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Some Statistics on the Hypercubes of Catalan Permutations

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Abstract

For a permutation σ of length 3, we define the oriented graph $Q_n(\sigma)$. The graph $Q_n(\sigma)$ is obtained by imposing edge constraints on the classical oriented hypercube Q_n, such that each path going from 0^n to 1^n in $Q_n(\sigma)$ bijectively encodes a permutation of size n avoiding the pattern σ. The orientation of the edges in $Q_n(\sigma)$ naturally induces an order relation $\preceq_\sigma$ among its nodes. First, we characterize $\preceq_\sigma$. Next, we study several enumerative statistics on $Q_n(\sigma)$, including the number of intervals, the number of intervals of fixed length k, and the number of paths (or permutations) intersecting a given node.

1 Introduction

In this paper, we study several structural and enumerative properties of the classical oriented hypercube Q_n when the hypercube satisfies certain additional edge constraints. The constraints are given in terms of permutation patterns, focusing on patterns of length 3.

The connection between the oriented hypercube Q_n defined over the set of n-binary words Σ^n and the set of permutations of size n is determined by considering each path in Q_n going from 0^n to 1^n as a permutation. There are exactly $n!$ paths from 0^n to 1^n and each one of these uniquely determines a permutation. More precisely, the permutation π_p associated with the path p has the entry i placed in position j if the i-th step of p creates an entry 1 in position j. See (3) and (4).

Taking only those permutations that avoid a pattern σ of length 3, it is possible to characterize those edges of Q_n that need to be removed to maintain the correspondence paths-permutations. In this way, a new ordered structure, $Q_n(\sigma)$, can be defined, where, as in Q_n, the partial order in $Q_n(\sigma)$ is naturally induced by the orientation of the edges (Fig. 1).

According to the classical definition of pattern avoidance, the set of permutations of size n that avoid a pattern σ is denoted by $Av_n(\sigma)$ and, when $|\sigma| = 3$, the cardinality of $Av_n(\sigma)$ is given by the well-known [10] sequence of *Catalan* numbers

$$c_n = |Av_n(\sigma)| = \frac{1}{n+1}\binom{2n}{n}; \tag{1}$$

see sequence A000108 of [9]. Several authors (see [2] and references therein) have focused on the bijective and enumerative properties of these permutations. To the best of our knowledge, their hypercube graph structure has not been investigated.

1.1 Motivation

Our general aim is to introduce — and hopefully motivate further studies of — certain subgraphs $Q_n(\sigma)$ of the oriented hypercube Q_n that can be defined for particular sets of permutation patterns σ. As a case study, we investigate patterns of length three for which $Q_n(\sigma)$ relates to the c_n permutations of $Av_n(\sigma)$.

By linking hypercubes to classes of permutations, we can explore how permutation pattern constraints can affect the different combinatorial statistics defined over the poset and oriented graph Q_n. In this spirit, we study the number of intervals (Section 3) and number of paths intersecting a given node (Section 4) in the oriented graphs $Q_n(\sigma)$. We consider the number of intervals in order to investigate the underlying poset structure of $Q_n(\sigma)$ and compare it with that of the classic hypercube. The second statistic measures instead how the symmetries of the hypercube Q_n are modified in $Q_n(\sigma)$.

The effect of the introduced constraints is quite strong, both from a quantitative and a qualitative point of view. For instance, we show that the expectation of the number of nodes that lie at Hamming distance k and above a randomly selected node depends on both k and n in the unconstrained Q_n, whereas in $Q_n(\sigma)$ it depends only on k if n is large enough. Another interesting new structural property of the oriented hypercube that emerges when pattern constraints are imposed is the loss of symmetry. More precisely, whereas in the unconstrained case Q_n each node of given rank D (= number of 0's) is intersected by a fixed number $n!/\binom{n}{D}$ of paths, other parameters (Table 1) become important when pattern avoidance is considered. Section 4 is indeed dedicated to the computation, for $\sigma \in \{123, 132\}$, of the number of paths/permutations that intersect a node w in $Q_n(\sigma)$. This value does not only depend on the rank of the node and it is strongly affected by the choice of the pattern σ.

Due to their topological structure, hypercubes find applications in the modelling of physical phenomena, with examples ranging from computer science to evolutionary biology. Changing the geometric structure of a model can be of help in understanding how results can be affected by deviations from the standard scenario. For instance, in evolutionary biology, the oriented graph structure of Q_n is commonly used to model accessibility phenomena in random *fitness landscapes* [5, 6]: paths through the nodes of the hypercube represent the possible evolutionary histories of a gene. In the model, the gene can be affected by binary mutations $0, 1$ at its n positions (*loci*). Each node of Q_n is assigned a random variable that determines the fitness value of that particular outcome. Researchers study the probability that there is an accessible (i.e. fitness increasing) path from a global minimum (typically 0^n) to a global maximum (typically 1^n). Recently, investigators have explored the relationships of graph topology to the availability of accessible paths by defining fitness landscapes over different geometric structures (e.g., [8]). The enumerative results provided in this paper can be used in the accessibility computations for models of fitness landscapes defined over the constrained hypercubes $Q_n(\sigma)$.

Finally, it is worth mentioning that the approach presented here to define $Q_n(\sigma)$ for patterns σ of length 3 can be extended to include other permutation patterns (Section 6).

1.2 Summary of the results

In this study, we focus on patterns $\sigma = 132, 123$ as representatives of the two clusters $\{132, 231, 312, 213\}$ and $\{123, 321\}$ that are not equivalent when considered under the standard operators of mirror image, complement, and inverse.

For each pattern σ, in Section 2 we define the oriented (sub)graph $Q_n(\sigma)$, which is obtained by removing particular edges from the classic oriented hypercube Q_n. For both $\sigma = 132, 123$ the number of nodes in the graph $Q_n(\sigma)$ is 2^n (as in Q_n) and the number of paths going from 0^n to 1^n is given by the n-th Catalan number c_n. Indeed, each path of $Q_n(\sigma)$ encodes a permutation of $Av_n(\sigma)$. However, though the considered two classes of permutations $Av_n(132)$ and $Av_n(123)$ are equinumerous, $Q_n(132)$ and $Q_n(123)$ are *not* isomorphic graphs. We conclude the section by characterizing the order relation $\preceq_\sigma$ induced on the nodes of $Q_n(\sigma)$ by the orientation of the edges.

