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Arithmetic Properties of Partition k-tuples with Odd Parts Distinct

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Abstract

Let $\text{pod}_{-k}(n)$ denote the number of partition k-tuples of n wherein odd parts are distinct (and even parts are unrestricted). We establish some interesting infinite families of congruences and internal congruences modulo 4, 16, and 5 for $\text{pod}_{-2}(n)$, $\text{pod}_{-4}(n)$, and $\text{pod}_{-6}(n)$, respectively. We also find Ramanujan-type congruences modulo 5 for $\text{pod}_{-3}(n)$ and densities of $\text{pod}_{-2}(n)$, $\text{pod}_{-3}(n)$, $\text{pod}_{-4}(n)$, and $\text{pod}_{-6}(n)$ modulo 4, 5, 16, and 5, respectively.

1 Introduction

For $|q| < 1$, Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty \tag{1}$$

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and

$$\psi(q) := \sum_{n=0}^{\infty} q^{(n^2+n)/2} = \sum_{n=-\infty}^{\infty} q^{2n^2+n} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Let $\mathrm{pod}(n)$ denote the number of partitions of n wherein odd parts are distinct (and even parts are unrestricted). The generating function of $\mathrm{pod}(n)$ is

$$\sum_{n=0}^{\infty} \mathrm{pod}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{1}{\psi(-q)}.$$

In 2010, Hirschhorn and Sellers [5] proved that, for all $\alpha \geq 0$ and $n \geq 0$,

$$\mathrm{pod}\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 3.$$

They also found some internal congruences such as

$$\mathrm{pod}(81n+17) \equiv 5\mathrm{pod}(9n+2) \pmod{27}.$$

Recently, Wang [10] established new congruences for $\mathrm{pod}(n)$. For example, for each $\alpha \geq 1$ and $n \geq 0$,

$$\mathrm{pod}\left(5^{2\alpha+2}n + \frac{11 \times 5^{2\alpha+1}+1}{8}\right) \equiv 0 \pmod 5.$$

Let $\mathrm{pod}_{-k}(n)$ denote the number of partition k-tuples of n wherein odd parts are distinct (and even parts are unrestricted). The generating function of $\mathrm{pod}_{-k}(n)$ is

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-k}(n)q^n = \frac{(-q;q^2)_\infty^k}{(q^2;q^2)_\infty^k} = \frac{1}{\psi(-q)^k}. \tag{3}$$

Chen and Lin [3] established congruences modulo 3 and 5 for $\mathrm{pod}_{-2}(n)$. For example, for $\alpha \geq 1$ and $n \geq 0$,

$$\mathrm{pod}_{-2}\left(5^{\alpha+1}n + \frac{11 \times 5^{\alpha}+1}{4}\right) \equiv 0 \pmod 5.$$

Wang [8, 9] has established congruences modulo 7, 9, and 11 satisfied by $\mathrm{pod}_{-3}(n)$ and congruences modulo 5, 9, and 81 satisfied by $\mathrm{pod}_{-4}(n)$ by employing theta function identities. For example, for $\alpha \geq 1$ and $n \geq 0$,

$$\mathrm{pod}_{-3}\left(3^{2\alpha+2}n + \frac{23 \times 3^{2\alpha+1}+3}{8}\right) \equiv 0 \pmod 9$$

and

$$\mathrm{pod}_{-4}\left(3^{\alpha+1}n + \frac{5 \times 3^{\alpha}+1}{2}\right) \equiv 0 \pmod 9.$$

He also found some internal congruences such as

$$\mathrm{pod}_{-4}(27n+5) \equiv -\mathrm{pod}_{-4}(9n+2) \pmod 9.$$

In this paper, we establish congruences modulo powers of 2 and modulo 5 for $\mathrm{pod}_{-k}(n)$ for $k \in \{2,3,4,6\}$. In this vein, in Section 3, we find infinite family of congruences and internal congruences modulo 4 satisfied by $\mathrm{pod}_{-2}(n)$ and we also find density of $\mathrm{pod}_{-2}(n)$ modulo 4. In Section 4, we prove Ramanujan-type congruences modulo 5 for $\mathrm{pod}_{-3}(n)$ and that $\mathrm{pod}_{-3}(n)$ is divisible by 5 at least 1/30 of the time. In Section 5, we establish infinite family of congruences and internal congruences modulo 16 satisfied by $\mathrm{pod}_{-4}(n)$ following density of $\mathrm{pod}_{-4}(n)$ modulo 16. In Section 6, we determine infinite family of congruences and internal congruences modulo 5 satisfied by $\mathrm{pod}_{-6}(n)$ and we also determine density of $\mathrm{pod}_{-6}(n)$ modulo 5.

