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An Aperiodic Subtraction Game of Nim-Dimension Two

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Abstract

In a recent manuscript, Fox studied infinite subtraction games with a finite (ternary) and aperiodic Sprague-Grundy function. Here we provide an elementary example of a game with the given properties, namely the game given by the subtraction set $\{F_{2n+1} - 1\}$, where F_i is the ith Fibonacci number, and n ranges over the positive integers.

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1 Introduction

In a recent preprint, Fox [2] studied infinite and aperiodic subtraction games [1, p. 84] with a finite, ternary, Sprague-Grundy function. For an impartial game, the Sprague-Grundy value is computed recursively as the least nonnegative integer not in the set of values of the move options, and starting with the terminal position(s) which have Sprague-Grundy value zero [9, 3]. In this note we provide an elementary example of a game with the given properties. In particular, this means our game has nim-dimension two[2].

Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. Let $A(n) = \lfloor n\phi \rfloor$, $B(n) = \lfloor n\phi^2 \rfloor$, and $AB(n) = A(B(n)) = A(n) + B(n) = 2\lfloor n\phi \rfloor + n$ for all nonnegative integers n; see also Kimberling's paper [4]. Then, define sets $A = \{A(n)\}_{n\geqslant 1}$, $B = \{B(n)\}_{n\geqslant 1}$, and $AB = \{AB(n)\}_{n\geqslant 1}$. Further, let $B_0 \oplus 1 = \{B(n)+1\}_{n\geqslant 0}$, and $AB \oplus 1 = \{2\lfloor n\phi \rfloor + n + 1\}_{n\geqslant 1}$. (In general, we let $X_0 = X \cup \{0\}$ if X is a set of integers.) It is worth noting that if the sets defined here are thought of as sequences, they all appear in the OEIS [10]. A appears as A000201, B as A001950, AB as A003623, $B_0 \oplus 1$ as A026352, and $AB \oplus 1$ as A089910.

Throughout this paper, we will use F_i to denote the ith Fibonacci number ($F_1 = F_2 = 1$ and so on). We will frequently use the following famous numeration system: each positive integer is expressed uniquely as a sum of distinct non-consecutive Fibonacci numbers of index at least two. Though this representation has been discovered independently many times [5, 7, 13], it is typically referred to as the *Zeckendorf representation*. It is well known that $x \in A$ if and only if the smallest Fibonacci term in the Zeckendorf representation of x has an even index [9]. Let $z_i = z_i(x)$ denote the ith smallest index of a Fibonacci term in the Zeckendorf representation of the number x. Then, the set A contains all the numbers with $z_1 \geqslant 2$ even. Further, for all n, $B(n)$ is the *left-shift* of $A(n)$; that is, the set B contains all the numbers with $z_1 \geqslant 3$ odd. Another well-known Fibonacci-type representation of integers is the *least-odd* representation (which Silber [9] calls the second canonical representation), where the smallest index is odd $\geqslant 1$ and no two consecutive Fibonacci numbers are used. Let $\ell_i(x)$ denote the ith smallest index in the least-odd representation of x. Then ℓ_1 is odd. By using this representation we find that $A(n)$ is the left-shift of n for any positive integer n. That is, if $n = F_{\ell_j} + \cdots + F_{\ell_1}$, then $A(n) = F_{\ell_j+1} + \cdots + F_{\ell_1+1}$.

2 Our construction

In this section, we will construct our example of an aperiodic subtraction game. Let $S = \{F_{2n+1} - 1\} = \{1, 4, 12, \ldots\}$, where n ranges over the positive integers. The two-player subtraction game S is played as follows. The players alternate in moving. From a given position, a nonnegative integer, p, the current player moves to a new integer of the form $p - s \geqslant 0$, where $s \in S$. A player unable to move, because no number in S satisfies the

[2]The number of power-of-two-components defines the group of nim-values generated by the games; this group is of order four so the dimension is two. In the classical definition [8], this dimension would have been one.

inequality, loses. Our main result states that the sequence of Sprague-Grundy values for this game is a ternary, aperiodic sequence. First, we need the following lemma.