In Section 3, we study e_n, the number of intervals, and $e_{n,k}$, the number of intervals of fixed length k, in the posets $(Q_n(\sigma), \preceq_\sigma)$. These statistics are independent of the pattern σ and they thus show a new common feature of permutations avoiding patterns of length 3. We provide a closed formula for e_n,

$$e_n = 2^{n-3}(n^2 + 3n + 8).$$

We determine a recursion on k for the value $e_{n,k}$ (24). For instance, if $0 \leq k \leq 5$, we find

$$\begin{aligned}
e_{n,0} &= 1 \cdot 2^n \\
e_{n,1} &= 2 \cdot 2^n - n - 2 \\
e_{n,2} &= 3 \cdot 2^n - n^2 - 2n - 3 \\
e_{n,3} &= 4 \cdot 2^n - \frac{1}{2}n^3 - \frac{1}{2}n^2 - 3n - 4 \\
e_{n,4} &= 5 \cdot 2^n - \frac{1}{6}n^4 + \frac{1}{6}n^3 - \frac{11}{6}n^2 - \frac{19}{6}n - 5 \\
e_{n,5} &= 6 \cdot 2^n - \frac{1}{24}n^5 + \frac{1}{6}n^4 - \frac{23}{24}n^3 - \frac{2}{3}n^2 - \frac{9}{2}n - 6.
\end{aligned}$$

In general, we show that for every k

$$e_{n,k} = (k+1) \cdot 2^n + \mathcal{O}(n^k).$$

In Section 4, we find closed formulas to compute $T_\sigma(w)$, the number of paths (or permutations) intersecting a given node $w \in Q_n(\sigma)$. In the notation of Table 1, we find

$$\begin{aligned}
T_{132}(w) &= \binom{D+d}{D} \cdot \frac{D-d+1}{D+1} \cdot \prod_{i=0}^{\ell} c_{u_i}, \\
T_{123}(w) &= \binom{D+d}{D} \cdot \frac{D-d+1}{D+1} \cdot \binom{U+u}{U} \cdot \frac{U-u+1}{U+1}.
\end{aligned} \tag{2}$$

In general, $T_{132}(w) \neq T_{123}(w)$, and we show by simulations that requiring w to satisfy certain constraints can increase or decrease the value of $T_{132}(w)$ with respect to the value of $T_{123}(w)$.

We conclude in Section 5 by introducing a new problem related to path intersection in $Q_n(\sigma)$ and solving some preliminary instances.

2 Constrained hypercubes

2.1 Hypercube and permutations

We let Q_n denote the classic oriented hypercube on the set of binary words $\Sigma^n = \{0,1\}^n$. Each node of the oriented graph Q_n is a binary word of length n. Q_n has exactly 2^n nodes.

If $w_1, w_2 \in \Sigma^n$, their *Hamming distance* $h(w_1, w_2)$ is defined as the number of positions at which the two strings w_1, w_2 are different. For instance, $h(0010, 1001) = 3$ because at the first, third and fourth position the two considered words do not match each other.

If $w_1, w_2 \in \Sigma^n$, we find an oriented edge of Q_n going from w_1 to w_2 if and only if the Hamming distance $h(w_1, w_2) = 1$ and $h(w_1, 0^n) < h(w_2, 0^n)$. The edge orientation naturally induces an order relation $\preceq$ among the nodes of Q_n. We thus consider Q_n as a poset in what follows.

With respect to the strict order relation $\prec$, the number of distinct increasing paths through the nodes of Q_n from 0^n to 1^n is given by $n!$. If S_n denotes the set of permutations of size n, the bijective correspondence holding between a path p of Q_n and a permutation $\pi_p \in S_n$ is sketched in the following example given for size $n = 5$:

$$p = 00000 \to 00010 \to 01010 \to 01011 \to 11011 \to 11111 \tag{3}$$

$$\pi_p = 00000 \to 00010 \to 02010 \to 02013 \to 42013 \to 42513. \tag{4}$$

The rule that leads from p to π_p is such that, at each step, we place the lowest available entry in the permutation in the position specified by the new entry 1 in the path.

2.1.1 Edge constraints of the hypercube

For each pattern $\sigma \in \{132, 123\}$, we define the oriented graph $Q_n(\sigma)$ as a constrained version of Q_n such that the previously described bijection $p \mapsto \pi_p$ maps paths of $Q_n(\sigma)$ onto permutations of $Av_n(\sigma)$. For $\sigma = 132$ and $\sigma = 123$ the definition of $Q_n(\sigma)$ is given as follows:

(i) The constrained hypercube $Q_n(132)$ (Fig. 1 (right)) is obtained by removing from Q_n those edges $w_1 \to w_2$ connecting two words $w_1 \prec w_2$ such that there exist indices $j_1 < j_2 < j_3$ with $w_2(j_3) = 1$, $w_1(j_3) = 0$, $w_1(j_1) = 1$, and $w_1(j_2) = 0$. In other words, we remove edges of type

$$w_1 = \alpha\, 1\, \beta\, 0\, \gamma\, 0\, \delta \nrightarrow \alpha\, 1\, \beta\, 0\, \gamma\, 1\, \delta = w_2.$$

(ii) The constrained hypercube $Q_n(123)$ (Fig. 1 (left)) is obtained by removing from Q_n those edges of type

$$w_1 = \alpha\, 1\, \beta\, 0\, \gamma\, 0\, \delta \nrightarrow \alpha\, 1\, \beta\, 1\, \gamma\, 0\, \delta = w_2.$$

Note that, in both cases (i) and (ii), all the nodes in $\Sigma^n \setminus \{0^n, 1^n\}$ have outdegree and indegree greater than or equal to one in $Q_n(\sigma)$. Therefore, for any fixed $\sigma \in \{132, 123\}$, each node of Σ^n has at least one path of $Q_n(\sigma)$ that intersects it. In particular, the set of nodes of $Q_n(\sigma)$ corresponds to the entire set of binary words Σ^n as it is in Q_n.

2.2 Structural properties of the constrained hypercubes

Inspection of Fig. 1 (circled nodes) shows that the two depicted graphs are not isomorphic. Indeed, suppose there exists an isomorphism $\phi : Q_4(123) \mapsto Q_4(132)$. The isomorphism ϕ would give $\phi(0000) = 0000$ and $\phi(1000) = 1000$, because 1000 is the only node of outdegree 1 that covers 0000. Furthermore, because 1001 (resp., 1100) is the only node that covers

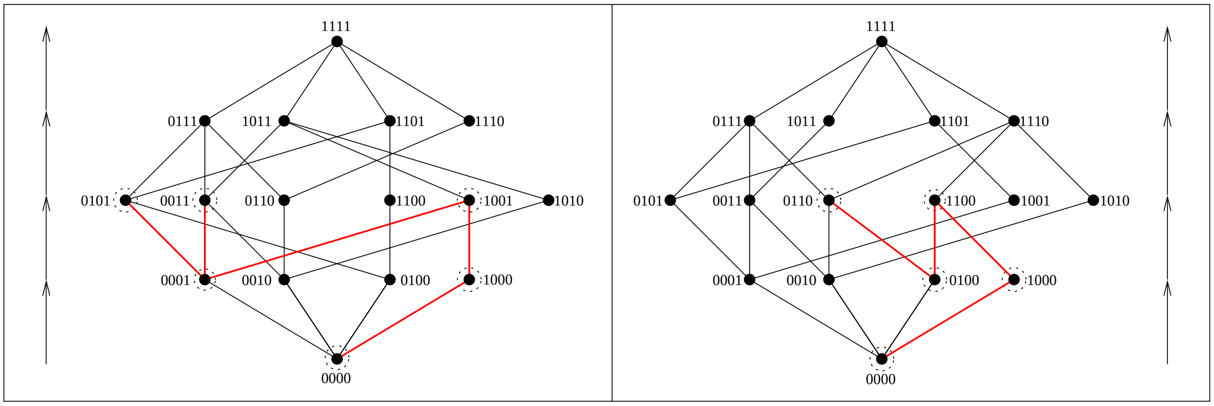

Figure 1: The oriented graphs $Q_4(123)$ (left) and $Q_4(132)$ (right). There are $c_4 = 14$ possible paths from 0000 to 1111 following the orientation specified by the arrows. Circled nodes result in two non-isomorphic regions. This fact shows that $Q_4(123)$ is not isomorphic to $Q_4(132)$ (see text in Section 2.2).

