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On a Conjecture on the Representation of Positive Integers as the Sum of Three Terms of the Sequence $\left\lfloor \frac{n^2}{a} \right\rfloor$

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Abstract

We prove some cases of a conjecture by Farhi on the representation of every positive integer as the sum of three terms of the sequence $\left\lfloor \frac{n^2}{a} \right\rfloor$. This is done by generalizing a method used by Farhi in his original paper.

1 Introduction

In the following we let $\mathbb{N}$ denote the set of non-negative integers, $\lfloor \cdot \rfloor$ denote the greatest integer function, and $\langle \cdot \rangle$ denote the fractional part function.

A classical result by Legendre [3] states that every natural number not of the form $4^s(8t+7), s, t \in \mathbb{N}$ can be written as the sum of three squares.

In relation to this, Farhi recently conjectured the following:

Conjecture 1 (Farhi [2])**.** Let $a \geq 3$ be an integer. Then every natural number can be represented as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a} \right\rfloor\right)_{n\in\mathbb{N}}$.

The conjecture was confirmed by Farhi [1] and Mezroui, Azizi, and Ziane [4] for $a \in \{3, 4, 8\}$.

In this paper we generalize the method used by Farhi for $a = 4$, and partially for $a = 3$, to prove that the conjecture holds for $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}$. The method uses Legendre's three-square theorem and properties of quadratic residues.

We also note that the set of integers a such that Conjecture 1 holds is closed under multiplication by a square.

2 Method and results

We start by introducing the following sets:

Definition 2. For any nonzero $a \in \mathbb{N}$ we define

$$\mathcal{Q}_a = \{0 < \varphi < a \mid \exists x \in \mathbb{Z} \colon \varphi \equiv x^2 \pmod{a}\}.$$

Therefore, $\mathcal{Q}_a$ is the set of quadratic residues modulo a.

Definition 3. For any nonzero $a \in \mathbb{N}$ we define

$$\mathcal{A}_a = \{\varphi \in \mathbb{N} \mid \exists x, y, z \in \mathcal{Q}_a \cup \{0\} \colon \varphi = x + y + z\}.$$

Thus, $\mathcal{A}_a$ is the set of integers that can be written as the sum of three elements of $\mathcal{Q}_a \cup \{0\}$.

Definition 4. For any nonzero $a \in \mathbb{N}$ we define

$$\mathcal{R}_a = \{\varphi \in \mathcal{A}_a \mid \forall \psi \in \mathcal{A}_a \colon \varphi \equiv \psi \pmod{a} \Rightarrow \varphi = \psi\}.$$

So, $\mathcal{R}_a$ is the set of integers that can be written as the sum of three elements of $\mathcal{Q}_a \cup \{0\}$, and such that no other integer in the same residue class modulo a has this property.

Now we are ready to formulate the main result.

Theorem 5. *Let $a \in \mathbb{N}$ be nonzero and assume that for every $k \in \mathbb{N}$ there exists an $r \in \mathcal{R}_a$ such that $ak + r \neq 4^s(8t+7)$ for any $s, t \in \mathbb{N}$. Then every $N \in \mathbb{N}$ can be written as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a} \right\rfloor\right)_{n\in\mathbb{N}}$.*

Proof. Let $N \in \mathbb{N}$ be fixed. By assumption we can choose $r \in \mathcal{R}_a$ such that $aN + r \neq 4^s(8t+7)$ for any $s, t \in \mathbb{N}$. By Legendre's theorem it follows that $aN + r$ can be written of the form

$$aN + r = A^2 + B^2 + C^2 \tag{1}$$

for some $A, B, C \in \mathbb{N}$. Now we have

$$r \equiv A^2 + B^2 + C^2 \pmod{a},$$

so

$$r = (A^2 \bmod a) + (B^2 \bmod a) + (C^2 \bmod a), \tag{2}$$

since $r \in \mathcal{R}_a$. Dividing by a and separating the integer and fractional parts of the right hand side in (1), we get

