Journal of Integer Sequences, Vol. 20 (2017),
Article 17.6.4

Explicit Bounds for the Sum of Reciprocals of Pseudoprimes and Carmichael Numbers

Jonathan Bayless and Paul Kinlaw
Husson University
1 College Circle
Bangor, ME 04401
USA
baylessj@husson.edu
kinlawp@husson.edu

Abstract

From a 1956 paper of Erdős, we know that the base-two pseudoprimes and the Carmichael numbers both have a convergent sum of reciprocals. We prove that the values of these sums are less than 33 and 28, respectively.

1 Introduction

By Fermat's little theorem, if p is a prime number then for all $a \in \mathbb{Z}$ we have $a^p \equiv a \pmod{p}$. However, a number p satisfying this property need not be a prime. For all $a \in \mathbb{Z}$, a *base-a Fermat pseudoprime* (or briefly, an *a-pseudoprime*) is a composite number n such that $\gcd(a, n) = 1$ and $a^n \equiv a \pmod{n}$. A *Carmichael number* is an odd composite number n for which $a^n \equiv a \pmod{n}$ for all integers a, and so is an a-pseudoprime for all a with $\gcd(a, n) = 1$.

Let $\mathscr{P}_2 = \{341, 561, 645, 1105, \ldots\}$ be the set of 2-pseudoprimes, also called *Poulet* or *Sarrus* numbers, and let $P_2(x) = |\{n \in \mathscr{P}_2 : n \leq x\}|$ be the corresponding counting function. Let $\mathscr{C} = \{561, 1105, 1729, \ldots\}$ be the set of Carmichael numbers and write its counting function as $C(x) = |\{n \in \mathscr{C} : n \leq x\}|$. Erdős [11] proved that for sufficiently large x, we have

$$P_2(x) < x/\exp(c_1\sqrt{\log x \log\log x})$$

and

$$C(x) < x/\exp\left(\frac{c_2 \log x \log\log\log x}{\log\log x}\right)$$

for constants $c_1, c_2 > 0$. This implies that both sets have asymptotic density zero, as well as the stronger statement that both sets have a bounded sum of reciprocals. Pomerance [18] improved these bounds, showing that

$$P_2(x) < x/\exp\left(\frac{\log x \log\log\log x}{2\log\log x}\right)$$

for all sufficiently large x, and that the constant c_2 in the above bound on $C(x)$ may be taken as 1.

In the other direction, it is well known that there are infinitely many pseudoprimes with respect to a given base. In 1994 Alford, Granville and Pomerance [1] proved that there are infinitely many Carmichael numbers. In particular, they proved that $C(x) > x^{2/7}$ for sufficiently large x. Their work has since been improved by Harman [13] to show that for large x, the inequality $C(x) > x^{\alpha}$ holds for some constant $\alpha > 1/3$.

In this paper we determine explicit upper and lower bounds for the sum of reciprocals of 2-pseudoprimes, as well as for the sum of reciprocals of Carmichael numbers. This extends previous work [4, 5, 3, 14] on reciprocal sums. See [5] for a discussion of reciprocal sums and their importance in number theory.

2 Preliminary lemmas

We state several preliminary lemmas. Throughout the paper m, n and k denote positive integers, p denotes a prime number, $\pi(x)$ is the prime counting function, $\log x$ denotes the natural logarithm and $P(n)$ denotes the largest prime factor of n. Put $x_0 = \exp(100)$ and $y_0 = \exp(10)$. We begin with Korselt's criterion [7, Thm. 3.4.6] which provides a useful characterization of Carmichael numbers.

Lemma 1 (Korselt's Criterion)**.** *A positive integer n is a Carmichael number if and only if it is composite, squarefree, and for each prime $p|n$ we have $p-1|n-1$.*

It follows from Korselt's criterion that Carmichael numbers are odd and have at least three different prime factors.

The following lemma involves certain divisibility properties possessed by all 2-pseudoprimes. In particular, it shows that if a 2-pseudoprime n is divisible by p^2 for a prime p, then p must be a *Wieferich prime*, that is,

$$2^{p-1} \equiv 1 \pmod{p^2}.$$

It follows that p is greater than or equal to 1093, the smallest Wieferich prime, see [7, p. 31].