1000 in $Q_4(123)$ (resp., $Q_4(132)$), we would have $\phi(1001) = 1100$. Besides covering node 1000, node 1001 (resp., 1100) covers node 0001 (resp., 0100) in $Q_4(123)$ (resp., $Q_4(132)$). Therefore, we would obtain

$$\phi(0001) = 0100. \tag{5}$$

At the same time, equality (5) is not compatible with an isomorphism ϕ because node 0001 of $Q_4(123)$ has outdegree 3, whereas node 0100 of $Q_4(132)$ has outdgree 2. It follows that the two graphs $Q_n(132)$ and $Q_n(123)$ are *not* in general isomorphic.

2.2.1 Order relation in $Q_n(132)$ and $Q_n(123)$

As for the classic hypercube Q_n, when $\sigma \in \{132, 123\}$, the orientation of the edges in $Q_n(\sigma)$ determines an order relation $\preceq_\sigma$ among the 2^n nodes of $Q_n(\sigma)$. By the definition of $Q_n(\sigma)$, the order relation $\preceq_\sigma$ is clearly a restriction of $\preceq$. It is interesting to note that, even if the two graphs $Q_n(132)$ and $Q_n(123)$ are not in general isomorphic (Fig. 1), there is a duality holding between the order relations $\preceq_{132}$ and $\preceq_{123}$. The duality can be shown introducing some further notation as follows.

Let w be a binary word. We denote by $D = D(w)$ the total number of entries 0 in w and we denote by $d = d(w) \leq D$ the length of the maximal (left) prefix of w containing only 0's (Table 1). We define the word $w^{(i)}$ ($D \geq i \geq 0$) as the word obtained by replacing the *first* i entries 0 of w with 1. For instance, given $w = 001101$, we have $w^{(0)} = 001101, w^{(1)} = 101101, w^{(2)} = 111101$, and $w^{(3)} = 111111$.

The definition of $w^{(i)}$ can be extended to $i \leq 0$ as follows: if $D \geq j \geq 0$, then $w^{(-j)}$ is the word obtained by replacing the *last* j entries 0 of w with 1. Taking as above $w = 001101$, we have $w^{(0)} = 001101, w^{(-1)} = 001111, w^{(-2)} = 011111$, and $w^{(-3)} = 111111$.

Table 1: Key parameters defined over the words in Σ^n. For instance, if $w = 00101110$, then $d = 2, u = 0, \ell = 2, u_1 = 1$, and $u_2 = 3$, whereas, setting $w = 1101011$, we obtain $d = 0, u = 2, \ell = 3, u_1 = 2, u_2 = 1$, and $u_3 = 2$.

$D(w)$	number of 0's in w
$d(w)$	length of the maximal *prefix* of w containing only 0's
$U(w)$	number of 1's in w
$u(w)$	length of the maximal *suffix* of w containing only 1's
$u_i(w)$	length of the i-th (from left to right) block of 1's in w
$\ell(w)$	number of blocks of consecutive 1's in w

With this notation, the next result describes the order relation $\preceq_\sigma$ existing among the nodes of $Q_n(\sigma)$ as it is induced by the orientation of the edges of $Q_n(\sigma)$.

Proposition 1. *We have the following:*

(i) If $w_1, w_2 \in \Sigma^n$ and $d = d(w_1)$, then $w_1 \preceq_{132} w_2$ if and only if

$$w_2 = w \cdot \big(w_1(d+1)w_1(d+2)\cdots w_1(n)\big)^{(i)}, \tag{6}$$

where $i \geq 0$, w is any word of length d, and $w_1(m)$ denotes the m-th letter of w_1.

(ii) If $w_1, w_2 \in \Sigma^n$ and $d = d(w_1)$, then $w_1 \preceq_{123} w_2$ if and only if

$$w_2 = w \cdot \big(w_1(d+1)w_1(d+2)\cdots w_1(n)\big)^{(-j)}, \tag{7}$$

where $j \geq 0$, w is any word of length d, and $w_1(m)$ denotes the m-th letter of w_1.

Proof. The result follows directly from the characterization of the covering relation in the posets $Q_n(132)$ and $Q_n(123)$. In $Q_n(132)$ (resp., $Q_n(123)$), a node v_2 *covers* a node v_1 if and only if v_2 can be obtained from v_1 either by replacing *any* 0 belonging to the maximal prefix of 0's in v_1 with an entry 1 or by replacing the *first* 0 (resp., *last* 0) to the right of the leftmost 1 in v_1 with an entry 1. □

Remark 2. As a corollary, Proposition 1 allows to define an inductive procedure ψ that, given a path p of $Q_n(132)$, creates a dual path $\psi(p)$ of $Q_n(123)$. This mapping is in fact a bijection between $Av_n(132)$ and $Av_n(123)$ that, as far as we know, has not been previously described in the framework of hypercubes.

Given a path p of $Q_n(132)$, such as

$$p = w_0 \to w_1 \to \cdots \to w_n,$$

we define

$$\psi(p) = w'_0 \to w'_1 \to \cdots \to w'_n$$

inductively as follows. Set $w'_0 = 0^n$ and assume that we have already defined w'_i for each $0 \le i \le j$. According to Proposition 1 (i), the word w_{j+1} that covers w_j is obtained from w_j as

$$w_{j+1} = w \cdot \big(w_j(d+1)w_j(d+2)\cdots w_j(n)\big)^{(+i)}, \tag{8}$$

where $d = d(w_j)$, w is a word of length d such that $h(w, 0^d) \le 1$, and $i = 1 - h(w, 0^d) \in \{0, 1\}$. Given (8), we set

$$w'_{j+1} = w \cdot \big(w'_j(d+1)\, w'_j(d+2)\, \cdots\, w'_j(n)\big)^{(-i)}, \tag{9}$$

where w is as in (8).

In particular, because $d(w_0) = d(0^n) = n$, we have $w_1 = w'_1$ and, more in general, $d(w_j) = d(w'_j)$ for all $0 \le j \le n$. Therefore, because the word w placed at the beginning of w'_{j+1} has length $d(w'_j)$, the word w'_{j+1} covers w'_j in agreement with statement (*ii*) of Proposition 1.

The bijection ψ acts on single paths, and it does not imply isomorphism properties of the two considered constrained hypercubes. In particular, if a pair of paths (p_1, p_2) share (or do not share) certain nodes in $Q_n(132)$, the same does not necessarily hold in $Q_n(123)$ for the pair $(\psi(p_1), \psi(p_2))$. For instance, $(2341, 1234)$ do not intersect in $Q_4(132)$, but both $\psi(2341) = 2431$ and $\psi(1234) = 1432$ pass through the node 1011 in $Q_4(123)$.