2 Preliminaries

The following results are useful in proving our main results.

Lemma 1. *[2, pp. 40–49] We have*

$$\begin{aligned}
\varphi(q) &= \varphi(q^4) + 2q\psi(q^8), && (4)\\
\varphi(q)^2 &= \varphi(q^2)^2 + 4q\psi(q^4)^2, && (5)\\
\psi(q) &= f(q^3,q^6) + q\psi(q^9) && (6)\\
&= f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25}), && (7)\\
\psi(q)^2 &= \varphi(q)\psi(q^2). && (8)
\end{aligned}$$

Lemma 2. *[1, Eq. 1.6.7, p. 26] We have*

$$f(q,q^4)f(q^2,q^3) = \psi(q)^2 - q\psi(q^5)^2. \tag{9}$$

Lemma 3. *Let* $\sum_{n=0}^{\infty} h(n)q^n = q\psi(q)^4$. *Then*

$$\sum_{n=0}^{\infty} h(5n+3)q^n \equiv \psi(q)^4 \pmod 5. \tag{10}$$

Proof. From (7), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} h(n)q^n &= q\psi(q)^4 \\ &= q\left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right)^4 \\ &= 12q^5 f(q^{10},q^{15})^2 f(q^5,q^{20})\psi(q^{25}) + 4q^{10} f(q^{10},q^{15})\psi(q^{25})^3 \\ &\quad + 12q^8 f(q^{10},q^{15}) f(q^5,q^{20})\psi(q^{25})^2 + 6q^7 f(q^{10},q^{15})^2\psi(q^{25})^2 \\ &\quad + 4q^2 f(q^{10},q^{15})^3 f(q^5,q^{20}) + 4q^4 f(q^{10},q^{15})^3\psi(q^{25}) \\ &\quad + 6q^3 f(q^{10},q^{15})^2 f(q^5,q^{20})^2 + q^5 f(q^5,q^{20})^4 + q^{13}\psi(q^{25})^4 \\ &\quad + 4q^4 f(q^{10},q^{15}) f(q^5,q^{20})^3 + 6q^9 f(q^5,q^{20})^2\psi(q^{25})^2 \\ &\quad + 4q^7 f(q^5,q^{20})^3\psi(q^{25}) + 12q^6 f(q^{10},q^{15}) f(q^5,q^{20})^2\psi(q^{25}) \\ &\quad + 4q^{11} f(q^5,q^{20})\psi(q^{25})^3 + qf(q^{10},q^{15})^4,\end{aligned}$$

which yields

$$\begin{aligned}\sum_{n=0}^{\infty} h(5n+3)q^n \equiv\, & 2qf(q^2,q^3)f(q,q^4)\psi(q^5)^2 + f(q^2,q^3)^2 f(q,q^4)^2 \\ & + q^2\psi(q^5)^4 \pmod{5}.\end{aligned}$$

Using (9) in the above equation, we arrive at (10). □

Lemma 4. *Let* $\sum_{n=0}^{\infty} g(n)q^n = \psi(q)^2$. *Then*

$$\sum_{n=0}^{\infty} g(5n+1)q^n = 2\psi(q)^2 - q\psi(q^5)^2. \tag{11}$$

Proof. It follows from (7) that

$$\begin{aligned}\sum_{n=0}^{\infty} g(n)q^n &= \psi(q)^2 \\ &= \left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right)^2 \\ &= q^6\psi(q^{25})^2 + 2q^4\psi(q^{25})f(q^5,q^{20}) + 2q^3\psi(q^{25})f(q^{10},q^{15}) \\ &\quad + q^2 f(q^5,q^{20})^2 + 2qf(q^{10},q^{15})f(q^5,q^{20}) + f(q^{10},q^{15})^2,\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} g(5n+1)q^n = 2f(q^2,q^3)f(q,q^4) + q\psi(q^5)^2.$$

Using (9) in the above equation, we arrive at (11). □

Lemma 5. *[4, Theorem 2.1] For any odd prime, p,*

$$\psi(q)=\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{12}$$

Furthermore, $\frac{m^2+m}{2}\not\equiv\frac{p^2-1}{8} \pmod p$, *for* $0\leq m\leq\frac{p-3}{2}$.

