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The 2-adic Order of the Tribonacci Numbers and the Equation $T_n = m!$

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Abstract

Let $(T_n)_{n\geq 0}$ be the Tribonacci sequence defined by the recurrence $T_{n+2} = T_{n+1} + T_n + T_{n-1}$, with $T_0 = 0$ and $T_1 = T_2 = 1$. In this paper, we characterize the 2-adic valuation of T_n and, as an application, we completely solve the Diophantine equation $T_n = m!$.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. The p-adic order of a Fibonacci number was completely characterized; see [9, 17, 18, 26, 28]. For instance, from the main theorem of Lengyel [17], we extract the following facts:

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

To prove this, many congruences involving Fibonacci numbers were used (see for instance [10]).

Among the several generalizations of Fibonacci numbers, one of the most known is the *Tribonacci* sequence $(T_n)_{n\geq 0}$, which is defined by the recurrence $T_{n+1} = T_n + T_{n-1} + T_{n-2}$, with initial values $T_0 = 0$ and $T_1 = T_2 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, \ldots$$

Recall that the Tribonacci numbers have a long history. They were first studied in 1914 by Agronomof [1] and subsequently by many others. The name "Tribonacci" was coined in 1963 by Feinberg [6]. The basic properties of Tribonacci numbers can be found in [11, 22, 27, 29]. For recent papers, we refer the reader to [2, 12, 13, 24, 25] and to a collection [14, 15, 16].

In this paper, we provide a complete description of the 2-adic order of Tribonacci numbers. More precisely, we prove the following result:

Theorem 1. *For $n \geq 1$, we have*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 4; \\ 1, & \text{if } n \equiv 3, 11 \pmod{16}; \\ 2, & \text{if } n \equiv 4, 8 \pmod{16}; \\ 3, & \text{if } n \equiv 7 \pmod{16}; \\ \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod{16}; \\ \nu_2(n+4) - 1, & \text{if } n \equiv 12 \pmod{16}; \\ \nu_2((n+1)(n+17)) - 3, & \text{if } n \equiv 15 \pmod{16}. \end{cases}$$

A number of mathematicians have been interested in Diophantine equations involving factorial and Fibonacci numbers. For example, in 1999, Luca [19] proved that F_n is a product of factorials only when $n = 1, 2, 3, 6, 12$. Also, the largest product of distinct Fibonacci numbers which is a product of factorials is $F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$; see [21] (a proof of this fact can be achieved by applying the primitive divisor theorem [5]).

Also, in [8] it is shown that if k is fixed, then there are only finitely many positive integers n such that

$$F_n = m_1! + m_2! + \cdots + m_k!$$

holds for some positive integers $m_1, \ldots, m_k$; furthermore, all the solutions for the case $k \leq 2$ have been determined. Later the case $k = 3$ was also solved; see [3]. In a recent paper, Luca and Siksek [20] found all factorials expressible as the sum of at most three Fibonacci numbers.

However, we point out that, to the best of the authors' knowledge, the problem of finding all factorials among Tribonacci numbers has not yet been solved. For this problem, the approach of using the primitive divisor theorem does not work, simply because such a theorem for higher order recurrence sequences seems to be out of reach.

Here we use Theorem 1 to solve completely the equation $T_n = m!$. We have

Theorem 2. *The only solutions of the Diophantine equation*

$$T_n = m! \tag{1}$$

in positive integers n, m are

$$(n, m) \in \{(1, 1), (2, 1), (3, 2), (7, 4)\}.$$

2 Auxiliary results

Before proceeding further, some considerations will be needed for the convenience of the reader.

From [4, Lemma 1], we can extract the following result.

Lemma 3. *For all $n \geq 1$, we have that*

$$\phi^{n-2} \leq T_n \leq \phi^{n-1}, \tag{2}$$

where $\phi = 1.839286\cdots$.

The next result provides an addition formula for Tribonacci numbers (see [7, Section 3]); it plays a crucial role in proving Theorem 1.

Lemma 4. *For all integers n, m, with $n \geq 0$ and $m \geq 2$, we have that*

$$T_{n+m} = T_{m-2}T_n + (T_{m-3} + T_{m-2})T_{n+1} + T_{m-1}T_{n+2}.$$

We require one last fact about ν_p in order to complete our proof of Theorem 2.

Lemma 5. *For any integer $k \geq 1$ and p prime, we have*

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_p(k!) \leq \frac{k-1}{p-1}, \tag{3}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

We refer the reader to [23, Lemma 2.4] for a proof of this result.

Now we are ready to deal with the proof of the theorems.

3 Proof of Theorem 1

In order to prove the first 4 cases of the 2-adic valuation, it suffices to prove the following congruences

(i) $T_n \equiv 1 \pmod{2}$, if $n \equiv 1, 2 \pmod{4}$.