3 Number of intervals

In this section, we study the number of intervals of $Q_n(\sigma)$. We are thus interested in those pairs $(w_1, w_2) \in \Sigma^n \times \Sigma^n$ with $w_1 \preceq_\sigma w_2$. By Proposition 1, the statistic number of intervals does not depend on the pattern σ and we denote by e_n the number of intervals in $Q_n(\sigma)$ for each $\sigma \in \{132, 123\}$.

3.1 Enumeration of e_n

Fix a word w_1. By Proposition 1, the number of words w_2 greater than or equal to w_1 in $Q_n(\sigma)$ is given by

$$2^d(D - d + 1), \tag{10}$$

where $D = D(w_1)$ and $d = d(w_1)$.

Summing over all possible words w_1, e_n can be computed as

$$e_n = 2^n + \sum_{D=0}^{n-1} \sum_{d=0}^{D} 2^d (D - d + 1) \binom{n - d - 1}{n - D - 1}. \tag{11}$$

The sequence $(e_n)_{n \geq 1}$ starts as

$$3, 9, 26, 72, 192, 496, 1248, 3072, 7424, 17664.$$

Furthermore, observe that

$$\tilde{e}_n = e_n - 2^n = \sum_{D=0}^{n-1} \sum_{d=0}^{D} 2^d (D - d + 1) \binom{n - d - 1}{n - D - 1} \tag{12}$$

counts the number of *strict* intervals of $Q_n(\sigma)$. Strict intervals are those pairs (w_1, w_2) with $w_1 \prec_\sigma w_2$.

The next proposition gives a closed formula for $\tilde{e}_n$ (and thus for e_n).

Proposition 3. *For all $n \geq 1$, we have*

$$\tilde{e}_n = \sum_{D=0}^{n-1} \sum_{d=0}^{D} 2^d (D - d + 1) \binom{n - d - 1}{n - D - 1} = 2^{n-3}(n^2 + 3n).$$

Proof. Observe that by performing the substitution $A = n - d$ and $B = n - D$, we obtain

$$\tilde{e}_n = 2^n \sum_{B=1}^{n} \sum_{A=B}^{n} \left(\frac{1}{2}\right)^A (A - B + 1) \binom{A - 1}{B - 1}.$$

By induction on n, we have

$$\begin{aligned}
\tilde{e}_n &= 2^n \left[\sum_{B=1}^{n-1} \left[\sum_{A=B}^{n-1} \left(\frac{1}{2}\right)^A (A - B + 1) \binom{A - 1}{B - 1} \right] + \left(\frac{1}{2}\right)^n (n - B + 1) \binom{n - 1}{B - 1} \right] + 1 \\
&= 2\tilde{e}_{n-1} + 2^n \left[\sum_{B=1}^{n-1} \left(\frac{1}{2}\right)^n (n - B + 1) \binom{n - 1}{B - 1} \right] + 1 \\
&= 2\tilde{e}_{n-1} + 2^n \left(\frac{1}{4} - 2^{-n} + \frac{n}{4} \right) + 1 \\
&= 2\tilde{e}_{n-1} + 2^{n-2}(n + 1) \\
&= 2 \cdot 2^{n-4}((n - 1)^2 + 3(n - 1)) + 2^{n-2}(n + 1) = 2^{n-3}(n^2 + 3n).
\end{aligned}$$

This concludes the proof. $\square$

Proposition 3 shows that $(\tilde{e}_n)_n$ provides a new interpretation of sequence A001793 of [9]. Furthermore, we have the following corollary.

Corollary 4. *A node of $Q_n(\sigma)$ has on average*

$$\frac{\tilde{e}_n}{2^n} = \frac{n^2 + 3n}{8}$$

nodes strictly above it.

It is interesting to observe that, in the unconstrained hypercube Q_n, the number of intervals $w_1 \preceq w_2$ is given by

$$\sum_{D=0}^{n} 2^D \binom{n}{D} = 3^n. \tag{13}$$

Thus, on average, a random node of Q_n has an exponential

$$\frac{3^n - 2^n}{2^n} \sim (3/2)^n \tag{14}$$

number of nodes strictly above it.

3.2 The number of intervals of given length k

In this section, we refine our previous results by considering the number of intervals with length equal to k; i.e., we count those word pairs (w_1, w_2) such that $w_1 \preceq_\sigma w_2$ and $h(w_1, w_2) = k$. We denote by $e_{n,k}$ the number of intervals of length k, and so $e_n = \sum_{k=0}^{n} e_{n,k}$. Note that Proposition 1 ensures, also in this case, that the statistic $e_{n,k}$ does not depend on the pattern $\sigma \in \{132, 123\}$.

By (6) and (7), for a given word w_1, the number of words w_2 greater than or equal to w_1 in $Q_n(\sigma)$ and such that $h(w_1, w_2) = k$ is given by

$$\sum_{i=0}^{D-d} \binom{d}{k-i}, \tag{15}$$

where $D = D(w_1)$ and $d = d(w_1)$.

Indeed, in the notation of Proposition 1, if w_2 is obtained from w_1 as

$$w_2 = w \cdot \big(w_1(d+1) w_1(d+2) \cdots w_1(n)\big)^{(i)}, \tag{16}$$

then their Hamming distance $k = h(w_1, w_2)$ equals the sum of i and $U(w)$, the latter being the number of entries in w equal to 1. That is, from (16) we have

$$k = h(w_1, w_2) = i + U(w)$$

Table 2: Values of $e_{n,k}$ for $1 \leq n \leq 10$ and $0 \leq k \leq 10$.

$e_{n,k}$	k										
	0	1	2	3	4	5	6	7	8	9	10
$n=1$	2	1	0	0	0	0	0	0	0	0	0
$n=2$	4	4	1	0	0	0	0	0	0	0	0
$n=3$	8	11	6	1	0	0	0	0	0	0	0
$n=4$	16	26	21	8	1	0	0	0	0	0	0
$n=5$	32	57	58	34	10	1	0	0	0	0	0
$n=6$	64	120	141	108	50	12	1	0	0	0	0
$n=7$	128	247	318	291	180	69	14	1	0	0	0
$n=8$	256	502	685	708	535	278	91	16	1	0	0
$n=9$	512	1013	1434	1612	1406	906	406	116	18	1	0
$n=10$	1024	2036	2949	3512	3400	2568	1442	568	144	20	1

and (15) easily follows. Note that if $D < k$ — that is, there are not enough 0's in w_1 — then formula (15) returns 0 because $k - i \geq k - (D - d) = d + (k - D) > d$.

Summing over all possible nodes w_1 of $Q_n(\sigma)$, we thus obtain

$$e_{n,k} = \binom{n}{k} + \sum_{D=0}^{n-1} \sum_{d=0}^{D} \binom{n-d-1}{n-D-1} \sum_{i=0}^{D-d} \binom{d}{k-i}. \tag{17}$$

Using the sum in (17), we can compute the first terms of the sequences $((e_{n,k})_n)_k$. These terms are shown in the table 2 for $1 \leq n \leq 10$ and $0 \leq k \leq 10$. Note that the first column corresponds to entry A000079 of [9] while the second and the third appear respectively as sequences A000295 (*Eulerian* numbers) and A047520.