Lemma 1. *The sets B_0, $B_0 \oplus 1$, $AB \oplus 1$ partition the nonnegative integers.*

Proof. By the work of Wythoff [12], it suffices to prove that the sets $B \oplus 1$ and $AB \oplus 1$ partition the set A.

Claim: For numbers in $AB \oplus 1$, we get $z_2 \geqslant 4$ even and $z_1 = 2$. (Hence $AB \oplus 1 \subset A$.) The claim is proved by noting that the least-odd representation coincides with the Zeckendorf representation for numbers of the form $B(n)$. Hence $AB(n)$ is the left-shift of $B(n)$, which proves the claim, since $z_1(B(n)) \geqslant 3$.

We must also show that $B_0 \oplus 1 \subset A$ contains all representatives with $z_1 \geqslant 4$ even. This follows, since B contains all representatives with $z_1 = 3$ odd (since $F_4 = F_3 + 1$, $F_6 = F_5 + F_3 + 1$ and so on). Further, since B contains all representatives with $z_1 \geqslant 5$ odd, $B \oplus 1$ contains all representatives with $z_2 \geqslant 5$ odd and $z_1 = 2$. Finally, this set also contains the representative with just $z_1 = 2$. □

Note that because the golden ratio is an irrational number, the sets in Lemma 1 are aperiodic when thought of as sequences (in fact they follow a beautiful fractal pattern [6, Thm. 2.1.13, p. 51] related to the Fibonacci morphism).

We can now prove our main theorem.

Theorem 2. *The Sprague-Grundy value of the subtraction game S is $g(p) = 0$ if $p \in B_0, g(p) = 1$ if $p \in B_0 \oplus 1$ and $g(p) = 2$ if $p \in AB \oplus 1$.*

Proof. We begin by showing that, if $p \in B_0$, then no follower of p is in B_0, which corresponds to showing that $g(p) = 0$. This holds for $p = 0$. Thus, it suffices to show that $x = x(i) = p - F_{2i+1} + 1 \in A$, for all $i > 0$ such that $p \geqslant F_{2i+1}$, which is true if and only if the Zeckendorf representation's smallest term is even indexed, i.e. $z_1(x)$ is even. It holds trivially unless $p - F_{2i+1}$ has as the smallest term F_3 or F_2. In case the former, then we compute $F_3 + F_2$ and get F_4. Unless F_5 is contained in the representation we are done. Continuing this argument gives the claim in the first case.

We show next that $z_1(p - F_{2i+1}) > 2$. Observe that

$$z_1(p) \geqslant 3 \text{ is odd.} \tag{1}$$

If $z_1(p) > 2i + 1$, that is, if the smallest Zeckendorf term, say F_{2j+1}, in p has index greater than $2i + 1$, then

$$F_{2j+1} - F_{2i+1} = F_{2j} + \cdots + F_{2i+2}. \tag{2}$$

Hence, in this case, $z_1(x) \geqslant 3$, so F_2 is not the smallest term. The case $i = j$ is trivial. Hence $j < i$, i.e. $z_1(p) < 2i + 1$, which implies $z_1(p - F_{2i+1}) \geqslant 2j + 1 > 2$, by (1).

Suppose next that $p \in B_0 \oplus 1$. We need to show that there is a follower in B_0, but no follower in $B_0 \oplus 1$. Let $b = p - 1 \in B_0$. Then $b + 1 - (F_{2i+1} - 1) = b - F_{2i+1} + F_3 \in B$

if $i=1$ (which solves the first part). Suppose now, that p has a follower in $B_0 \oplus 1$. Then $b+1-(F_{2i+1}-1) \in B_0 \oplus 1$, that is $b-(F_{2i+1}-1) \in B_0$, which is contradictory by the first paragraph.

At last we prove that if $p \in AB \oplus 1$ then p has both a follower in B_0 and in $B_0 \oplus 1$, but no follower in $AB \oplus 1$. We begin with the latter. Note that $z_1(p)=2$.