$$N + \frac{r}{a} = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor + \left\langle \frac{A^2}{a} \right\rangle + \left\langle \frac{B^2}{a} \right\rangle + \left\langle \frac{C^2}{a} \right\rangle,$$

and from (2) we have

$$\frac{r}{a} = \left\langle \frac{A^2}{a} \right\rangle + \left\langle \frac{B^2}{a} \right\rangle + \left\langle \frac{C^2}{a} \right\rangle,$$

so

$$N = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor.$$

□

Since we can find the sets $\mathcal{R}_a$ by computation, we can now apply the main theorem to get the following corollary.

Corollary 6. *Conjecture 1 is satisfied for* $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}$.

Proof. Consider the following table:

a	$\mathcal{R}_a$
4	$\{0, 1, 2, 3\}$
7	$\{4, 6\}$
8	$\{2, 3, 5, 6\}$
9	$\{1, 4, 7, 8\}$
20	$\{11, 15, 18, 19\}$
24	$\{11, 14, 19, 21, 22\}$
40	$\{27, 38\}$
104	$\{99\}$
120	$\{107\}$

Calculating modulo 8 it can be checked fairly easily that for each $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}$ and every $k \in \mathbb{N}$ there exists an $r \in \mathcal{R}_a$ such that $ak + r$ is not of the form $4^s(8t+7), s, t \in \mathbb{N}$, and thus every natural number can be written as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a} \right\rfloor\right)_{n\in\mathbb{N}}$.

To demonstrate this, we show the case $a = 7$. All the other cases are done in exactly the same way.

For $k \equiv 1, 2, 3, 6$ or $7 \pmod 8$ we have $7k + 4 \equiv 3, 2, 1, 6$ and $5 \pmod 8$, respectively, and for $k \equiv 0, 4$ or $5 \pmod 8$ we have $7k + 6 \equiv 6, 2$ and $1 \pmod 8$, respectively. Since $4^s(8t + 7) \equiv 0, 4$ or $7 \pmod 8$, $s, t \in \mathbb{N}$, we conclude that for every $k \in \mathbb{N}$ we can write $7k + r$, for $r \in \mathcal{R}_7 = \{4, 6\}$, such that it is not of the form $4^s(8t + 7), s, t \in \mathbb{N}$. The case now follows from Theorem 5. □

Further, one should note that the set of integers satisfying Conjecture 1 is closed under multiplication by a square.

Observation 7. *Let $\mathcal{M}$ be the set of integers satisfying Conjecture 1. If $a \in \mathcal{M}$, then $ak^2 \in \mathcal{M}$ for any integer $k > 0$.*

Proof. This follows easily since for any n we can find $A, B, C \in \mathbb{N}$ such that

$$\begin{aligned} n &= \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor \\ &= \left\lfloor \frac{(Ak)^2}{ak^2} \right\rfloor + \left\lfloor \frac{(Bk)^2}{ak^2} \right\rfloor + \left\lfloor \frac{(Ck)^2}{ak^2} \right\rfloor. \end{aligned}$$

□

Knowing this, we see that since Conjecture 1 is satisfied for $a = 3, 9, 4$, and 8, it must also hold for $a = 3^k$ for any positive integer k and for $a = 2^k, k > 1$.

Finally, using Observation 7, Corollary 6, and the fact [4] that Conjecture 1 holds for $a = 3$, we get that the conjecture holds for the following values up to 120.

$$\begin{aligned} a \in \{&3, 4, 7, 8, 9, 12, 16, 20, 24, 27, 28, 32, 36, 40, 48, \\ &63, 64, 72, 75, 80, 81, 96, 100, 104, 108, 112, 120\}. \end{aligned}$$

Unfortunately, it seems that the method deployed in Theorem 5 is not extendable to other cases, since its success relies on $\mathcal{R}_a$, and in general $\mathcal{R}_a$ does not contain the necessary elements for the condition in the theorem to be satisfied.

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