Using (17), we can write that $\forall k \geq 0$ and $\forall n \geq k+1$

$$\begin{aligned} e_{n,k+1} - e_{n,k} &= \binom{n}{k+1} - \binom{n}{k} + \sum_{D=0}^{n-1} \sum_{d=0}^{D} \binom{n-d-1}{n-D-1} \left[\binom{d}{k+1} - \binom{d}{k-(D-d)} \right] \\ &= \binom{n}{k+1} - \binom{n}{k} + a_{n,k} - b_{n,k}, \end{aligned} \tag{18}$$

where, with $1 \leq k+1 \leq n$,

$$a_{n,k} \equiv \sum_{D=0}^{n-1} \sum_{d=0}^{D} \binom{n-d-1}{n-D-1} \binom{d}{k+1} = 2^n - \sum_{i=0}^{k+1} \binom{n}{i} \quad \text{and} \tag{19}$$

$$b_{n,k} \equiv \sum_{D=0}^{n-1} \sum_{d=0}^{D} \binom{n-d-1}{n-D-1} \binom{d}{k-(D-d)} = (n-k)\binom{n}{k}. \tag{20}$$

To prove the equality in (19), observe that

$$\begin{aligned}
a_{n,k} &= \sum_{D=0}^{n-1}\sum_{d=0}^{n-1}\binom{n-d-1}{n-D-1}\binom{d}{k+1} = \sum_{d=0}^{n-1}\binom{d}{k+1}\sum_{D=0}^{n-1}\binom{n-d-1}{n-D-1} \\
&= \sum_{d=0}^{n-1}\binom{d}{k+1}\sum_{D=d}^{n-1}\binom{n-d-1}{n-D-1} = \sum_{d=0}^{n-1}\binom{d}{k+1}2^{n-d-1} \\
&= 2^{n-1}\sum_{d=k+1}^{n-1}\binom{d}{k+1}\left(\frac{1}{2}\right)^{d} \qquad (21)
\end{aligned}$$

Thus, from (21), the recursion

$$a_{n,k} = 2a_{n-1,k} + \binom{n-1}{k+1}. \qquad (22)$$

Now, by induction on n, assuming that (19) holds for $a_{n-1,k}$, by substituting in (22) we have

$$\begin{aligned}
a_{n,k} &= 2a_{n-1,k} + \binom{n-1}{k+1} = 2\left(2^{n-1} - \sum_{i=0}^{k+1}\binom{n-1}{i}\right) + \binom{n-1}{k+1} \\
&= 2^n - \sum_{i=0}^{k+1}\binom{n-1}{i} - \sum_{i=0}^{k}\binom{n-1}{i} = 2^n - \left[\sum_{i=0}^{k+1}\binom{n}{i} - \binom{n-1}{i-1}\right] - \sum_{i=0}^{k}\binom{n-1}{i} \\
&= 2^n - \sum_{i=0}^{k+1}\binom{n}{i} - \left[\sum_{i=0}^{k}\binom{n-1}{i} - \sum_{i=0}^{k+1}\binom{n-1}{i-1}\right] = 2^n - \sum_{i=0}^{k+1}\binom{n}{i}
\end{aligned}$$

Checking (19) for $a_{k+1,k}$ completes the proof.

To prove the equality in (20), note that

$$b_{n,k} = \sum_{D=k}^{n-1}\sum_{d=0}^{D}\binom{n-d-1}{n-D-1}\binom{d}{k-(D-d)}.$$

Thus, setting $A = n-d$ and $B = n-D$, we have the recursion

$$
\begin{aligned}
b_{n+1,k} &= \sum_{B=1}^{n+1-k}\sum_{A=B}^{n+1}\binom{A-1}{B-1}\binom{n+1-A}{k-A+B} \\
&= \left[\sum_{B=1}^{n-k}\sum_{A=B}^{n+1}\binom{A-1}{B-1}\binom{n+1-A}{k-A+B}\right] + \sum_{A=B=n+1-k}^{n+1}\binom{A-1}{n-k}\cdot 1 \\
&= \binom{n+1}{k} + \sum_{B=1}^{n-k}\sum_{A=B}^{n+1}\binom{A-1}{B-1}\binom{n+1-A}{k-A+B} \\
&= \binom{n+1}{k} + \left[\sum_{B=1}^{n-k}\sum_{A=B}^{n}\binom{A-1}{B-1}\binom{n+1-A}{k-A+B}\right] + \sum_{B=1}^{n-k}\binom{n}{B-1}\binom{0}{k-n-1+B} \\
&= \binom{n+1}{k} + \sum_{B=1}^{n-k}\sum_{A=B}^{n}\binom{A-1}{B-1}\left[\binom{n-A}{k-A+B} + \binom{n-A}{k-1-A+B}\right] \\
&= \binom{n+1}{k} + b_{n,k} + \sum_{B=1}^{n-k}\sum_{A=B}^{n}\binom{A-1}{B-1}\binom{n-A}{k-1-A+B} \\
&= \binom{n+1}{k} + b_{n,k} + \left[\sum_{B=1}^{n-(k-1)}\sum_{A=B}^{n}\binom{A-1}{B-1}\binom{n-A}{k-1-A+B}\right] - \sum_{A=B=n+1-k}^{n}\binom{A-1}{n-k}\cdot 1 \\
&= \binom{n+1}{k} + b_{n,k} + b_{n,k-1} - \binom{n}{k-1} = b_{n,k} + b_{n,k-1} + \binom{n}{k}. \qquad (23)
\end{aligned}
$$

Now, by induction on n and k, assuming that (20) holds for $b_{n,k}$ and $b_{n,k-1}$, by substituting in (23),

$$
\begin{aligned}
b_{n+1,k} &= (n-k)\binom{n}{k} + (n-k+1)\binom{n}{k-1} + \binom{n}{k} \\
&= (n-k)\left[\binom{n}{k} + \binom{n}{k-1}\right] + \binom{n+1}{k} \\
&= (n+1-k)\binom{n+1}{k}.
\end{aligned}
$$

Checking (20) for $b_{n,0}$ ($\forall n \geq 1$) and $b_{k+1,k}$ ($\forall k \geq 0$) completes the proof.

Finally, plugging (19) and (20) in (18) we obtain a recursion for the number of intervals of a given length.