(ii) $T_n \equiv 2 \pmod{4}$, if $n \equiv 3, 11 \pmod{16}$.

(iii) $T_n \equiv 4 \pmod{8}$, if $n \equiv 4, 8 \pmod{16}$.

(iv) $T_n \equiv 8 \pmod{16}$, if $n \equiv 7 \pmod{16}$.

To avoid unnecessary repetitions we shall prove only that $T_n \equiv 4 \pmod{8}$ if $n \equiv 4 \pmod{16}$ (the other cases can be handled in much the same way and we leave them as an exercise to the reader).

Thus, we want to prove that $T_{16t+4} \equiv 4 \pmod{8}$, for all $t \geq 0$. For that, we shall proceed by induction on t. The basis case, $t = 0$, follows because $T_4 = 4$. Thus, we may suppose that $T_{16t+4} \equiv 4 \pmod{8}$. Therefore, we use Lemma 4 to $T_{16(t+1)+4} = T_{(16t+4)+16}$ to obtain

$$T_{16(t+1)+4} = 1705T_{16t+4} + 2632T_{16t+5} + 3136T_{16t+6}.$$

Then $T_{16(t+1)+4} \equiv 1705 \cdot 4 \equiv 4 \pmod{8}$, where we used that 8 divides both 2632 and 3136.

Now, we shall split our proof into three cases:

Case 1. $n \equiv 0 \pmod{16}$. To deal with this case, we shall prove the following result:

Lemma 6. *For all $s \geq 1$ and $t \geq 6$, we have that $T_{2^{t-3}s} \equiv s2^{t-4} \pmod{2^{t-3}}$.*

Proof. We shall use induction on s to prove simultaneously the following congruences

$$T_{2^{t-3}s} \equiv s2^{t-4} \pmod{2^{t-3}}, T_{2^{t-3}s-1} \equiv 0 \pmod{2^{t-3}}, T_{2^{t-3}s+1} \equiv 1 \pmod{2^{t-3}}. \tag{4}$$

First, let us deal with the basis case $s = 1$. So, we desire to prove that, for all $t \geq 6$, we have that

$$T_{2^{t-3}} \equiv 2^{t-4} \pmod{2^{t-3}}, T_{2^{t-3}-1} \equiv 0 \pmod{2^{t-3}}, T_{2^{t-3}+1} \equiv 1 \pmod{2^{t-3}}. \tag{5}$$

We will use again induction on t. Clearly the congruences hold for $t = 6$. So, suppose that they are true for t. Then we use Lemma 4 for $T_{2^{t-2}} = T_{(2^{t-3}-1)+(2^{t-3}+1)}$ to obtain

$$T_{2^{t-2}} = T_{2^{t-3}-1}^2 + (T_{2^{t-3}-2} + T_{2^{t-3}-1})T_{2^{t-3}} + T_{2^{t-3}}T_{2^{t-3}+1}. \tag{6}$$

By using the Tribonacci recurrence, we obtain $T_{2^{t-3}-2} \equiv 2^{t-4} + 1 \pmod{2^{t-3}}$. Now, we write $T_{2^{t-3}-2} = 1 + 2^{t-4} + a2^{t-3}$, $T_{2^{t-3}-1} = b2^{t-3}$, $T_{2^{t-3}} = 2^{t-4} + c2^{t-3}$ and $T_{2^{t-3}+1} = 1 + d2^{t-3}$. Then, we get

$$\begin{aligned} T_{2^{t-2}} &= b^2 2^{2t-6} + (1 + 2^{t-4} + (a+b)2^{t-3})(2^{t-4} + c2^{t-3}) + (2^{t-4} + c2^{t-3})(1 + d2^{t-3}) \\ &\equiv 2^{t-4} + c2^{t-3} + 2^{t-4} + c2^{t-3} \pmod{2^{t-2}} \\ &\equiv 2^{t-3} \pmod{2^{t-2}}, \end{aligned}$$

as desired. Here we used that $2t-8 \geq t-2$, since $t \geq 6$. We proceed similarly to prove the other congruences in (5). Now, by induction hypothesis, we suppose that the congruences in (4) hold for s. Then, we use exactly the same procedure as before (and Lemma 4) for $T_{2^{t-3}(s+1)} = T_{(2^{t-3}s-1)+(2^{t-3}+1)}$. We omit the details. □

Since $16 \mid n$, then $n = 2^{t-3}s$, with s odd and $t \geq 7$. By Lemma 6, we obtain that $\nu_2(T_{2^{t-3}s}) = t-4$ and then

$$\nu_2(T_n) = \nu_2(T_{2^{t-3}s}) = t-4 = \nu_2(2^{t-3}s) - 1 = \nu_2(n) - 1$$

as desired.