Lemma 2. *Let n be a 2-pseudoprime. If $p^2|n$ then p is a Wieferich prime. Furthermore, if $7|n$ then $n \equiv 1 \pmod 3$.*

Proof. Suppose that $p^2|n$ and let k be the order of 2 in $(\mathbb{Z}/p^2\mathbb{Z})^\times$. Thus $k|\varphi(p^2) = p(p-1)$. Now $2^{n-1} \equiv 1 \pmod{n}$, so $2^{n-1} \equiv 1 \pmod{p^2}$, and thus $k|n-1$, so k does not divide p. Thus $k|p-1$, so that $2^{p-1} \equiv 1 \pmod{p^2}$. This completes the proof of the first assertion.

For the second assertion, we have $2^{n-1} \equiv 1 \pmod{n}$, so if $7|n$ then $2^{n-1} \equiv 1 \pmod{7}$. The order of 2 modulo 7 is 3, so $3|n-1$. □

We will also use explicit versions of several classical theorems. The following modification of [2, Thm. 3.2] gives fairly sharp explicit bounds on the partial sums of the harmonic series.

Lemma 3. *For all $x \geq 1$,*

$$\left| \sum_{n \leq x} \frac{1}{n} - (\log x + \gamma) \right| < \frac{1}{x}$$

where $\gamma = 0.5772156649\ldots$ denotes Euler's constant.

We will also use the following modification of [3, Lem. 6.2] to bound the sum of reciprocals of numbers in a certain interval and residue class.

Lemma 4. *Let $a \in \mathbb{Z}$ and $d = 2\prod_{i=2}^{9} p_i^2$, where p_i denotes the i-th prime number. We have*

$$\sum_{\substack{n \equiv a (d) \\ 10^{19} < n \leq x_0}} \frac{1}{n} < \frac{56.25587}{d}.$$

Proof. Let $b = (10^{19} - d + 1)/d$ and $c = x_0/d$. Without loss of generality, we may assume that $a \in \{0, \ldots, d-1\}$. By Lemma 3 we have

$$\sum_{b < k \leq c} \frac{1}{dk + a} \leq \frac{1}{d}\left(100 - \log(10^{19} - d + 1) + \frac{1}{c} + \frac{1}{b}\right) < \frac{56.25587}{d}.$$ □

We will use Dusart's bounds [10, Thm. 5.6] on the sum of reciprocals of prime numbers up to x.

Lemma 5. *For all $x \geq 2278383$, we have*

$$\left| \sum_{p \leq x} \frac{1}{p} - (\log\log x + B) \right| < \frac{0.2}{\log^3 x},$$

where $B = 0.2614972128476\ldots$ denotes the Mertens constant.

We will also use Dusart's bounds [10, Cor. 5.2 & Thm. 5.9].

Lemma 6. *For all $x > 1$ we have*

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.53816}{\log^2 x}\right),$$

and for all $x \geq 599$ we have

$$\pi(x) \geq \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right).$$

Lemma 7. *For all $x \geq 2278382$, we have*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) \geq \frac{\exp(-\gamma)}{\log x}\left(1 - \frac{0.2}{\log^3 x}\right).$$

We will also use the following result [9, Lem. 9.6].

Lemma 8. *Let f be a multiplicative function such that $f(n) \geq 0$ for all n, and such that there exist constants A and B such that for all $x > 1$, we have*

$$\sum_{p \leq x} f(p) \log p \leq Ax \qquad \text{and} \qquad \sum_{p} \sum_{\alpha \geq 2} \frac{f(p^\alpha)}{p^\alpha} \log p^\alpha \leq B. \tag{1}$$

Then, for $x > 1$, we have

$$\sum_{n \leq x} f(n) \leq (A + B + 1)\frac{x}{\log x} \sum_{n \leq x} \frac{f(n)}{n}.$$

The following lemma makes the implied constants in [9, Lem. 9.7] explicit and modifies [3, Lem. 2.4].

Lemma 9. *Let f be a multiplicative function such that $0 \leq f(p^\alpha) \leq \exp\left(\frac{2\alpha}{3}\right)$ for all primes p and integers $\alpha \geq 1$, and such that $f(p^\alpha) = p^{2\alpha/(3\log y)}$ for all $p \leq y$. Then for all $x \geq x_0$ and $y \geq y_0$, we have*

$$\sum_{n \leq x} f(n) \leq 9.68765388x \prod_{p \leq x}\left(\left(1 - \frac{1}{p}\right)\sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha}\right).$$