3 Arithmetic properties of $\mathrm{pod}_{-2}(n)$

In this section, we prove the infinite family of congruences and internal congruences modulo 4 for $\mathrm{pod}_{-2}(n)$.

3.1 Infinite family of congruences modulo 4

Theorem 6. *Let p be any odd prime such that* $\left(\frac{-2}{p}\right)=-1$ *and* $\alpha\geq 0$. *Then*

$$\sum_{n=0}^{\infty}\mathrm{pod}_{-2}\left(2p^{2\alpha}n+\frac{3p^{2\alpha}+1}{4}\right)q^n\equiv 2\psi(q)\psi(q^2) \pmod 4 \tag{13}$$

and, for all $n\geq 0$ *and* $1\leq\xi\leq p-1$,

$$\mathrm{pod}_{-2}\left(2p^{2\alpha+1}(pn+\xi)+\frac{3p^{2\alpha+2}+1}{4}\right)\equiv 0 \pmod 4. \tag{14}$$

Proof. We have

$$\sum_{n=0}^{\infty}\mathrm{pod}_{-2}(n)q^n=\frac{1}{\psi(-q)^2}. \tag{15}$$

Invoking (8) and (15),

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{pod}_{-2}(n)q^n&=\frac{1}{\psi(q^2)\varphi(-q)}\\&=\frac{(1-(1-\varphi(-q)))^{-1}}{\psi(q^2)}\\&=\frac{1+(1-\varphi(-q))+(1-\varphi(-q))^2+\cdots}{\psi(q^2)}\\&\equiv\frac{2-\varphi(-q)}{\psi(q^2)} \pmod 4 \quad \text{from (1).}\end{aligned}$$

Using (4) in the above equation, we find that

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(n)q^n \equiv \frac{2-\varphi(q^4)+2q\psi(q^8)}{\psi(q^2)} \pmod 4,$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(2n+1)q^n \equiv 2\frac{\psi(q^4)}{\psi(q)} \pmod 4. \tag{16}$$

From the binomial theorem, we can see that for any prime p and for each positive integer ℓ,

$$(q;q)_\infty^{p^\ell} \equiv (q^p;q^p)_\infty^{p^{\ell-1}} \pmod{p^\ell}. \tag{17}$$

In view of (17), (16) can be expressed as

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(2n+1)q^n \equiv 2\psi(q)\psi(q^2) \pmod 4, \tag{18}$$

which is the $\alpha = 0$ case of (13). If we assume that (13) holds for some $\alpha \geq 0$, then, substituting (12) in (13),

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{pod}_{-2}\left(2p^{2\alpha}n+\frac{3p^{2\alpha}+1}{4}\right)q^n \\
&\equiv 2\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \\
&\quad\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{m^2+m} f\left(q^{p^2+(2m+1)p}, q^{p^2-(2m+1)p}\right) + q^{\frac{p^2-1}{4}}\psi(q^{2p^2})\right) \pmod 4.
\end{aligned} \tag{19}$$

For any odd prime, p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 2\times\frac{m_2^2+m_2}{2} \equiv \frac{3p^2-3}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 2(2m_2+1)^2 \equiv 0 \pmod p. \tag{20}$$

Since $\left(\frac{-2}{p}\right) = -1$, the only solution of the congruence (20) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{3p^2-3}{8}}$ from both sides of (19), dividing throughout by $q^{\frac{3p^2-3}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}\left(2p^{2\alpha+1}n+\frac{3p^{2\alpha+2}+1}{4}\right)q^n \equiv 2\psi(q^p)\psi(q^{2p}) \pmod 4. \tag{21}$$

Equating the coefficients of q^{pn} on both sides of (21) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}\left(2p^{2\alpha+2}n + \frac{3p^{2\alpha+2}+1}{4}\right) q^n \equiv 2\psi(q)\psi(q^2) \pmod 4,$$

which is the $\alpha+1$ case of (13).