We want to show that $p - F_{2i+1} + 1 \notin AB \oplus 1$, for all i. Thus, it suffices to show $\alpha = p - F_{2i+1} \notin AB$. We may assume that there is a smallest k such that $F_k \geqslant F_{2i+1}$, and where F_k is a term in the Zeckendorf representation of p. Claim: If k is odd, then $\alpha \in A \setminus AB$, and otherwise $\alpha \in B \cup (A \setminus AB)$. It suffices to prove this claim to prove this case. For the first part it is easy to see that $z_1(\alpha)=2$, since $z_1(p)=2$ and by (2). If k is even, then we study the greatest Zeckendorf term in p, smaller than F_{2i+1}, say F_ℓ with existence of $\ell \leqslant 2i$ clear by definition of p. If $\ell = 2i$, then $F_k + F_\ell - F_{2i+1} = y + 2F_{2i} = y + F_{2i+1} + F_{2i-2}$, where y has no terms smaller than F_{2i+3}. If $\ell = 2i-1$, then similarly $F_k + F_\ell - F_{2i+1} = y + F_{2i} + F_{2i-1} = y + F_{2i+1}$, and if $\ell < 2i-1$ then $F_k + F_\ell - F_{2i+1} = y + F_{2i} + F_\ell$. In these latter two cases the Zeckendorf representation of α is already clear, and $z_1(\alpha)=2$ which gives $\alpha \in A \setminus AB$. In case $\ell = 2i$, we may need to repeat the argument, in particular if F_{2i-2} belongs to the Zeckendorf representation of p, and possibly further repetition of this form will terminate with a representation of the form $y + 2F_2 = y + F_3$ with Zeckendorf indexes in y greater than 5. This is the unique case where $z_1(\alpha)$ is odd and hence $\alpha \in B$. Any other case will give $z_1(\alpha)=2$ which gives $\alpha \in A \setminus AB$.

Next, we find an i such that $p-(F_{2i+1}-1) \in B_0 \oplus 1$. Take $i=1$. We show that $p - F_3 \in B_0$. Write $p = a + F_2$ and show that $a - F_2 \in B_0$, where $z_1(a) = 2k \geqslant 4$ is even, by the definition of the set AB and by $a = p-1$. By the identity $F_{2k} - F_2 = F_{2k-1} + \cdots + F_3$, the result follows.

It remains to find an i such that $\alpha = p - (F_{2i+1}-1) \in B_0$. With $a = p-1$, and since $p+1 = a + F_3$, we may define $z_1(a) = F_{2k+2}$, with $k \geqslant 1$. With the Zeckendorf representation $a = y + F_{2k+2}$, we must show that $\alpha = y + F_{2k+2} + F_3 - F_{2i+1} \in B_0$, for some i. If $k>1$, then we let $i=k$; if $k=2$, then $\alpha = y + F_6 + F_3 - F_5 = y + F_5$, so $z_1(\alpha)=5$ and otherwise $z_1(\alpha)=3$. If $k=1$, then $z_1(a+F_3) = 2\ell+1 > 3$. In case $a + F_3 = F_{2\ell+1}$, then we choose $i=\ell$, and so $\alpha = 0 \in B_0$. Otherwise there is a smallest Zeckendorf term in y, say $F_m > F_5 > F_{2k+2}$. Hence $\alpha = y' + F_m + F_5 - F_{2i+1}$. If m is odd, we let $i=(m-1)/2$, which gives $z_1(\alpha)=5$. Suppose m is even, then, if $m \geqslant 8$, we let $2i+1 = m-1$, which gives either $z_1(\alpha)=5$ or, in case $m=8$, $z_1(\alpha)=7$ (since the smallest Zeckendorf term in y' is greater than $m+1=9$). □

Note that this example is also studied in Fox's manuscript [2] but with a less elementary proof. The sequence of Sprague-Grundy values for the game S appears as sequence A242082 in OEIS.

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