Proof. We wish to apply Lemma 8, and so we first establish values of A and B to use in inequality (1) above. We have

$$\sum_{p \leq x} f(p) \log p \leq \exp(2/3)\theta(x) \leq 1.00000075 \exp(2/3)x \leq 1.94773551x, \tag{2}$$

using the bound $\theta(x) < 1.00000075x$ for all $x > 0$ [17, Cor. 2]. We also have $y \geq y_0$ and

$$f(p^\alpha) = p^{2\alpha/(3\log y)}$$

whenever $p \le y$. For brevity, let $g(t) = (f(t) \cdot \log t)/t$. We have

$$\sum_{p} \sum_{\alpha \ge 2} g\left(p^{\alpha}\right) = \sum_{p < y_0} \sum_{\alpha \ge 2} g\left(p^{\alpha}\right) + \sum_{p > y_0} \sum_{\alpha \ge 2} g\left(p^{\alpha}\right).$$

Starting to bound the sum with $p = 2$, observe that

$$\begin{aligned}\sum_{\alpha \ge 2} g\left(2^{\alpha}\right) \le \sum_{\alpha \ge 2} \frac{\alpha(\log 2) \cdot 2^{\alpha/15}}{2^{\alpha}} &= \log 2 \sum_{\alpha \ge 2} \alpha \left(2^{1/15-1}\right)^{\alpha} \\ &< 1.23661725.\end{aligned}$$

Here we used the fact that

$$\sum_{\alpha \ge 2} \alpha r^{\alpha - 1} = \frac{2r - r^2}{(1-r)^2}$$

for $|r| < 1$. By similar reasoning,

$$\sum_{3 \le p < y_0} \sum_{\alpha \ge 2} g\left(p^{\alpha}\right) < 1.25448156.$$

Let $h(t) = (2 - e^{2/3}/t)(\log t)/(t - e^{2/3})^2$. By partial summation we have

$$\begin{aligned}\sum_{p > y_0} \sum_{\alpha \ge 2} g\left(p^{\alpha}\right) &\le e^{4/3} \sum_{p > y_0} \frac{(2 - e^{2/3}/p) \log p}{(p - e^{2/3})^2} \\ &= e^{4/3} \left(-h(y_0)\pi(y_0) - \int_{y_0}^{\infty} \pi(t) h'(t)\ dt \right) \\ &< e^{4/3} \left(-h(y_0)\pi(y_0) + \int_{y_0}^{\infty} \frac{\pi(t) \cdot 2(2 - \frac{e^{2/3}}{t}) \log t\ dt}{(t - e^{2/3})^3} \right) \\ &< e^{4/3} \left(-h(y_0)\pi(y_0) + \int_{y_0}^{\infty} \frac{2t \cdot 1.1253816(2 - \frac{e^{2/3}}{t})\ dt}{(t - e^{2/3})^3} \right) \\ &= e^{4/3} \left(-0.00010166 + \frac{1.1253816 \cdot \left(4e^{10} - 3e^{2/3}\right)}{\left(e^{2/3} - e^{10}\right)^2} \right) \\ &< 0.00038973.\end{aligned}$$

Here we used Lemma 6. Furthermore, for all $x \ge x_0$ we have

$$1.781072775 \prod_{p \le x} \left(1 - \frac{1}{p}\right) \ge \frac{1}{\log x}.$$

by Lemma 7. Applying Lemma 8 with $A = 1.94773551$ and $B = 2.49148854$, and noting that $A + B + 1 = 5.43922405$ and $5.43922405 \cdot 1.781072775 < 9.68765388$, we complete the proof of the lemma. $\square$

We now prove the following explicit upper bound on de Bruijn's function $\Psi(x, y)$, defined as the number of numbers up to x that are *y-smooth* (that is, free of prime factors exceeding y).

Lemma 10. *For all $x \geq x_0$ and $y \geq y_0$, we have*

$$\Psi(x, y) \leq 31.928253 x^{1-1/(2\log y)}.$$

Proof. We follow [9, Thm. 9.5]. By Rankin's method, if $\beta > 0$, then

$$\Psi(x, y) \leq x^{3/4} + \sum_{n \leq x} \left(\frac{n}{x^{3/4}}\right)^{\beta} \chi_y(n), \tag{3}$$

where $\chi_y(n) = 1$ if $P(n) \leq y$ and $\chi_y(n) = 0$ if $P(n) > y$. Set $\beta = \frac{2}{3\log y}$. By Lemma 9, we have that

$$\sum_{n \leq x} n^{\beta} \chi_y(n) \leq 9.68765388 x \prod_{p \leq y} \left(\left(1 - p^{-1}\right) \sum_{\alpha \geq 0} p^{\alpha(\beta - 1)} \right). \tag{4}$$