Equating the coefficients of $q^{pn+\xi}$ for $1 \le \xi \le p-1$ from (21), we arrive at (14). □

Corollary 7. *Let p be any odd prime such that $\left(\frac{-2}{p}\right) = -1$. Then $\mathrm{pod}_{-2}(n)$ is divisible by 4 for at least $\frac{1}{2(p+1)}$ of all nonnegative integers n.*

Proof. The arithmetic sequences $\left\{2p^{2\alpha+1}(pn+\xi) + \frac{3p^{2\alpha+2}+1}{4} : \alpha \ge 0\right\}$ for $1 \le \xi \le p-1$, on which $\mathrm{pod}_{-2}(\cdot)$ is 0 modulo 4, do not intersect. These sequences account for

$$(p-1)\left(\frac{1}{2p^2} + \frac{1}{2p^4} + \frac{1}{2p^6} + \cdots\right) = \frac{1}{2(p+1)}$$

of all nonnegative integers. □

3.2 Some internal congruences

Theorem 8. *For each $n \ge 0$,*

$$\mathrm{pod}_{-2}(54n+25) \equiv \mathrm{pod}_{-2}(6n+3) \pmod 4, \tag{22}$$
$$\mathrm{pod}_{-2}(54n+43) \equiv \mathrm{pod}_{-2}(6n+5) \pmod 4, \tag{23}$$
$$\mathrm{pod}_{-2}(162n+7) \equiv 2\mathrm{pod}_{-2}(18n+1) \pmod 4, \tag{24}$$
$$\mathrm{pod}_{-2}(162n+115) \equiv 2\mathrm{pod}_{-2}(18n+13) \pmod 4. \tag{25}$$

Proof. If $\sum_{n=0}^{\infty} a(n)q^n = \psi(q)\psi(q^2)$, then the authors [6] found that

$$\sum_{n=0}^{\infty} a(3n)q^n = \psi(q)\varphi(q) - q\psi(q^3)\psi(q^6). \tag{26}$$

Using (26), we can express (18) as

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(6n+1)q^n \equiv 2\psi(q)\varphi(q) - 2q\psi(q^3)\psi(q^6) \pmod 4. \tag{27}$$

Invoking (1) and (27),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(6n+1)q^n \equiv 2\psi(q) + 2q\psi(q^3)\psi(q^6) \pmod 4. \tag{28}$$

Substituting (6) into (28) and extracting the terms involving q^{3n+1},

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(18n+7)q^n \equiv 2\psi(q^3) + 2\psi(q)\psi(q^2) \pmod 4,$$

which is equivalent to

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(18n+7)q^n \equiv 2\psi(q^3) + \sum_{n=0}^{\infty} \mathrm{pod}_{-2}(2n+1)q^n \pmod 4. \tag{29}$$

Equating the coefficients of q^{3n+1} and q^{3n+2} from (29), we arrive at (22) and (23), respectively.

Equating the coefficients of q^{3n} and then replacing q^3 by q from (29),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(54n+7)q^n \equiv 2\psi(q) + \sum_{n=0}^{\infty} \mathrm{pod}_{-2}(6n+1)q^n \pmod 4. \tag{30}$$

Invoking (28) and (30),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(54n+7)q^n \equiv 2\sum_{n=0}^{\infty} \mathrm{pod}_{-2}(6n+1)q^n - 2q\psi(q^3)\psi(q^6) \pmod 4, \tag{31}$$

Equating the coefficients of q^{3n} and q^{3n+2} from (31), we arrive at (24) and (25), respectively. □

4 Ramanujan-type congruences for $\mathrm{pod}_{-3}(n)$

In this section, we prove the Ramanujan-type congruences modulo 5 for $\mathrm{pod}_{-3}(n)$.

Theorem 9. *For each $\alpha \geq 1$,*

$$\mathrm{pod}_{-3}\left(5^{2\alpha+1}n + \frac{\mu \times 5^{2\alpha} + 3}{8}\right) \equiv 0 \pmod 5, \tag{32}$$

where $\mu = 13,\ 21,\ 29,$ and 37.

