obtained from the ξ_n in *jumps* of 1 or 2.

Lemma 11. Let $\xi_0 = (a_0, a_1, \ldots)$ be an RCF expansion. Then if $[x]$ denotes the nearest integer to x, we have

$$[\xi_n] = \begin{cases} a_n, & \text{if } a_{n+1} > 1; \\ a_n + 1, & \text{if } a_{n+1} = 1. \end{cases}$$

Proof. If $[\xi_n] = a_n + 1$, then $\xi_n > a_n + \frac{1}{2}$ and hence $a_{n+1} = 1$, whereas if $[\xi_n] = a_n$, then $\xi_n < a_n + \frac{1}{2}$ and hence $a_{n+1} > 1$. □

Lemma 12. *Let* $\xi_0 = \frac{P_0 + \sqrt{D}}{Q_0}$ *have NICF-P expansion*

$$\xi_0 = a'_0 + \frac{\epsilon_1|}{|a'_1} + \cdots .$$

Then

$$A'_m = \begin{cases} A_{f(m)} & \textit{if } \epsilon_{m+1} = 1; \\ A_{f(m)+1} & \textit{if } \epsilon_{m+1} = -1, \end{cases} \tag{28}$$

where $f(m)$ *is defined by* (24)*. Equivalently, in the notation of Bosma [2, p. 372], if* $n(k) = f(k+1) - 1$ *for* $k \geq -1$*, then*

$$n(k) = \begin{cases} n(k-1) + 1, & \textit{if } \epsilon_{k+1} = 1; \\ n(k-1) + 2, & \textit{if } \epsilon_{k+1} = -1 \end{cases} \tag{29}$$

and (28) *has the simpler form*

$$A'_k = A_{n(k)} \textit{ for } k \geq 0. \tag{30}$$

Remark 13. From (25) and (29), we see that $\epsilon_{m+1} = -1$ implies $a_{n(m)} = 1$.

Proof. (by induction). We first prove (30) for $k = 0$. We use Lemma 11.

$$\begin{aligned} \epsilon_1 = 1 \implies a_1 > 1 &\implies [\xi_0] = a_0 \\ &\implies A'_0 = A_0. \\ \epsilon_1 = -1 \implies a_1 = 1 &\implies [\xi_0] = a_0 + 1 = a_0 a_1 + 1 \\ &\implies A'_0 = A_1. \end{aligned}$$

We next prove (30) for $k = 1$. We have to prove

$$A_1' = \begin{cases} A_{f(1)} & \text{if } \epsilon_2 = 1; \\ A_{f(1)+1} & \text{if } \epsilon_2 = -1, \end{cases}$$

where

$$f(1) = \begin{cases} 1 & \text{if } \epsilon_1 = 1; \\ 2 & \text{if } \epsilon_1 = -1. \end{cases}$$

i.e.,

$$A_1' = \begin{cases} A_1 & \text{if } \epsilon_1 = 1, \epsilon_2 = 1; \\ A_2 & \text{if } \epsilon_1\epsilon_2 = -1; \\ A_3 & \text{if } \epsilon_1 = -1, \epsilon_2 = -1. \end{cases}$$

Now $A_1' = a_0'a_1' + \epsilon_1$. We have

$$a_0' = \begin{cases} a_0 & \text{if } \epsilon_1 = 1; \\ a_0 + 1 & \text{if } \epsilon_1 = -1 \end{cases}$$

and

$$a_1' = \begin{cases} a_{f(1)} & \text{if } \epsilon_1 = 1 = \epsilon_2; \\ a_{f(1)} + 1 & \text{if } \epsilon_1\epsilon_2 = -1; \\ a_{f(1)} + 2 & \text{if } \epsilon_1 = -1 = \epsilon_2. \end{cases}$$

Hence

$$a_1' = \begin{cases} a_1 & \text{if } \epsilon_1 = 1 = \epsilon_2; \\ a_1 + 1 & \text{if } \epsilon_1 = 1, \epsilon_2 = -1; \\ a_2 + 1 & \text{if } \epsilon_1 = -1, \epsilon_2 = 1; \\ a_2 + 2 & \text{if } \epsilon_1 = -1, \epsilon_2 = -1. \end{cases}$$

Case 1. $\epsilon_1 = 1 = \epsilon_2$. Then $A_1' = a_0a_1 + 1 = A_1$.
Case 2. $\epsilon_1 = 1, \epsilon_2 = -1$. Then $a_2 = 1$ and

$$\begin{aligned} A_1' &= a_0(a_1 + 1) + 1, \\ A_2 &= (a_0a_1 + 1)a_2 + a_0 = a_0a_1 + 1 + a_0 = A_1'. \end{aligned}$$

Case 3. $\epsilon_1 = -1, \epsilon_2 = 1$. Then $a_1 = 1$ and

$$\begin{aligned} A_1' &= (a_0 + 1)(a_2 + 1) - 1 \\ &= a_0a_2 + a_0 + a_2, \\ A_2 &= (a_0a_1 + 1)a_2 + a_0 \\ &= (a_0 + 1)a_2 + a_0 = A_1'. \end{aligned}$$

Case 4. $\epsilon_1 = -1 = \epsilon_2$. Then $a_1 = 1 = a_3$ and

$$\begin{aligned} A_1' &= (a_0 + 1)(a_2 + 2) - 1 \\ &= a_0a_2 + a_2 + 2a_0 + 1. \end{aligned}$$

Also

$$\begin{aligned} A_3 &= a_3A_2 + A_1 \\ &= A_2 + A_1 \\ &= ((a_0a_1+1)a_2+a_0) + (a_0a_1+1) \\ &= ((a_0+1)a_2+a_0) + (a_0+1) = A'_1. \end{aligned}$$