Now, for any given value of p,

$$\left(1 - \frac{1}{p}\right) \sum_{\alpha \geq 0} p^{\alpha(\beta-1)} = \left(1 - \frac{1}{p}\right) \frac{1}{1 - p^{\beta - 1}} = \frac{p - 1}{p - p^{\beta}} = 1 + \frac{p^{\beta} - 1}{p - p^{\beta}}.$$

For $p \leq 13$, the product contributes a factor less than 1.18817106 to the product in inequality (4). Bounding this second term for $p > 13$, we see that

$$\frac{p^{\beta} - 1}{p - p^{\beta}} = \frac{p}{p - p^{\beta}} \cdot \frac{p^{\beta} - 1}{p} < 1.07648742 \cdot \frac{p^{\beta} - 1}{p}$$

since for $p \geq 17$,

$$\frac{p}{p - p^{\beta}} \leq \left(\frac{17}{17 - 17^{1/15}}\right) < 1.07648742.$$

For $0 < t \leq \frac{2}{3}$ we have $e^t - 1 \leq 1.4216011 t$ and therefore we may bound

$$\begin{aligned}
\sum_{p=17}^{y} 1.07648742 \cdot \frac{p^{\beta} - 1}{p} &= \sum_{p=17}^{y} 1.07648742 \cdot \frac{\exp\left(\log p^{\beta}\right) - 1}{p} \\
&\leq 1.4216011 \cdot 1.07648742 \beta \sum_{p \leq y} \frac{\log p}{p} \\
&< 1.4216011 \cdot 1.07648742 \beta \log y.
\end{aligned}$$

Here we used Rosser and Schoenfeld's bound (3.23), [19, p. 70],

$$\sum_{p \leq y} \frac{\log p}{p} < \log y - 1.3325 + \frac{1}{\log y} < \log y$$

for $y \geq 32$. We therefore have inequality (4) bounded above by

$$9.68765388 \cdot 1.18817106 e^{1.4216011 \cdot 1.07648742 \beta \log y} \leq 31.9282527.$$

Using this in inequality (3) for $x \geq x_0$ gives the bound

$$\Psi(x, y) \leq x^{3/4} + 31.9282527 x^{1-1/(2 \log y)} \leq 31.928253 x^{1-1/(2 \log y)}. \qquad \square$$

3 The sum of reciprocals of base-two pseudoprimes

We prove explicit bounds on the sum of reciprocals of 2-pseudoprimes by expanding Luca and De Koninck's proof [9, Prop. 9.11] of the following result.

Proposition 11. *For all $c < 1/(2\sqrt{2})$ we have $P_2(x) \ll x \exp\left(-c\sqrt{\log x}\right)$.*

By partial summation, it follows that the sum of reciprocals is convergent.

Theorem 12. *The sum of the reciprocals of base-two pseudoprimes satisfies*

$$0.0152608 < \sum_{n \in \mathscr{P}_2} \frac{1}{n} < 33.$$

To prove Theorem 12 we split the 2-pseudoprimes into three ranges.

3.1 The small range

We first compute the reciprocal sum over $n \leq 10^{19}$. Feitsma [12] has computed an exhaustive list of all 2-pseudoprimes $n \leq 10^{19}$. We use this information to directly compute the reciprocal sum from the 2-pseudoprimes $n \leq 10^{12}$ to seven decimal places as

$$\sum_{\substack{n \in \mathscr{P}_2 \\ n \leq 10^{12}}} \frac{1}{n} = 0.0152608\ldots.$$

For each $k = 12, \ldots, 18$, the contribution to the reciprocal sum from 2-pseudoprimes n such that $10^k < n \leq 10^{k+1}$ is bounded above by the number of such n times 10^{-k}. We thus obtain

$$\sum_{\substack{n \in \mathscr{P}_2 \\ n \leq 10^{19}}} \frac{1}{n} \leq 0.0152612.$$