Proof. We have

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-3}(n)(-1)^n q^n = \frac{1}{\psi(q)^3}.$$

It follows from (17) that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{pod}_{-3}(n)(-1)^n q^n &\equiv \frac{\psi(q)^2}{\psi(q^5)} \pmod 5 \\ &\equiv \frac{1}{\psi(q^5)} \sum_{n=0}^{\infty} g(n)q^n \pmod 5.\end{aligned}$$

Extracting the terms involving q^{5n+1}, dividing throughout by q and then replacing q^5 by q,

$$\begin{aligned}\sum_{n=0}^{\infty}\text{pod}_{-3}(5n+1)(-1)^{n+1}q^n &\equiv \frac{1}{\psi(q)}\sum_{n=0}^{\infty}g(5n+1)q^n \pmod 5\\ &\equiv \frac{1}{\psi(q)}\left(2\psi(q)^2-q\psi(q^5)^2\right) \pmod 5 \quad \text{from (11)}\\ &\equiv 2\psi(q)-q\psi(q^5)\psi(q)^4 \pmod 5 \quad \text{using (17)}. \end{aligned} \tag{33}$$

Substituting (7) into (33) and from the Lemma (3) , we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\text{pod}_{-3}(5n+1)(-1)^{n+1}q^n \equiv\, & 2f(q^{10},q^{15})+2qf(q^5,q^{20})+2q^3\psi(q^{25})\\ & -\psi(q^5)\sum_{n=0}^{\infty}h(n)q^n \pmod 5,\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{n=0}^{\infty}\text{pod}_{-3}(25n+16)(-1)^{n}q^n &\equiv 2\psi(q^5)-\psi(q)\sum_{n=0}^{\infty}h(5n+3)q^n \pmod 5\\ &\equiv 2\psi(q^5)-\psi(q)^5 \pmod 5 \quad \text{using (10)}.\end{aligned} \tag{34}$$

Using (17), (34) can be expressed as

$$\sum_{n=0}^{\infty}\text{pod}_{-3}(25n+16)(-1)^{n}q^n \equiv \psi(q^5) \pmod 5. \tag{35}$$

Extracting the terms involving q^{5n} from (35),

$$\begin{aligned}\sum_{n=0}^{\infty}\text{pod}_{-3}(125n+16)(-1)^{n}q^n &\equiv \psi(q) \pmod 5\\ &\equiv f(q^{10},q^{15})+qf(q^5,q^{20})+q^3\psi(q^{25}) \pmod 5,\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty}\text{pod}_{-3}(625n+391)(-1)^{n+1}q^n \equiv \psi(q^5) \pmod 5. \tag{36}$$

From (35), (36), and by induction, we find that for each $\alpha \geq 1$,

$$\sum_{n=0}^{\infty}\text{pod}_{-3}\left(5^{2\alpha}n+\frac{5^{2\alpha+1}+3}{8}\right)(-1)^{n+1}q^n \equiv (-1)^{\alpha}\psi(q^5) \pmod 5. \tag{37}$$

Equating the coefficients of $q^{5n+\xi}$ for $1 \leq \xi \leq 4$ from (37), we arrive at (32). □

Corollary 10. *The function* $\mathrm{pod}_{-3}(n)$ *is divisible by* 5 *for at least* $\frac{1}{30}$ *of all nonnegative integers* n.

Proof. The arithmetic sequences $\left\{5^{2\alpha+1}n + \frac{\mu\times 5^{2\alpha}+3}{8} : \alpha \geq 1\right\}$ for $\mu = 13$, 21, 29, and 37, on which $\mathrm{pod}_{-3}(\cdot)$ is 0 modulo 5, do not intersect. These sequences account for

$$4\left(\frac{1}{5^3} + \frac{1}{5^5} + \frac{1}{5^7} + \cdots\right) = \frac{1}{30}$$

of all nonnegative integers. □

5 Arithmetic properties of $\mathrm{pod}_{-4}(n)$

In this section, we prove the infinite family of congruences and internal congruences modulo 16 for $\mathrm{pod}_{-4}(n)$.

5.1 Infinite family of congruences modulo 16

Theorem 11. *Let* p *be any prime such that* $p \equiv 3 \pmod 4$ *and* $\alpha \geq 0$. *Then*

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}\left(2p^{2\alpha}n + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv 4\psi(q)^2 \pmod{16} \tag{38}$$

and, for all nonnegative integers n *and* $1 \leq \xi \leq p-1$,

$$\mathrm{pod}_{-4}\left(2p^{2\alpha+1}(pn+\xi) + \frac{p^{2\alpha+2}+1}{2}\right) \equiv 0 \pmod{16}. \tag{39}$$

Proof. We have

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(n)q^n = \frac{1}{\psi(-q)^4}. \tag{40}$$