Finally, let $k \geq 0$ and assume (30) holds for k and $k+1$ and use the equation

$$A'_{k+2} = a'_{k+2}A'_{k+1} + \epsilon_{k+2}A'_k.$$

Then from (27), with $j = n(k+1)+1$, we have

$$a'_{k+2} = \begin{cases} a_j & \text{if } \epsilon_{k+2} = 1, \epsilon_{k+3} = 1; \\ a_j + 1 & \text{if } \epsilon_{k+2}\epsilon_{k+3} = -1; \\ a_j = 2 & \text{if } \epsilon_{k+2} = -1 = \epsilon_{k+3}. \end{cases} \tag{31}$$

Case 1. Suppose $\epsilon_{k+2} = 1 = \epsilon_{k+3} = 1$. Then

$$n(k+1) = n(k) + 1 = j - 1, \quad n(k+2) = n(k+1) + 1 = j$$

and

$$A'_{k+2} = a_jA_{j-1} + A_{j-2} = A_j = A_{n(k+2)}.$$

Case 2. Suppose $\epsilon_{k+2} = 1, \epsilon_{k+3} = -1$. Then

$$n(k+1) = n(k) + 1 = j - 1, \quad n(k+2) = n(k+1) + 2 = j + 1$$

and

$$A'_{k+2} = (a_j + 1)A_{j-1} + A_{j-2}.$$

Now $\epsilon_{k+3} = -1$ implies $1 = a_{n(k+2)} = a_{j+1}$, so $A_{j+1} = A_j + A_{j-1}$. Hence

$$A'_{k+2} = A_{j+1} = A_{n(k+2)}.$$

Case 3. Suppose $\epsilon_{k+2} = -1, \epsilon_{k+3} = 1$. Then

$$n(k+1) = n(k) + 2 = j - 1, \quad n(k+2) = n(k+1) + 1 = j$$

and

$$A'_{k+2} = (a_j + 1)A_{j-1} - A_{j-3}.$$

Now $\epsilon_{k+2} = -1$ implies $1 = a_{n(k+1)} = a_{j-1}$, so $A_{j-1} = A_{j-2} + A_{j-3}$. Hence

$$A'_{k+2} = a_jA_{j-1} + A_{j-2} = A_j = A_{n(k+2)}.$$

Case 4. Suppose $\epsilon_{k+2} = -1 = \epsilon_{k+3}$. Then

$$n(k+1) = n(k) + 2 = j - 1, \quad n(k+2) = n(k+1) + 2 = j + 1$$

and

$$A'_{k+2} = (a_j + 2)A_{j-1} - A_{j-3}.$$

Now $\epsilon_{k+3} = -1 \implies 1 = a_{n(k+2)} = a_{j+1}$ and $\epsilon_{k+2} = -1 \implies 1 = a_{n(k+1)} = a_{j-1}$, so

$$A_{j+1} = A_j + A_{j-1} \text{ and } A_{j-1} = A_{j-2} + A_{j-3}.$$

Hence

$$\begin{aligned} A'_{k+2} &= (a_j + 2)A_{j-1} - (A_{j-1} - A_{j-2}) \\ &= A_j + A_{j-1} = A_{j+1} = A_{n(k+2)}. \end{aligned}$$

□

Lemma 14. *Each $a_n > 1$, $n \geq 1$ will be visited by the algorithm of Lemma 9, i.e., there exists an m such that $n = f(m)$.*

Proof. Let $a_n > 1$ and $f(m) \leq n < f(m+1)$. If $f(m) < n$, then $f(m+1) = f(m) + 2$ and $\epsilon_{m+1} = -1$; also $n = f(m) + 1$. Then from (25), $a_n = a_{f(m)+1} = 1$. □

Note that in the RCF to NICF-P transformation, we have $f(k) = p$, where k is the NICF-P period-length.

Lemma 15. *Suppose that RCF partial quotients a_r and a_s satisfy $a_r > 1, a_s > 1, r < s$. Then the number J of jumps in the RCF to NICF-P transformation when starting from a_r and finishing at a_s is $J = (s - r + E)/2$, where E is the number of even unisequences in the interval $[a_r, a_s]$. Here we include zero unisequences $[a_i, a_{i+1}]$, where $a_i > 1$ and $a_{i+1} > 1$.*

Proof. Suppose the unisequences in $[a_r, a_s]$ have lengths $m_1, \ldots, m_N$ and let j_i be the number of jumps occurring in a unisequence of length m_i. Then

$$j_i = \frac{m_i + 1 + e_i}{2}, \text{ where } e_i = \begin{cases} 0 & \text{if } m_i \text{ is odd;} \\ 1 & \text{if } m_i \text{ is even.} \end{cases}$$

Then

$$\begin{aligned} J &= \sum_{i=1}^{N} j_i = \sum_{i=1}^{N} \frac{m_i + 1 + e_i}{2} \\ &= \tfrac{1}{2}(N + \sum_{i=1}^{N} m_i) + \frac{E}{2} \\ &= \frac{s - r}{2} + \frac{E}{2} = \frac{s - r + E}{2}. \end{aligned}$$

□

Corollary 16. *The number of jumps in the interval $[a_0, a_r]$, $a_r > 1$, equals the number in the interval $[a_{p-r}, a_p]$, where p is the period-length and $r \leq p/2$.*

Proof. This follows from Lemma 15 and the symmetry of an RCF period. □

6 Proof of Theorem 8

We use the notation of Lemmas 9 and 12.