Proposition 5. *Starting with $e_{n,0} = 2^n$, for every $k \geq 0$ and for every $n \geq k+1$, we have*

$$
e_{n,k+1} = e_{n,k} + 2^n - \sum_{i=0}^{k}\binom{n}{i} - (n-k+1)\binom{n}{k}. \qquad (24)
$$

For $0 \le k \le 5$ and $n \ge k$, formulas for $e_{n,k}$ are shown below

$$
\begin{aligned}
e_{n,0} &= 1 \cdot 2^n \\
e_{n,1} &= 2 \cdot 2^n - n - 2 \\
e_{n,2} &= 3 \cdot 2^n - n^2 - 2n - 3 \\
e_{n,3} &= 4 \cdot 2^n - \frac{1}{2}n^3 - \frac{1}{2}n^2 - 3n - 4 \\
e_{n,4} &= 5 \cdot 2^n - \frac{1}{6}n^4 + \frac{1}{6}n^3 - \frac{11}{6}n^2 - \frac{19}{6}n - 5 \\
e_{n,5} &= 6 \cdot 2^n - \frac{1}{24}n^5 + \frac{1}{6}n^4 - \frac{23}{24}n^3 - \frac{2}{3}n^2 - \frac{9}{2}n - 6.
\end{aligned}
$$

By iterating (24), we obtain the next corollary.

Corollary 6. *For a fixed $k \ge 0$, when $n \ge k$, we have*

$$e_{n,k} = (k+1) \cdot 2^n + \mathrm{Pol}_k(n), \tag{25}$$

where $\mathrm{Pol}_k(n)$ is a polynomial of degree k in n. Therefore, the average number of nodes at distance k from a randomly selected one is given by

$$\frac{e_{n,k}}{2^n} = k + 1 + \mathcal{O}\left(\frac{n^k}{2^n}\right) \quad (n \to \infty). \tag{26}$$

Formula (26) says that, if n is large enough, each node of $Q_n(\sigma)$ has on average $k+1$ nodes at distance k above it. In the unconstrained hypercube Q_n, the number of intervals of length k is given by

$$\sum_{D=0}^{n} \binom{D}{k}\binom{n}{D} = 2^{n-k}\binom{n}{k}; \tag{27}$$

also see sequences A038207 and A065109 of [9]. Thus, on average, a random node of Q_n has

$$\binom{n}{k} \Big/ 2^k \tag{28}$$

nodes above at distance k. This value strongly depends on n (and k), whereas in $Q_n(\sigma)$, for n sufficiently large, the only parameter is k (26).

4 Number of permutations intersecting a node

In this section, we compute the number of permutations belonging to $Av_n(\sigma)$ that intersect a given node w of $Q_n(\sigma)$. This number is denoted by $T_\sigma(w)$ or simply by $T(w)$. For a fixed node w, in general $T_{132}(w) \ne T_{123}(w)$. The statistic $T_\sigma(w)$ thus depends on the pattern σ. To start, let us focus on $\sigma = 132$.

Table 3: Values of $\alpha_{d,D}$ for $0 \leq d \leq 10$ and $1 \leq D \leq 10$.

$\alpha_{d,D}$	d										
	0	1	2	3	4	5	6	7	8	9	10
$D = 1$	1	1	0	0	0	0	0	0	0	0	0
$D = 2$	1	2	2	0	0	0	0	0	0	0	0
$D = 3$	1	3	5	5	0	0	0	0	0	0	0
$D = 4$	1	4	9	14	14	0	0	0	0	0	0
$D = 5$	1	5	14	28	42	42	0	0	0	0	0
$D = 6$	1	6	20	48	90	132	132	0	0	0	0
$D = 7$	1	7	27	75	165	297	429	429	0	0	0
$D = 8$	1	8	35	110	275	572	1001	1430	1430	0	0
$D = 9$	1	9	44	154	429	1001	2002	3432	4862	4862	0
$D = 10$	1	10	54	208	637	1638	3640	7072	11934	16796	16796

4.1 Case $\sigma = 132$

For a node w, we define $\alpha(w)$ as the number of paths in $Q_n(132)$ going from w to 1^n. Similarly, $\beta(w)$ counts those paths from 0^n to w. Then $T(w) = \alpha(w) \cdot \beta(w)$.

Let us start by computing $\alpha(w)$. Using the notation of Table 1, we have recursion (29), which is obtained by summing $\alpha(w')$ over the nodes w' that cover w:

$$\alpha(w) = \alpha_{d,D} = \alpha_{d,D-1} \cdot (1 - \delta_{d,D}) + \left(\sum_{i=0}^{d-1} \alpha_{i,D-1} \right), \tag{29}$$

where $\alpha_{0,1} = \alpha_{1,1} = 1$.

Formula (29) allows the calculation of the terms $\alpha_{d,D}$, for the first values of d and D. Results are collected in Table 3.

As it easily follows from (29), each entry of Table 3 above is computed by summing the entries in the previous row with a lesser or equal d-value. This results in the equivalent recursion

$$\alpha_{d,D} = \alpha_{d-1,D} + \alpha_{d,D-1},$$

where $\alpha_{0,1} = \alpha_{1,1} = 1$ and $\alpha_{d,D} = 0$ if $d > D$. Table 3 thus corresponds to a well-known [1] Catalan triangle A009766 whose entries also determine the distribution of important statistics defined for other Catalan structures, such as the length of the last descent or the number of primitive subpaths [4] in Dyck paths.

Defining the generating function

$$A(x, y) = \sum_{D \geq 1} \sum_{d=0}^{D} \alpha_{d,D} x^d y^D,$$

then

$$
\begin{aligned}
A(x,y) &= \frac{1}{1-y} - 1 + xy + +y \sum_{D\geq 1}\sum_{d=1}^{D} \alpha_{d,D} x^d y^D + x \sum_{D\geq 2}\sum_{d=0}^{D-1} \alpha_{d,D} x^d y^D \\
&= \frac{1}{1-y} - 1 + xy + y\left(A(x,y) - \frac{1}{1-y} + 1\right) \\
&\quad +x\left[A(x,y) - y - xy - \left(\frac{1-\sqrt{1-4xy}}{2xy} - 1 - xy\right)\right].
\end{aligned}
$$

Solving gives

$$
A(x,y) = \frac{1 - 2xy - 2y^2 - \sqrt{1-4xy}}{2y(x+y-1)}
$$

and coefficients $\alpha_{d,D}$ are therefore

$$
\alpha(w) = \binom{D+d}{D} \cdot \frac{D-d+1}{D+1}. \tag{30}
$$

To compute the number of paths going from 0^n to a node w, we consider, as in Table 1, the parameters $u_1, u_2, \ldots, u_\ell$, where $\ell = \ell(w)$ is the number of blocks of consecutive 1's in w and $u_i = u_i(w)$ is the length of the i-th block taken from left to right. Indeed, observe that each path connecting 0^n to $w = w(u_1, \ldots, u_\ell)$ first creates the sequence of 1's corresponding to u_ℓ, followed by the sequence corresponding to $u_{\ell-1}$, and so on up to u_1. Each step u_i can be completed in exactly c_{u_i} ways and therefore $\beta(w)$ is given by

$$
\beta(w) = \prod_{i=0}^{\ell} c_{u_i}. \tag{31}
$$

We can now provide a new combinatorial interpretation of the following well-known recursive relation

$$
c_n = \beta(1^n) = \sum_{i=0}^{n-1} \beta(1^i 0 1^{n-1-i}) = \sum_{i=0}^{n-1} c_i \cdot c_{n-1-i}.
$$

The next proposition determines a formula for the computation of $T(w) = \alpha(w) \cdot \beta(w)$.