3.2 The middle range

We now bound the reciprocal sum over n such that $10^{19} < n \le x_0$. By Lemma 2, all 2-pseudoprimes are odd and not divisible by the square of any prime $p < 1093$. Therefore, they must lie in one of

$$D = \prod_{i=2}^{9} (p_i^2 - 1)$$

residue classes modulo d, where d is as defined in Lemma 4. Also by Lemma 2, no 2-pseudoprime is congruent to 0 or 14 modulo 21. Therefore we may rule out 30 more residue classes modulo $3^2 7^2$. Thus 2-pseudoprimes must lie in one of

$$D' = 354 \cdot 24 \cdot 120 \cdot 168 \cdot 288 \cdot 360 \cdot 528$$

residue classes modulo d. Therefore by Lemma 4, we have

$$\sum_{\substack{n \in \mathscr{P} \\ 10^{19} < n \le x_0}} \frac{1}{n} < \frac{56.25587 D'}{d} < 21.196317.$$

We may also remove the contribution to the sum from prime numbers. By Lemma 5, we have

$$\sum_{10^{19} < p \le x_0} \frac{1}{p} > 0.826696.$$

The sum in the middle range is therefore bounded above by

$$21.196317 - 0.826696 = 20.369621.$$

3.3 The large range

Finally, we bound the reciprocal sum over 2-pseudoprimes $n > x_0$. Let $x > x_0$ and define $y = \exp(\sqrt{\log x})$, with $y_0 = y(x_0) = \exp(10)$. For ease of notation, write $p = P(n)$ and define t_p as the multiplicative order of 2 modulo p. Define the set $\mathcal{Q} = \{p : t_p < p^{1/4}\}$, and let $Q(x) = |\{p \in \mathcal{Q} : p \le x\}|$. Each 2-pseudoprime $n > x_0$ falls into exactly one of the following categories:

1. $p \le y$,
2. $p > y$ and $p \in \mathcal{Q}$,
3. $p > y$ and $p \notin \mathcal{Q}$.

For each $1 \le i \le 3$, write $\mathscr{A}_i$ for the set of 2-pseudoprimes $n \le x$ satisfying property i above and put $A_i(x) = |\{n \in \mathscr{A}_i : n \le x\}|$.

Observe that $A_1(x) = \Psi(x, y)$. By Lemma 10 we have

$$\Psi(x, y) \leq 31.928253x/\exp(u/2)$$

where $u = \dfrac{\log x}{\log y} = \dfrac{\log x}{\sqrt{\log x}} = \sqrt{\log x}$. We thus have

$$A_1(x) \leq \frac{31.928253x}{\exp\left(0.5\sqrt{\log x}\right)}$$

for $x > x_0$. Therefore, by partial summation,

$$\sum_{\substack{n\in\mathscr{A}_1 \\ n>x_0}} \frac{1}{n} \leq \int_{x_0}^{\infty} \frac{31.928253\ dt}{t\exp\left(0.5\sqrt{\log t}\right)} = \int_{100}^{\infty} \frac{31.928253\ dw}{\exp(\sqrt{w/4})}$$

$$= 31.928253\left[4e^{-\sqrt{w}/2}(\sqrt{w}+2)\right]_{\infty}^{100} < 10.326283.$$

Here we used substitution followed by integration by parts.

We now consider the second set, $\mathscr{A}_2$. We first show that

$$Q(x) < 0.01588\sqrt{x}$$

for all x. A computer check shows that the claim holds for $x \leq e^{22}$, that $Q(e^{22}) = 26$, and that

$$\prod_{\substack{p\in\mathcal{Q} \\ p\leq e^{22}}} p > e^{496.34447}.$$

Assume that $x > e^{22}$. We have

$$\prod_{\substack{p\in\mathcal{Q} \\ p\leq x}} p < \prod_{t<x^{1/4}} 2^t = \exp\left(\log 2 \sum_{t<x^{1/4}} t\right) \leq \exp\left(0.34799x^{1/2}\right),$$

while also

$$\prod_{\substack{p\in\mathcal{Q} \\ p\leq x}} p = \prod_{\substack{p\in\mathcal{Q} \\ p\leq e^{22}}} p \prod_{\substack{p\in\mathcal{Q} \\ e^{22}<p\leq x}} p > e^{496.34447}\left(e^{22}\right)^{Q(x)-26} = e^{22Q(x)-75.65553},$$

so that $Q(x) < 0.01588x^{1/2}$ for all x as claimed. We thus have

$$A_2(x) < \sum_{2\leq m<x/y} 0.01588\left(\frac{x}{m}\right)^{1/2} \leq 0.01588x^{1/2}\int_1^{x/y} \frac{dt}{t^{1/2}} < \frac{0.03176x}{y^{1/2}}.$$