Case (1)(i). Assume $p = 2h-1, h > 1$ and there is an even length unisequence, or no unisequence, on each side of $a_{h-1} > 1, a_h > 1$, e.g., $\sqrt{73} = [8, \underset{*}{1}, 1, 5, 5, 1, 1, \underset{*}{16}]$ or $\sqrt{89} = [9, \underset{*}{2}, 3, 3, 2, \underset{*}{18}]$. Let $f(m) = h-1$, where m is the number of jumps in $[a_0, a_{h-1}]$. Then $f(m+1) = h$ and

$$\begin{aligned}
\xi'_m &= \xi_{h-1},\ \epsilon_m = 1, \\
\xi'_{m+1} &= \xi_h,\ \epsilon_{m+1} = 1.
\end{aligned}$$

By Corollary 16, m is also the number of jumps in $[a_h, a_p]$. There is also one jump in $[a_{h-1}, a_h]$. Hence $k = 2m+1$. As $Q_{h-1} = Q_h$, we have $Q'_m = Q'_{m+1}$, which is criterion 3) of Theorem 6. Also

$$\begin{aligned}
\epsilon_m = 1 &\implies f(m) = f(m-1)+1,\ A'_{m-1} = A_{f(m-1)}, \\
\epsilon_{m+1} = 1 &\implies f(m+1) = f(m)+1,\ A'_m = A_{f(m)}.
\end{aligned}$$

Hence $A'_{m-1} = A_{h-2},\ A'_m = A_{h-1},\ B'_{m-1} = B_{h-2},\ B'_m = B_{h-1}$ and

$$\begin{aligned}
A_{p-1} &= A_{h-1}B_{h-1} + A_{h-2}B_{h-2} \\
&= A'_m B'_m + A'_{m-1}B'_{m-1} \\
&= A'_m B'_m + \epsilon_{m+1}A'_{m-1}B'_{m-1}.
\end{aligned}$$

Also

$$\begin{aligned}
B_{p-1} &= B^2_{h-1} + B^2_{h-2} \\
&= {B'}^2_m + {B'}^2_{m-1} \\
&= {B'}^2_m + \epsilon_{m+1}{B'}^2_{m-1}.
\end{aligned}$$

If $p = 1$, $\sqrt{D} = [a, \overline{2a}]$ and the NICF and RCF expansions are identical. Also $Q'_0 = Q'_1 = 1, \epsilon_1 = 1$ and we have criterion 3).

Case (1)(ii). Assume $p = 2h-1$, with an odd length unisequence on each side of $a_{h-1} > 1, a_h > 1$, e.g., $\sqrt{113} = [10, \underset{*}{1}, 1, 1, 2, 2, 1, 1, 1, \underset{*}{20}]$. Let $f(m) = h-1$. Then $f(m+1) = h$ and

$$\begin{aligned}
\xi'_m &= \xi_{h-1}+1,\ \epsilon_m = -1, \\
\xi'_{m+1} &= \xi_h,\ \epsilon_{m+1} = 1,
\end{aligned}$$

and as in Case (1)(i), $k = 2m+1$. As $Q_{h-1} = Q_h$, we have $Q'_m = Q'_{m+1}$, which is criterion 3). Also

$$\begin{aligned}
\epsilon_m = -1 &\implies f(m) = f(m-1)+2,\ A'_{m-1} = A_{f(m-1)+1}, \\
\epsilon_{m+1} = 1 &\implies f(m+1) = f(m)+1,\ A'_m = A_{f(m)}.
\end{aligned}$$

Hence $A'_{m-1} = A_{h-2},\ A'_m = A_{h-1}$ and $B'_{m-1} = B_{h-2},\ B'_m = B_{h-1}$. Then as in Case (1)(i), we get

$$A'_m B'_m + \epsilon_{m+1} A'_{m-1} B'_{m-1} = A_{p-1} \text{ and } {B'}^2_m + \epsilon_{m+1} {B'}^2_{m-1} = B_{p-1}.$$

Case (2) Assume $p = 2h, h > 1, a_{h-1} > 1, a_h > 1$, e.g., $\sqrt{92} = [9, \underset{*}{1}, 1, 2, 4, 2, 1, 1, \underset{*}{18}]$.
Let $f(m) = h$. Then $f(m+1) = h+1$ and

$$\begin{aligned} \xi'_m &= \xi_h,\ \epsilon_m = 1, \\ \xi'_{m+1} &= \xi_{h+1},\ \epsilon_{m+1} = 1, \end{aligned}$$

Also by Corollary 16, m is the number of jumps in $[a_h, a_p]$. Hence $k = 2m$. Then $P_h = P_{h+1}$ gives $P'_m = P'_{m+1}$ and we have criterion 1) of Theorem 6. Also

$$\begin{aligned} \epsilon_m = 1 &\implies f(m) = f(m-1) + 1,\ A'_{m-1} = A_{f(m-1)}, \\ \epsilon_{m+1} = 1 &\implies f(m+1) = f(m) + 1,\ A'_m = A_{f(m)}. \end{aligned}$$

Hence $A'_{m-1} = A_{h-1},\ A'_m = A_h$ and $B'_{m-1} = B_{h-1},\ B'_m = B_h$. Then

$$\begin{aligned} A_{p-1} &= A_h B_{h-1} + A_{h-1} B_{h-2} \\ &= A'_m B'_{m-1} + A'_{m-1} B_{h-2}. \end{aligned} \tag{32}$$

But $B_{h-2} = B_h - a_h B_{h-1} = B'_m - a'_m B_{m-1} = B'_{m-2}$. Hence (32) gives

$$\begin{aligned} A_{p-1} &= A'_m B'_{m-1} + A'_{m-1} B'_{m-2} \\ &= A'_m B'_{m-1} + \epsilon_m A'_{m-1} B'_{m-2}. \end{aligned}$$