Proposition 7. *For a node w of $Q_n(132)$, with the notation of Table 1, we have*

$$
T_{132}(w) = \binom{D+d}{D} \cdot \frac{D-d+1}{D+1} \cdot \prod_{i=0}^{\ell} c_{u_i}. \tag{32}
$$

4.2 Case $\sigma = 123$

In the case $\sigma = 123$, the value of $\alpha(w)$ — the number of paths from the node w to 1^n — can be computed exactly as when $\sigma = 132$ (30). What is different is the computation of $\beta(w)$: the number of paths from 0^n to w. In the easiest case, node w ends with an entry equal to 0, and then $\beta(w) = 1$. In general, if u denotes the length of the maximal suffix of w containing only entries equal to 1 (Table 1), then, similarly to (29), we have the recursion

$$\beta(w) = \beta_{u,U} = \beta_{u,U-1} \cdot (1 - \delta_{U,u}) + \sum_{i=0}^{u-1} \beta_{i,U-1},$$

where $\beta_{0,1} = \beta_{1,1} = 1$ and the parameter U is as in Table 1. The same kind of computation that led from (29) to (30) now gives

$$\beta(w) = \binom{U+u}{U} \cdot \frac{U-u+1}{U+1}. \tag{33}$$

Combining the results together, we obtain a formula for $T(w) = \alpha(w) \cdot \beta(w)$.

Proposition 8. *For a node w of $Q_n(123)$, with the notation of Table 1, we have*

$$T_{123}(w) = \binom{D+d}{D} \cdot \frac{D-d+1}{D+1} \cdot \binom{U+u}{U} \cdot \frac{U-u+1}{U+1}. \tag{34}$$

4.3 Expected number of intersections for a random node

Assuming a uniform distribution over the nodes w of $Q_n(\sigma)$, the expected value of $T_\sigma(w)$ coincides in the two cases $\sigma = 132, 123$. The expectation can be computed by considering the natural ranking of $Q_n(\sigma)$, where two nodes w_1, w_2 have the same rank r if $D(w_1) = D(w_2) = r$. Indeed, summing the value of $T_\sigma(w)$ over all the possible nodes w rank by rank,

$$\sum_{w \in Q_n(\sigma)} T_\sigma(w) = (n+1)c_n = \binom{2n}{n}.$$

It follows that on average, both in $Q_n(132)$ and in $Q_n(123)$, we have

$$\mathrm{E}\big(T_\sigma(w)\big) = \frac{1}{2^n} \cdot \binom{2n}{n}. \tag{35}$$

Similarly, we obtain the expectation of $T_\sigma(w)$ at each rank r as

$$\mathrm{E}\big(T_\sigma(w) \big| D(w) = r\big) = c_n \Big/ \binom{n}{r}. \tag{36}$$

The results obtained in (35) and (36) can be rephrased by saying that, removing a random node chosen uniformly from the entire hypercube $Q_n(\sigma)$ or from a given rank $\{w : D(w) = r\}$

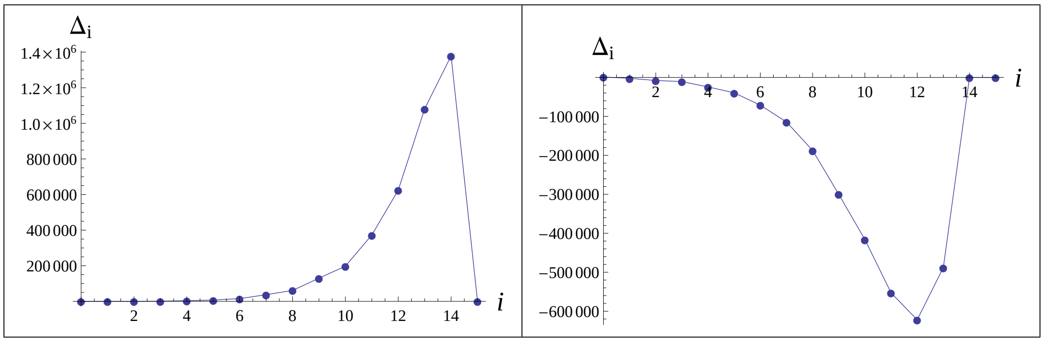

Figure 2: Picking random nodes from Σ_i^n (left) and Σ_{-i}^n (right) , for $n = 15$. The x-axis determines the value of i. For each $0 \leq i \leq 15$ we randomly select $2 \cdot 10^3$ nodes w from Σ_i^n (resp. Σ_{-i}^n). For each selected node w, we compute $\Delta_{i,w} = T_{132}(w) - T_{123}(w)$ and we take Δ_i as the average of $\Delta_{i,w}$ over all the selected w's. Δ_i is shown on the vertical axis.

of $Q_n(\sigma)$, the number of paths (or permutations) that consequently cease to exist is on average the same in the two scenarios $\sigma = 132$ and $\sigma = 123$.

This equivalence does not hold in general. Using random simulations, we show that if we remove nodes from certain regions of the hypercube $Q_n(\sigma)$, the number of permutations that remain in the graph can strongly depend on the pattern σ. As an example (Fig. 2), we can indeed consider the two regions

$$\begin{aligned} \Sigma_i^n &= \{w \in \Sigma^n : w \text{ starts with } j \geq i \text{ entries } 1\} \\ \Sigma_{-i}^n &= \{w \in \Sigma^n : w \text{ ends with } j \geq i \text{ entries } 1\}. \end{aligned}$$

In Fig. 2 (left), we pick random nodes from Σ_i^n, whereas on the right, we take nodes from Σ_{-i}^n. In both cases, the vertical axis gives the value of the difference $T_{132}(w) - T_{123}(w)$ averaged over several randomly selected nodes w.

5 Open problem: intersecting permutations

In Section 4, we have studied the number of permutations of $Av_n(\sigma)$ that intersect a given node $w \in \Sigma^n$. Going a step further, one can consider the number of permutations of $Av_n(\sigma)$ that intersect a given permutation $\pi \in S_n$. In other words, it is interesting to address the following general question:

Problem. Given a path p of Q_n, say

$$p = 0^n \to w_1 \to \cdots \to w_{n-1} \to 1^n, \tag{37}$$

how many paths of $Q_n(\sigma)$ intersect p only in the extreme points 0^n and 1^n?

When

$$p = p_{id} = 0^n \to 10^{n-1} \to 110^{n-2} \to \cdots \to 1^n \tag{38}$$

is the path associated with the identity permutation $\pi_{id} = (1\,2\,3\,\ldots\,n)$, the problem reduces to the computation of the number of *indecomposable* permutations in $Av_n(\sigma)$. Indeed, a permutation $\pi = (\pi_1 \cdots \pi_n)$ is indecomposable [3, 7, 10] when there is no index $i \in [1, n)$ such that $\{\pi_1, \ldots, \pi_i\} = \{1, \ldots, i\}$.