Invoking (8) and (40),

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(n)q^n &= \frac{1}{\psi(q^2)^2\varphi(-q)^2} \\
&= \frac{(1-(1-\varphi(-q)^2))^{-1}}{\psi(q^2)^2} \\
&= \frac{1+(1-\varphi(-q)^2)+(1-\varphi(-q)^2)^2+\cdots}{\psi(q^2)^2} \\
&\equiv \frac{2-\varphi(-q)^2}{\psi(q^2)^2} \pmod{16} \quad \text{using (1)} \\
&\equiv \frac{2-\varphi(q^2)^2+4q\psi(q^4)^2}{\psi(q^2)^2} \pmod{16} \quad \text{from (5)},
\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1)q^n \equiv 4\frac{\psi(q^2)^2}{\psi(q)^2} \pmod{16}, \tag{41}$$

In view of (17), (41) can be expressed as

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1)q^n \equiv 4\psi(q)^2 \pmod{16}, \tag{42}$$

which is the $\alpha = 0$ case of (38). If we assume that (38) holds for some $\alpha \geq 0$, then, substituting (12) into (38),

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{pod}_{-4}\left(2p^{2\alpha}n + \frac{p^{2\alpha}+1}{2}\right) q^n \\
&\equiv 4\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right)^2.
\end{aligned} \tag{43}$$

For any odd prime, p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + \frac{m_2^2+m_2}{2} \equiv \frac{2p^2-2}{8} \pmod{p},$$

which implies that

$$(2m_1+1)^2 + (2m_2+1)^2 \equiv 0 \pmod{p}. \tag{44}$$

Since $\left(\frac{-1}{p}\right) = -1$ for $p \equiv 3 \pmod{4}$, the only solution of the congruence (44) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{2p^2-2}{8}}$ from both sides of (43), dividing throughout by $q^{\frac{2p^2-2}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}\left(2p^{2\alpha}\left(pn + \frac{2p^2-2}{8}\right) + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv 4\psi(q^p)^2 \pmod{16}. \tag{45}$$

Equating the coefficients of q^{pn} on both sides of (45) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}\left(2p^{2\alpha+2}n + \frac{p^{2\alpha+2}+1}{2}\right) q^n \equiv 4\psi(q)^2 \pmod{16},$$

which is the $\alpha+1$ case of (38).

Equating the coefficients of $q^{pn+\xi}$ for $1 \leq \xi \leq p-1$ from (45), we arrive at (39). $\square$

Corollary 12. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then $\mathrm{pod}_{-4}(n)$ is divisible by 16 for at least $\frac{1}{2(p+1)}$ of all nonnegative integers n.*

Proof. The arithmetic sequences $\left\{2p^{2\alpha+1}(pn+\xi)+\frac{p^{2\alpha+2}+1}{2} : \alpha \geq 0\right\}$ for $1 \leq \xi \leq p-1$, on which $\mathrm{pod}_{-4}(\cdot)$ is 0 modulo 16, do not intersect. These sequences account for

$$(p-1)\left(\frac{1}{2p^2}+\frac{1}{2p^4}+\frac{1}{2p^6}+\cdots\right)=\frac{1}{2(p+1)}$$

of all nonnegative integers. □

5.2 Some internal congruences

Theorem 13. *For each $n \geq 0$,*

$$\begin{aligned}
\mathrm{pod}_{-4}(50n+3) &\equiv 2\mathrm{pod}_{-4}(10n+1) \pmod{16},\\
\mathrm{pod}_{-4}(50n+23) &\equiv 2\mathrm{pod}_{-4}(10n+5) \pmod{16},\\
\mathrm{pod}_{-4}(50n+33) &\equiv 2\mathrm{pod}_{-4}(10n+7) \pmod{16},\\
\mathrm{pod}_{-4}(50n+43) &\equiv 2\mathrm{pod}_{-4}(10n+9) \pmod{16}.
\end{aligned}$$

Proof. If $\sum_{n=0}^{\infty} g(n)q^n = \psi(q)^2$, then (42) can be expressed as

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1)q^n \equiv 4\sum_{n=0}^{\infty} g(n)q^n \pmod{16},$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(10n+3)q^n \equiv 4\sum_{n=0}^{\infty} g(5n+1)q^n \pmod{16}. \tag{46}$$

Invoking (11) and (46),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(10n+3)q^n \equiv 8\psi(q)^2 - 4q\psi(q^5)^2 \pmod{16}. \tag{47}$$

Substituting (42) into (47),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(10n+3)q^n \equiv 2\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1)q^n - 4q\psi(q^5)^2 \pmod{16},$$

equating the coefficients of q^{5n+i} for $i = 0$, 2, 3, and 4 from the above equation, we obtain the desired results. □

6 Arithmetic properties of $\mathrm{pod}_{-6}(n)$

In this section, we prove the infinite family of congruences and internal congruences modulo 5 for $\mathrm{pod}_{-6}(n)$.