Also

$$\begin{aligned} B_{p-1} &= B_{h-1}(B_h + B_{h-2}) \\ &= B'_{m-1}(B'_m + B'_{m-2}). \end{aligned}$$

Case (3)(a). Assume $p = 2h$, with an even length unisequence each side of $a_h > 1$, e.g., $\sqrt{21} = [4, \underset{*}{1}, 1, 2, 1, 1, \underset{*}{8}]$. Let $f(m) = h$. Then $f(m+1) = h+2$. As in Case(2), $k = 2m$. Also

$$\begin{aligned} \xi'_m &= \xi_h,\ \epsilon_m = 1, \\ \xi'_{m+1} &= \xi_{h+2} + 1,\ \epsilon_{m+1} = -1. \end{aligned}$$

Then

$$\begin{aligned} \epsilon_m = 1 &\implies f(m) = f(m-1) + 1,\ A'_{m-1} = A_{f(m-1)}, \\ \epsilon_{m+1} = -1 &\implies f(m+1) = f(m) + 2,\ A'_m = A_{f(m)+1}. \end{aligned}$$

Hence $A'_{m-1} = A_{h-1},\ A'_m = A_{h+1}$ and $B'_{m-1} = B_{h-1},\ B'_m = B_{h+1}$.
We prove criterion 2) of Theorem 6, $P'_{m+1} = P'_m + Q'_m$, i.e., $P_{h+2} + Q_{h+2} = P_h + Q_h$.

We note from Theorem 10.19, Rosen [8], that $a_{h+1} = 1$ implies $P_{h+2} + P_{h+1} = Q_{h+1}$. Also

$$\begin{aligned} P_{h+2}^2 &= D - Q_{h+1}Q_{h+2}, \\ P_{h+1}^2 &= D - Q_hQ_{h+1}. \end{aligned}$$

Hence

$$\begin{aligned} P_{h+2}^2 - P_{h+1}^2 &= Q_{h+1}(Q_h - Q_{h+2}) \\ &= (P_{h+2} + P_{h+1})(Q_h - Q_{h+2}). \end{aligned}$$

Hence

$$\begin{aligned} P_{h+2} - P_{h+1} &= Q_h - Q_{h+2}, \\ P_{h+2} + Q_{h+2} &= P_{h+1} + Q_h \\ &= P_h + Q_h \qquad (33) \end{aligned}$$

We next prove

$$A_{p-1} = B'_{m-1}(A'_m - A'_{m-1}) + A'_{m-1}B'_{m-2}, \qquad (34)$$
$$B_{p-1} = B'_{m-1}(B'_m - B'_{m-1} + B'_{m-2}). \qquad (35)$$

First note that by equations (27), $\epsilon_m = 1$ and $\epsilon_{m+1} = 1$ imply

$$a'_m = a_{f(m)} + 1 = a_h + 1.$$

Also $a_{h+1} = 1$ implies $B_{h+1} = B_h + B_{h-1}$, i.e., $B'_m = B_h + B'_{m-1}$. Hence

$$\begin{aligned} B_{h-2} &= B_h - a_hB_{h-1} \\ &= (B'_m - B'_{m-1}) - (a'_m - 1)B'_{m-1} \\ &= B'_m - a'_mB'_{m-1} = B'_{m-2}. \end{aligned}$$

Then

$$\begin{aligned} A_{p-1} &= A_hB_{h-1} + A_{h-1}B_{h-2} \\ &= (A'_m - A'_{m-1})B'_{m-1} + A'_{m-1}B'_{m-2}, \end{aligned}$$

proving (34). Also

$$\begin{aligned} B_{p-1} &= B_{h-1}(B_h + B_{h-2}) \\ &= B'_{m-1}(B'_m - B'_{m-1} + B'_{m-2}), \end{aligned}$$

proving (35).

If $p = 2$ and $a_1 > 1$, then $D = a^2 + b, 1 < b < 2a, b$ dividing $2a$ (Rosen [8, p. 389]). Then $\sqrt{D} = [a, \underset{*}{2a}/b, \underset{*}{2a}]$ and the NICF and RCF expansions are identical. Then $\xi'_1 = \frac{a+\sqrt{D}}{b}, \xi'_2 = a + \sqrt{D}, P'_1 = P'_2 = a, \epsilon_1 = 1$ and we have criterion 1).

Case (3)(b). Assume $p = 2h$, with an odd length unisequence each side of $a_h > 1$, e.g., $\sqrt{14} = [3, \underset{*}{1}, 2, 1, \underset{*}{6}]$. Let $f(m) = h$. Then $f(m+1) = h+2$ and $k = 2m$. Then

$$\begin{aligned} \xi'_m &= \xi_h + 1, \ \epsilon_m = -1, \\ \xi'_{m+1} &= \xi_{h+2} + 1, \ \epsilon_{m+1} = -1. \end{aligned}$$

Then

$$\begin{aligned} \epsilon_m = -1 &\implies f(m) = f(m-1) + 2, \ A'_{m-1} = A_{f(m-1)+1}, \\ \epsilon_{m+1} = -1 &\implies f(m+1) = f(m) + 2, \ A'_m = A_{f(m)+1}. \end{aligned}$$

We have $A'_{m-1} = A_{h-1}, \ A'_m = A_{h+1}, B'_{m-1} = B_{h-1}, \ B'_m = B_{h+1}$. Then using (33), we get

$$P'_{m+1} = P_{h+2} + Q_{h+2} = P_h + Q_h = P'_m,$$

which is criterion 1) of Theorem 6.
As $\epsilon_m = -1$, it remains to prove

$$\begin{aligned} A_{p-1} &= B'_{m-1}A'_m - A'_{m-1}B'_{m-2} \\ B_{p-1} &= B'_{m-1}(B'_m - B'_{m-2}). \end{aligned}$$