Case 2. $n \equiv 12 \pmod{16}$. By (4) together with the Tribonacci recurrence we obtain that $T_{2^{t-3}s-4} \equiv s2^{t-4} \pmod{2^{t-3}}$ holds for all $s \geq 1$ and $t \geq 6$. Since $n \equiv -4 \pmod{16}$, then we can write $n = 2^{t-3}s - 4$, for some $t \geq 7$ and $s \equiv 1 \pmod 2$. Then, we apply the previous congruence to get

$$\nu_2(T_n) = t-4 = \nu_2(2^{t-3}s - 4 + 4) - 1 = \nu_2(n+4) - 1.$$

Case 3. $n \equiv 15 \pmod{16}$. In this case, we know that 32 divides exactly one among $n+1$ and $n+17$. Suppose that $32 \mid n+a$, for some $a \in \{1, 17\}$. Then $\nu_2(n+b) = 4$, for $b \in \{1, 17\} \backslash \{a\}$, and so, we desire to prove that

$$\nu_2(T_n) = \nu_2(n+a) + 1.$$

For that, we proceed as in the other cases to prove that the following congruence holds, for all $s \geq 1, t \geq 8$ and $a \in \{1, 17\}$:

$$T_{2^{t-3}s-a} \equiv s2^{t-2} \pmod{2^{t-1}}$$

with n being in the form $n = 2^{t-3}s - a$. Therefore,

$$\nu_2(T_n) = \nu_2(T_{2^{t-3}s-a}) = t-2 = \nu_2(n+a) + 1.$$

This completes the proof. □

Remark 7. We note that as a starting point one can use a multisection approach (cf. [18]) to discover the underlying structure of the above analysis. Here we applied the 4-section of the ordinary generating function of the sequence $(T_n)_{n\geq 0}$.

We conclude with a

Conjecture 8. For $n \geq n(p)$ with some multiple $\pi'(p)$ of the modulo p period $\pi(p)$ of the Tribonacci sequence T_n and some integer constants $n(p) > 0$, $0 \leq r = r(p) \leq p$, $n_i = n_i(p)$, $c_i' = c_i'(p)$, $c_{i,j} = c_{i,j}(p), 1 \leq i \leq \pi'(p), 1 \leq j \leq r$, we have that

$$\nu_p(T_n) = c_i' + \nu_p\left(\prod_{j=1}^{r}(n + c_{i,j})\right) \quad \text{if} \quad n \equiv n_i \pmod{\pi'(p)}.$$

(Note that it is easy to pick $c_{i,j}$s for i so that $\nu_p(n + c_{i,j}) = 0$.)

4 Proof of Theorem 2

If $m \leq 4$, one can see that the only solutions are the ones listed in the Theorem 2. So we shall suppose that $m \geq 5$. By using Lemma 5 (for $p = 2$) together with Theorem 1, we deduce that

$$\begin{aligned} m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 1 &\leq \nu_2(m!) = \nu_2(T_n) \\ &< \nu_2(n(n+1)(n+4)(n+17)) - 5 \leq 4\nu_2(n+\delta) - 5, \end{aligned}$$

for some $\delta \in \{0, 1, 4, 17\}$. Thus $\nu_2(n+\delta) \geq (m - \lfloor \log m / \log 2 \rfloor + 4)/4$ and therefore, $2^{\lfloor (m - \lfloor \log m / \log 2 \rfloor + 4)/4 \rfloor}$ divides $n + \delta$. In particular, $2^{\lfloor (m - \lfloor \log m / \log 2 \rfloor + 4)/4 \rfloor} \leq n + \delta \leq n + 17$ and by applying the log function, we obtain

$$\left\lfloor \frac{1}{4} \left(m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor + 4 \right) \right\rfloor \leq \frac{\log(n+17)}{\log 2}. \tag{7}$$

On the other hand, by Lemma 3, $(1.83)^{n-2} < T_n = m! < (m/2)^m$ and so $n < 1.66m \log(m/2) + 2$. Substituting this in (7), we arrive at

$$\left\lfloor \frac{1}{4} \left(m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor + 4 \right) \right\rfloor \leq \frac{\log(1.66m \log(m/2) + 19)}{\log 2}.$$

This inequality yields $m \leq 32$ and then $n < 1.66 \cdot 32 \log(32/2) + 2 = 149.27 \cdots$. Now, we use a simple routine written in *Mathematica* which (in a few seconds) does not return us any solution in the range $5 \leq m \leq 32$ and $1 \leq n \leq 149$. The proof is complete. $\square$

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