As introductory examples, we provide the answer to the problem defined above for two instances of the path p. We indeed consider the case $p = p_{id}$ as in (38) and the case

$$p = \psi(p_{id}) = 0^n \to 10^{n-1} \to 10^{n-2}1 \to 10^{n-3}11 \to \cdots \to 101^{n-2} \to 1^n, \tag{39}$$

where ψ is the bijection that maps paths of $Q_n(132)$ onto paths of $Q_n(123)$ as defined in Section 2.2.1. Note that the path in (39) corresponds to the permutation $\pi_{\psi(p_{id})} = (1\,n\,n-1\,\ldots\,2)$.

If we denote by $j_n(p)$ the number of paths (permutations) of $Q_n(\sigma)$ that intersect p only in the extreme poits 0^n and 1^n, we have the following result:

Proposition 9. *If p_{id} is the path of the identity permutation, then we have*

$$j_n(p) = \begin{cases} c_n - c_{n-1}, & \text{if } p = p_{id} \text{ and } \sigma = 132; \\ c_n - n + 1, & \text{if } p = p_{id} \text{ and } \sigma = 123; \\ c_n - \sum_{i=1}^{n-1} c_{i-1}, & \text{if } p = \psi(p_{id}) \text{ and } \sigma = 132; \\ c_n - c_{n-1}, & \text{if } p = \psi(p_{id}) \text{ and } \sigma = 123. \end{cases}$$

Proof. We have four cases depending on the path p and the pattern σ.

i) Case $p = p_{id}$. Following the notation of (37), for $1 \leq i \leq n-1$ we thus have $w_i = 1^i 0^{n-i}$. Let us first focus on $\sigma = 123$. Note that $T_{123}(w_1) = T_{123}(w_2) = \cdots = T_{123}(w_{n-1}) = 1$. At the same time, for all indices $i \neq i'$ in $[1, n-1]$ nodes w_i and $w_{i'}$ are incomparable in $\preceq_{123}$ (Proposition 1) and therefore the path intersecting w_i does not intersect $w_{i'}$. Thus, by subtracting one path for each node w_i from the total number c_n of paths present in $Q_n(123)$, we have

$$j_n(p) = c_n - n + 1 \quad (\sigma = 123). \tag{40}$$

Take $\sigma = 132$. Note that all the paths that intersect w_i also pass through w_{i+1}, because w_{i+1} is the only node that covers w_i in $\preceq_{132}$ (Proposition 1). Thus

$$j_n(p) = c_n - T_{132}(w_{n-1}) = c_n - T_{132}(1^{n-1}0) = c_n - c_{n-1} \quad (\sigma = 132), \tag{41}$$

and $j_n(p)$ gives in this case a new interpretation of sequence A000245 of [9].

ii) Case $p = \psi(p_{id})$. We now consider p as in (39). Thus, in the notation of (37), for $1 \leq i \leq n-1$ we have $w_i = 10^{n-i}1^{i-1}$. Take first $\sigma = 123$. In this case, all the paths through w_i also intersect w_{i+1}, because w_{i+1} is the only node that covers w_i in $\preceq_{123}$ (Proposition 1). Therefore, as in (41), we have

$$j_n(p) = c_n - T_{123}(w_{n-1}) = c_n - T_{123}(101^{n-2}) = c_n - c_{n-1} \quad (\sigma = 123). \tag{42}$$

When $\sigma = 132$, we have $T_{132}(w_i) = c_{i-1}$. Furthermore, for all indices $i \neq i'$ in $[1, n-1]$ nodes w_i and $w_{i'}$ are incomparable in $\preceq_{132}$ (Proposition 1) and therefore the paths intersecting w_i do not intersect $w_{i'}$. Thus, by subtracting c_i paths for each node w_i from the total number c_n of paths present in $Q_n(132)$, we obtain

$$j_n(p) = c_n - \sum_{i=1}^{n-1} c_{i-1} \quad (\sigma = 132). \tag{43}$$

Considering (40), (41), (42), and (43) concludes the proof. □

6 Conclusions

We have introduced the oriented graphs $Q_n(\sigma)$ defined over the set of binary words Σ_n, where σ is a permutation pattern of length three: $\sigma \in \{132, 123\}$. The poset $Q_n(\sigma)$, with its order relation $\preceq_\sigma$, is obtained from the classical hypercube Q_n by requiring edge constraints such that each strictly increasing path in $Q_n(\sigma)$ from 0^n to 1^n bijectively encodes a permutation of $Av_n(\sigma)$.

We have investigated some of the combinatorial properties of the posets $Q_n(\sigma)$. In particular, we have studied their numbers of intervals, which are independent of the pattern σ. More precisely, Proposition 1 characterizes the order relations $\preceq_\sigma$ induced by the orientation of the edges in $Q_n(\sigma)$. Proposition 3 and Corollary 4 give closed formulas for the number of intervals in $Q_n(\sigma)$ and the associated expectation. Proposition 5 and Corollary 6 refine the result by considering the numbers of intervals of given length. We compared these results with those obtained for the unconstrained hypercube Q_n (see (13) and (27)).

To highlight some other differences between the non-isomorphic oriented graphs $Q_n(123)$ and $Q_n(132)$ and the unconstrained hypercube Q_n, in Section 4 we focused on the number of paths/permutations intersecting a given node. Formulas are given according to the simple parameters described in Table 1. Proposition 7 covers the case $\sigma = 132$, and Proposition 8 determines the result for $\sigma = 123$. Whereas the expected number of paths intersecting a random node is the same in the two scenarios $\sigma = 132, 123$, simulations showed that this is not true in general when we pick nodes from particular subsets of Σ^n.

Finally, in Section 5, we introduced a new problem related to path intersection in $Q_n(\sigma)$ and solved some preliminary instances.

6.1 Other patterns

It is quite natural to ask whether results similar to those shown here could be obtained for other types of constrained hypercubes, such as $Q_n(\sigma)$ where σ is a permutation-pattern of length greater than 3. If we take any *single* pattern σ, with $|\sigma| > 3$, it is easy to see that the approach used above to define $Q_n(\sigma)$ by removing single edges from the unconstrained Q_n does not apply anymore. Indeed, we cannot explicitly characterize those edges of Q_n that are responsible for the presence of those paths/permutations containing the pattern σ. For example, if we take $\sigma = 1243$, we cannot remove from Q_4 the edge $1100 \to 1101$ because, besides 1243, this would cancel also the permutation 2143. In other words, for single patterns $|\sigma| > 3$, there is no subset of the edges of Q_n that is responsible for the appearance of all and *only* the permutations containing σ.

Considering *sets* of patterns things change and, in some special cases, for sets of patterns of length greater than 3, we can still define the associated hypercube. For instance, considering the set of patterns $\sigma = \{1243, 2143\}$, $Q_n(\sigma)$ can be obtained by removing from Q_n edges of the form

$$\alpha\,1\,\beta\,1\,\gamma\,0\,\delta\,0\,\epsilon \to \alpha\,1\,\beta\,1\,\gamma\,0\,\delta\,1\,\epsilon.$$

A complete characterization of those sets of patterns σ for which the associated class of permutations $Av_n(\sigma)$ admits an hypercube representation is still missing. It would be of interest to see whether the classes of permutations with an hypercube representation share some common enumerative features that characterize them.

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