First, $a_{h+1} = 1$ implies $A_{h+1} = A_h + A_{h-1}$, i.e., $A'_m = A_h + A_{h-1}$. Also $\epsilon_m = -1 = \epsilon_{m+1}$ implies $a'_m = a_{f(m)} + 2 = a_h + 2$. Hence

$$\begin{aligned} -B'_{m-2} &= \epsilon_m B'_{m-2} \\ &= B'_m - a'_m B'_{m-1} \\ &= (B_h + B_{h-1}) - (a_h + 2)B_{h-1} \\ &= B_h - B_{h-1} - a_h B_{h-1} \\ &= B_{h-2} - B_{h-1}. \end{aligned}$$

Hence

$$\begin{aligned} B'_{m-1}A'_m - A'_{m-1}B'_{m-2} &= B_{h-1}(A_h + A_{h-1}) - A_{h-1}(B_{h-1} - B_{h-2}) \\ &= B_{h-1}A_h + A_{h-1}B_{h-2} \\ &= A_{p-1}. \end{aligned}$$

Finally,

$$\begin{aligned} B'_{m-1}(B'_m - B'_{m-2}) &= B_{h-1}((B_h + B_{h-1}) + (B_{h-2} - B_{h-1})) \\ &= B_{h-1}(B_h + B_{h-2}) \\ &= B_{p-1}. \end{aligned}$$

Case (4)(a). Assume $p = 2h$ with an M-unisequence, M odd, at the centre of a period. There are two cases:

$M = 4t+3$, e.g., $\sqrt{88} = [9, \underset{*}{2}, 1, 1, 1, 2, \underset{*}{18}]$. Let $f(m) = h$. Then $f(m+1) = h+2$ and

$$\begin{aligned} \xi'_m &= \xi_h + 1,\ \epsilon_m = -1; \\ \xi'_{m+1} &= \xi_{h+2} + 1,\ \epsilon_{m+1} = -1. \end{aligned}$$

and as with case (3)(b), we have criterion 1) of Theorem 6. Now m is the number of jumps in $[a_0, a_h]$. Then we have a central unisequence $[a_r, a_{p-r}]$ of length $4t+3$. There are $t+1$ jumps of 2 in $[a_r, a_h]$, so $r + 2t + 2 = h$. Let J be the number of jumps in $[a_0, a_r]$. Hence $m = J + (t+1)$. There are $t+1$ jumps of 2 in $[a_h, a_{p-r}]$ and J jumps in $[a_{p-r}, a_p]$. Hence

$$k = (J + t + 1) + (t + 1) + J = 2(J + t + 1) = 2m.$$

$M = 4t+1$, e.g., $\sqrt{91} = [9, \underset{*}{1}, 1, 5, 1, 5, 1, 1, \underset{*}{18}]$. Let $f(m) = h-1$. Then $f(m+1) = h+1$ and

$$\xi'_{m+1} = \xi_{h+1} + 1,\ \epsilon_{m+1} = -1.$$

Also $\xi'_m = \xi_{h-1}$ or $\xi_{h-1} + 1$. We have a central unisequence $[a_r, a_{p-r}]$ of length $4t+1$. There are t jumps of 2 in $[a_r, a_{h-1}]$, so $r + 2t = h - 1$. Let J be the number of jumps in $[a_0, a_r]$. Hence m, being the number of jumps in $[a_0, a_{h-1}]$ satisfies $m = J + t$. There is also one jump in $[a_{h-1}, a_{h+1}]$, t jumps of 2 in $[a_{h+1}, a_{p-r}]$ and J jumps in $[a_{p-r}, a_p]$. Hence

$$k = (J + t) + 1 + t + J = 2(J + t) + 1 = 2m + 1.$$

Then $Q'_{m+1} = Q_{h+1}$ and $Q'_m = Q_{h-1}$, so $Q_{h-1} = Q_{h+1}$ implies $Q'_m = Q'_{m+1}$ and we have criterion 3) of Theorem 6. We have $A'_m = A_{f(m)+1} = A_h$. Also, regardless of the sign of ϵ_m, we have $A'_{m-1} = A_{h-2}$. We now prove

$$A_{k-1} = A'_m B'_m - A'_{m-1} B'_{m-1}, \tag{36}$$

$$B_{k-1} = B'^2_m - B'^2_{m-1}. \tag{37}$$

Noting that $a_h = 1$ gives $A_h = A_{h-1} + A_{h-2}$ and $B_h = B_{h-1} + B_{h-2}$, we have

$$\begin{aligned} A'_m B'_m - A'_{m-1} B'_{m-1} &= A_h B_h - A_{h-2} B_{h-2} \\ &= A_h (B_{h-1} + B_{h-2}) - (A_h - A_{h-1}) B_{h-2} \\ &= A_h B_{h-1} + A_{h-1} B_{h-2} \\ &= A_{p-1}. \end{aligned}$$

Also

$$\begin{aligned} B'^2_m - B'^2_{m-1} &= B_h^2 - B_{h-2}^2 \\ &= (B_h - B_{h-2})(B_h + B_{h-2}) \\ &= B_{h-1}(B_h + B_{h-2}) \\ &= B_{p-1}. \end{aligned}$$

Case (4)(b) Assume $p = 2h - 1$ with an M-unisequence, M even, at the centre of a period. There are two cases:

$M=4t$, e.g., $\sqrt{13}=[3,\underset{*}{1},1,1,1,\underset{*}{6}]$. Let $f(m)=h-1$. Then $f(m+1)=h+1$ and

$$\begin{aligned}\xi'_m &= \xi_{h-1}+1,\ \epsilon_m=-1,\\ \xi'_{m+1} &= \xi_{h+1}+1,\ \epsilon_{m+1}=-1.\end{aligned}$$

We have a central unisequence $[a_r, a_{p-r}]$ of length $4t$ with $r+2t=h-1$. Let J be the number of jumps in $[a_0,a_r]$. There are t jumps of 2 in $[a_r,a_{h-1}]$. Hence m, being the number of jumps in $[a_0,a_{h-1}]$ satisfies $m=J+t$. There are also t jumps of 2 in $[a_{h-1},a_{p-r-1}]$, one jump in $[a_{p-r-1},a_{p-r}]$ and J jumps in $[a_{p-r},a_p]$. Hence

$$k=(J+t)+t+1+J=2(J+t)+1=2m+1.$$

Then

$$\begin{aligned}\epsilon_m=-1 &\implies f(m)=f(m-1)+2,\ A'_{m-1}=A_{f(m-1)+1},\\ \epsilon_{m+1}=-1 &\implies f(m+1)=f(m)+2,\ A'_m=A_{f(m)+1}.\end{aligned}$$

We have $A'_{m-1}=A_{h-2},\ A'_m=A_h$. We now verify criterion 4) of Theorem 6.

$$P'_{m+1}=Q'_m+\tfrac{1}{2}Q'_{m+1}. \tag{38}$$

We note that $a_h=1$ implies $P_{h+1}=Q_h-P_h$. Then

$$\begin{aligned}P'_{m+1} &= P_{h+1}+Q_{h+1}\\ &= Q_h-P_h+Q_{h+1}.\end{aligned}$$

Also

$$Q'_m+\tfrac{1}{2}Q'_{m+1}=Q_{h-1}+\tfrac{1}{2}Q_{h+1}.$$

Hence (38) holds if and only if

$$\begin{aligned}Q_h-P_h+Q_{h+1} &= Q_{h-1}+\tfrac{1}{2}Q_{h+1},\\ \text{i.e., } Q_{h-1}-P_h+Q_{h+1} &= Q_{h-1}+\tfrac{1}{2}Q_{h+1},\\ \text{i.e., } P_h &= \tfrac{1}{2}Q_{h+1}.\end{aligned}$$

However

$$\begin{aligned}P_h^2 &= D-Q_{h-1}Q_h,\\ P_{h+1}^2 &= D-Q_hQ_{h+1},\\ P_h^2-P_{h+1}^2 &= Q_h(Q_{h+1}-Q_{h-1}),\\ P_h-P_{h+1} &= Q_{h+1}-Q_{h-1},\\ P_h-(Q_h-P_h) &= Q_{h+1}-Q_h,\\ 2P_h &= Q_{h+1}.\end{aligned}$$

Next we prove

$$A_{p-1}=A'_mB'_m+2A'_{m-1}B'_{m-1}-(A'_mB'_{m-1}+B'_mA'_{m-1}), \tag{39}$$

$$B_{p-1}={B'}^2_m+2{B'}^2_{m-1}-2B'_mB'_{m-1}. \tag{40}$$

Let

$$T = A'_m B'_m + 2A'_{m-1}B'_{m-1} - (A'_m B'_{m-1} + B'_m A'_{m-1}).$$

Then

$$\begin{aligned}
T &= A_h B_h + 2A_{h-2}B_{h-2} - (A_h B_{h-2} + B_h A_{h-2}) \\
&= A_h(B_h - B_{h-2}) + A_{h-2}(B_{h-2} - B_h) + A_{h-2}B_{h-2} \\
&= A_h B_{h-1} - A_{h-2}B_{h-1} + A_{h-2}B_{h-2} \\
&= A_h B_{h-1} + A_{h-2}(B_{h-2} - B_{h-1}) \\
&= (A_{h-1} + A_{h-2})B_{h-1} + A_{h-2}(B_{h-2} - B_{h-1}) \\
&= A_{h-1}B_{h-1} + A_{h-2}B_{h-2} \\
&= A_{p-1}.
\end{aligned}$$

Also

$$\begin{aligned}
{B'}_m^2 + 2{B'}_{m-1}^2 - 2B'_m B'_{m-1} &= B_h^2 + 2B_{h-2}^2 - 2B_h B_{h-2} \\
&= B_h(B_h - 2B_{h-2}) + 2B_{h-2}^2 \\
&= (B_{h-1} + B_{h-2})(B_{h-1} - B_{h-2}) + 2B_{h-2}^2 \\
&= B_{h-1}^2 - B_{h-2}^2 + 2B_{h-2}^2 \\
&= B_{h-1}^2 + B_{h-2}^2 \\
&= B_{p-1}.
\end{aligned}$$

$M = 4t+2$, e.g., $\sqrt{29} = [5, \underset{*}{2}, 1, 1, 2, \underset{*}{10}]$. Let $f(m) = h$. Then $\epsilon_m = -1$ and $\xi'_m = \xi_h + 1$. Also $f(m) = f(m-1) + 2$, so $A'_{m-1} = A_{f(m-1)+1} = A_{h-1}$. Then we have a central unisequence $[a_r, a_{p-r}]$ of length $4t+2$. There are $t+1$ jumps in $[a_r, a_h]$, so $r + 2t + 2 = h$. Let J be the number of jumps in $[a_0, a_r]$. Hence m, being the number of jumps in $[a_0, a_h]$ satisfies $m = J + t + 1$. There are also t jumps of 2 in $[a_h, a_{p-r-1}]$, one jump in $[a_{p-r-1}, a_{p-r}]$ and J jumps in $[a_{p-r}, a_p]$. Hence

$$k = (J + t + 1) + t + 1 + J = 2(J + t + 1) = 2m.$$

We now verify Case 5) of the midpoint criteria:

$$P'_m = Q'_m + \tfrac{1}{2}Q'_{m-1}. \tag{41}$$

Then, as $\xi'_{m-1} = \xi_{h-2}$ or $\xi_{h-2} + 1$, we have $Q'_{m-1} = Q_{h-2}$ and

$$\begin{aligned}
P'_m = Q'_m + \tfrac{1}{2}Q'_{m-1} &\iff P_h + Q_h = Q_h + \tfrac{1}{2}Q_{h-2} \\
&\iff P_h = \tfrac{1}{2}Q_{h-2}.
\end{aligned}$$