6.1 Infinite family of congruences modulo 5

Theorem 14. *Let p be any prime such that $p \equiv 3 \pmod 4$ and $\alpha \geq 0$. Then*

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}\left(5p^{2\alpha}n + \frac{5p^{2\alpha}+3}{4}\right) q^n \equiv \psi(q)^2 \pmod 5$$

and, for each $n \geq 0$ and $1 \leq \xi \leq p-1$,

$$\mathrm{pod}_{-6}\left(5p^{2\alpha+1}(pn+\xi) + \frac{5p^{2\alpha+2}+3}{4}\right) \equiv 0 \pmod{5}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}(n) q^n = \frac{1}{\psi(-q)^6}. \tag{48}$$

In view of (17), (48) can be expressed as

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}(n)(-1)^n q^n \equiv \frac{\psi(q)^4}{\psi(q^5)^2} \pmod{5}. \tag{49}$$

Substituting (7) into (49),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}(n)(-1)^n q^n \equiv \frac{(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25}))^4}{\psi(q^5)^2} \pmod{5},$$

which yields

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}(5n+2)(-1)^n q^n \equiv \frac{f(q^2,q^3)^2 f(q,q^4)^2}{\psi(q)^2} + \frac{2qf(q^2,q^3)f(q,q^4)\psi(q^5)^2}{\psi(q)^2} + \frac{q^2\psi(q^5)^4}{\psi(q)^2} \pmod{5}. \tag{50}$$

Invoking (9) and (50),

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}(5n+2)(-1)^n q^n \equiv \psi(q)^2 + q^2\frac{\psi(q^5)^4}{\psi(q)^2} - 2q\psi(q^5)^2 + 2q\psi(q^5)^2 - 2q^2\frac{\psi(q^5)^4}{\psi(q)^2} + q^2\frac{\psi(q^5)^4}{\psi(q)^2} \pmod{5}, \tag{51}$$

which implies that

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-6}(5n+2)(-1)^n q^n \equiv \psi(q)^2 \pmod{5}. \tag{52}$$

The remainder of the proof is similar to that of Theorem 11, but rather than (42), we use (52). □

Corollary 15. *Let p be a prime such that $p \equiv 3 \pmod{4}$. Then $\mathrm{pod}_{-6}(n)$ is divisible by 5 for at least $\frac{1}{5(p+1)}$ of all nonnegative integers n.*

Proof. The arithmetic sequences $\left\{5p^{2\alpha+1}(pn+\xi)+\frac{5p^{2\alpha+2}+3}{4} : \alpha \geq 0\right\}$ for $1 \leq \xi \leq p-1$, on which $\mathrm{pod}_{-6}(\cdot)$ is 0 modulo 5, do not intersect. These sequences account for

$$(p-1)\left(\frac{1}{5p^2}+\frac{1}{5p^4}+\frac{1}{5p^6}+\cdots\right)=\frac{1}{5(p+1)}$$

of all nonnegative integers. □

6.2 Some internal congruences

Theorem 16. *For each $n \geq 0$,*

$$\begin{aligned}
\mathrm{pod}_{-6}(125n+7) &\equiv 3\mathrm{pod}_{-6}(25n+2) \pmod{5},\\
\mathrm{pod}_{-6}(125n+57) &\equiv 3\mathrm{pod}_{-6}(25n+12) \pmod{5},\\
\mathrm{pod}_{-6}(125n+82) &\equiv 3\mathrm{pod}_{-6}(25n+17) \pmod{5},\\
\mathrm{pod}_{-6}(125n+107) &\equiv 3\mathrm{pod}_{-6}(25n+22) \pmod{5}.
\end{aligned}$$

Proof. The proof is similar to that of Theorem 13, but rather than (42), we use (52). □

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