But $a_{h-1} = 1$ implies $P_h = Q_{h-1} - P_{h-1}$. Hence

$$\begin{aligned}
P_{h-1}^2 - P_h^2 &= (D - Q_{h-2}Q_{h-1}) - (D - Q_{h-1}Q_h), \\
(P_{h-1} - P_h)Q_{h-1} &= Q_{h-1}(Q_h - Q_{h-2}), \\
P_{h-1} - P_h &= Q_h - Q_{h-2}, \\
Q_{h-1} - 2P_h &= Q_{h-1} - Q_{h-2}, \\
2P_h &= Q_{h-2}.
\end{aligned}$$

Next we prove

$$A_{p-1} = 2A'_{m-1}B'_{m-1} + A'_{m-2}B'_{m-2} - (A'_{m-1}B'_{m-2} + B'_{m-1}A'_{m-2}), \tag{42}$$
$$B_{p-1} = 2{B'}^2_{m-1} + {B'}^2_{m-2} - 2B'_{m-1}B'_{m-2}. \tag{43}$$

Note that regardless of the sign of ϵ_{m-1}, we have $A'_{m-2} = A_{h-3}$.
To prove (42), let

$$T = 2A'_{m-1}B'_{m-1} + A'_{m-2}B'_{m-2} - (A'_{m-1}B'_{m-2} + B'_{m-1}A'_{m-2}).$$

Then

$$\begin{aligned} T &= 2A_{h-1}B_{h-1} + A_{h-3}B_{h-3} - (A_{h-1}B_{h-3} + B_{h-1}A_{h-3}) \\ &= 2A_{h-1}B_{h-1} + (A_{h-1} - A_{h-2})(B_{h-1} - B_{h-2}) \\ &\quad - A_{h-1}(B_{h-1} - B_{h-2}) - B_{h-1}(A_{h-1} - A_{h-2}) \\ &= A_{h-1}B_{h-1} + A_{h-2}B_{h-2} \\ &= A_{p-1}. \end{aligned}$$

Finally, we prove (43).

$$\begin{aligned} 2{B'}^2_{m-1} + {B'}^2_{m-2} - 2B'_{m-1}B'_{m-2} &= 2B^2_{h-1} + B^2_{h-3} - 2B_{h-1}B_{h-3} \\ &= 2B^2_{h-1} + (B_{h-1} - B_{h-2})^2 \\ &\quad - 2B_{h-1}(B_{h-1} - B_{h-2}) \\ &= B^2_{h-1} + B^2_{h-2} \\ &= B_{p-1}. \end{aligned}$$

This completes the proof of Theorem 8.

7 RCF periods with only odd length unisequences

Lemma 17. *Suppose there are no even length (≥ 2) unisequences in an RCF period of length p for $\sqrt{D}$. If k is the NICF period-length and $0 \leq t \leq k/2$, then*

$$f(k-t) = p - f(t), \tag{44}$$

Proof. Let $f(t) = r$. Then we have t jumps on $[a_0, a_r]$. Because of the symmetry of the partial quotients and the absence of even length unisequences, we get the same t jumps but in reverse order on $[a_{p-r}, a_p]$. There are k jumps on $[a_0, a_p]$ and hence $k - t$ jumps on $[a_0, a_{p-r}]$. Hence $f(k-t) = p - r = p - f(t)$. □

Theorem 18. *Suppose there are no even length (≥ 2) unisequences in a RCF period of length p for $\sqrt{D}$. Then if k is the NICF period-length, for $1 \leq t \leq k/2$, we have*

(i) $\epsilon_t = \epsilon_{k+1-t}$*;*

(ii) $P'_t = P'_{k+1-t}$;

(iii) $Q'_t = Q'_{k-t}$;

(iv) $a'_t = a'_{k-t}$,

where $\xi'_t = \frac{P'_t+\sqrt{D}}{Q'_t}$ *is the* t*-th complete quotient of the NICF-P expansion of* $\sqrt{D}$.

Remark 19. In particular, if $k = 2h$, then (ii) implies $P'_h = P'_{h+1}$ and we have criterion 1) of Theorem 6; while if $k = 2h+1$, then (iii) implies $Q'_h = Q'_{h+1}$ and we have criterion 3) of Theorem 6.

Proof. (i) We use (24) and Lemma 17

$$\begin{aligned}
\epsilon_t = 1 &\iff f(t) = f(t-1)+1 \\
&\iff p - f(k-t) = p - f(k-t+1)+1 \\
&\iff f(k-t+1) = f(k-t)+1 \\
&\iff \epsilon_{k-t+1} = 1.
\end{aligned}$$

(ii) (a) Assume $\epsilon_t = 1$. Then $\epsilon_{k+1-t} = 1$ and $f(t) = f(t-1)+1$. Hence $\xi'_t = \xi_{f(t)} = \frac{P_{f(t)}+\sqrt{D}}{Q_{f(t)}}$ and

$$\begin{aligned}
\xi'_{k+1-t} &= \xi_{f(k+1-t)} = \xi_{p-f(t-1)} = \xi_{p+1-f(t)} \\
&= \frac{P_{p+1-f(t)}+\sqrt{D}}{Q_{p+1-f(t)}} \\
&= \frac{P_{f(t)}+\sqrt{D}}{Q_{f(t)-1}}.
\end{aligned}$$

Hence $P'_t = P_{f(t)} = P'_{k+1-t}$.

(b) Assume $\epsilon_t = -1$. Then $\epsilon_{k+1-t} = -1$ and $f(t) = f(t-1)+2$.
Hence $\xi'_t = \xi_{f(t)}+1 = \frac{P_{f(t)}+Q_{f(t)}+\sqrt{D}}{Q_{f(t)}}$ and

$$\begin{aligned}
\xi'_{k+1-t} &= \xi_{f(k+1-t)}+1 = \xi_{p-f(t-1)}+1 \\
&= \frac{P_{p-f(t-1)}+\sqrt{D}}{Q_{p-f(t-1)}}+1 = \frac{P_{f(t-1)+1}+Q_{f(t-1)}+\sqrt{D}}{Q_{f(t-1)}}.
\end{aligned}$$

Hence

$$\begin{aligned}
P'_t &= P_{f(t)}+Q_{f(t)}, \\
P'_{k+1-t} &= P_{f(t-1)+1}+Q_{f(t-1)} \\
&= P_{f(t)-1}+Q_{f(t)-2}.
\end{aligned}$$

For brevity, write $r = f(t) - 1$. We have to prove $P'_t = P'_{k+1-t}$, i.e.,

$$P_r + Q_{r-1} = P_{r+1} + Q_{r+1}. \tag{45}$$

We have

$$\begin{aligned} P_{r+1}^2 - P_r^2 &= (D - Q_r Q_{r+1}) - (D - Q_{r-1} Q_r) \\ &= Q_r (Q_{r-1} - Q_{r+1}). \end{aligned} \tag{46}$$

Now $\epsilon_t = -1$ implies $a_{f(t-1)+1} = 1 = a_{f(t)-1} = a_r$ and hence $P_{r+1} + P_r = a_r Q_r = Q_r$. Then (46) gives

$$P_{r+1} - P_r = Q_{r-1} - Q_{r+1}$$

and hence (45).
(iii) To prove $Q'_t = Q'_{k-t}$, we observe that

$$\begin{aligned} \xi'_t &= \xi_{f(t)} \text{ or } \ \xi_{f(t)} + 1, \\ \xi'_{k-t} &= \xi_{f(k-t)} \text{ or } \ \xi_{f(k-t)} + 1. \end{aligned}$$

Then

$$Q'_{k-t} = Q_{f(k-t)} = Q_{p-f(t)} = Q_{f(t)} = Q'_t.$$

(iv)

$$a'_t = \begin{cases} a_{f(t)} \text{ if } (\epsilon_t, \epsilon_{t+1}) = (1,1); \\ a_{f(t)} + 1 \text{ if } (\epsilon_t, \epsilon_{t+1}) = (1,-1) \text{ or } (-1,1); \\ a_{f(t)} + 2 \text{ if } (\epsilon_t, \epsilon_{t+1}) = (-1,-1). \end{cases}$$

$$a'_{k-t} = \begin{cases} a_{f(k-t)} \text{ if } (\epsilon_{k-t}, \epsilon_{k-t+1}) = (1,1); \\ a_{f(k-t)} + 1 \text{ if } (\epsilon_{k-t}, \epsilon_{k-t+1}) = (1,-1) \text{ or } (-1,1); \\ a_{f(k-t)} + 2 \text{ if } (\epsilon_{k-t}, \epsilon_{k-t+1}) = (-1,-1). \end{cases}$$

Then as $(\epsilon_t, \epsilon_{t+1}) = (\epsilon_{k+1-t}, \epsilon_{k-t})$ and $a_{f(k-t)} = a_{p-f(t)} = a_{f(t)}$, it follows that $a'_t = a'_{k-t}$. $\square$

We give examples of even and odd NICF period-length in which only odd length unisequences occur.

(a) $$\sqrt{1532} = [39, \underset{*}{7}, 9, 1, 1, 1, 3, 1, 18, 1, 3, 1, 1, 1, 9, 7, \underset{*}{78}]$$
$$= 39 + \frac{1|}{|\underset{*}{7}} + \frac{1|}{|10} - \frac{1|}{|3} - \frac{1|}{|5} - \frac{1|}{|20} - \frac{1|}{|5} - \frac{1|}{|3} - \frac{1|}{|10} + \frac{1|}{|7} + \frac{1|}{|\underset{*}{78}}.$$

Here $p = 16, k = 10, P'_5 = P'_6$.

(b) $$\sqrt{277} = [16, \underset{*}{1}, 1, 1, 4, 10, 1, 7, 2, 2, 3, 3, 2, 2, 7, 1, 10, 4, 1, 1, 1, \underset{*}{32}] =$$
$$17 - \frac{1|}{|\underset{*}{3}} - \frac{1|}{|5} + \frac{1|}{|11} - \frac{1|}{|8} + \frac{1|}{|2} + \frac{1|}{|2} + \frac{1|}{|3} + \frac{1|}{|3} + \frac{1|}{|2} + \frac{1|}{|2} + \frac{1|}{|8} - \frac{1|}{|11} + \frac{1|}{|5} - \frac{1|}{|3} - \frac{1|}{|\underset{*}{34}}.$$

Here $p = 21, k = 15, Q'_7 = Q'_8$.

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