By partial summation, we therefore have

$$\sum_{\substack{n\in\mathscr{A}_2\\ n>x_0}} \frac{1}{n} \le \int_{x_0}^{\infty} \frac{0.03176\ dt}{t\exp(\sqrt{(\log t)/4})} = \int_{100}^{\infty} \frac{0.03176\ dw}{\exp(\sqrt{w/4})} < 0.0102719.$$

Let $n \in \mathscr{A}_3$ and let $p = P(n)$. Then $t_p \ge p^{1/4} > y^{1/4}$. Following [11] we show that for such n we have $n \equiv p \pmod{pt_p}$. We clearly have $n \equiv p \pmod{p}$. We also have $n \equiv p \pmod{t_p}$. To see this, note that $n - p = (n-1) - (p-1)$. We have $t_p | p-1$ by Fermat's little theorem, and $t_p | n-1$ since n is a 2-pseudoprime and $p|n$. Since $t_p < p$ we have $\gcd(p, t_p) = 1$, and thus also $n \equiv p \pmod{pt_p}$ as claimed. Thus the number of such n is bounded above by $x/(pt_p) + 1$. Also $n > p$ (since p is not a pseudoprime), so the number of such n is in fact bounded above by $x/(pt_p)$. Thus the number of 2-pseudoprimes in $\mathscr{A}_3$ satisfies

$$A_3(x) \le \sum_{p>y} \frac{x}{pt_p} \le x \sum_{p>y} \frac{1}{p^{5/4}}.$$

By Lemma 6 we have $t/(\log t) \cdot (1 + 1/\log t) \le \pi(t) \le 1.1253816t/\log t$ for $t \ge y_0$. Thus by partial summation we have

$$\begin{aligned}
A_3(x) &\le x\left(-\frac{\pi(y)}{y^{5/4}} + \frac{5}{4}\int_y^{\infty} \frac{\pi(t)}{t^{9/4}} dt\right) \\
&\le x\left(-\frac{1 + 1/\log y}{y^{1/4}\log y} + \frac{5(1.1253816)}{4}\int_y^{\infty} \frac{dt}{(\log t)t^{5/4}}\right) \\
&= x\left(-\frac{1 + 1/\log y}{y^{1/4}\log y} - \frac{5.626908}{4}\mathrm{Ei}\left(-\frac{\log y}{4}\right)\right) \\
&= x\left(-\frac{1 + 1/\sqrt{\log x}}{\sqrt{\log x}\cdot \exp\sqrt{(\log x)/16}} - 1.406727 \cdot \mathrm{Ei}\left(-\frac{\sqrt{\log x}}{4}\right)\right),
\end{aligned}$$

where

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$$

denotes the exponential integral function. By another application of partial summation, we thus have

$$\begin{aligned}
\sum_{\substack{n\in\mathscr{A}_3\\ n>x_0}} \frac{1}{n} &\le -\int_{x_0}^{\infty} \left(\frac{1 + 1/\sqrt{\log t}}{t\sqrt{\log t}\cdot \exp(\sqrt{(\log t)/16})} + \frac{1.406727 \cdot \mathrm{Ei}\left(-\frac{\sqrt{\log t}}{4}\right)}{t}\right) dt \\
&= -\int_{100}^{\infty} \frac{(1 + 1/\sqrt{w})\ dw}{\sqrt{w}\cdot \exp(\sqrt{w/16})} - 1.406727\int_{100}^{\infty} \mathrm{Ei}\left(-\frac{\sqrt{w}}{4}\right) dw \\
&= -8e^{-2.5} + 2\mathrm{Ei}(-2.5) + 1.406727\left(100\mathrm{Ei}(-2.5) + 56e^{-2.5}\right) \\
&< 2.2550278.
\end{aligned}$$

Here we substituted $w = \log t$ and used integration by parts. Adding the contributions from the small, middle and large ranges, we obtain

$$0.0152612 + 20.369621 + 12.5915827 < 32.9765,$$

completing the proof of Theorem 12.

4 The sum of reciprocals of the Carmichael numbers

Theorem 13. *The sum of reciprocals of Carmichael numbers satisfies*

$$0.004706 < \sum_{n \in \mathscr{C}} \frac{1}{n} < 27.8724.$$

To prove Theorem 13, we modify the small, middle and large ranges from the proof of Theorem 12 above.

4.1 The small range: $n \leq 10^{21}$

A table of all 10000 Carmichael numbers up to 1713045574801 has been computed by R. Pinch, [15, 16]. Their contribution to the reciprocal sum is easily computed to be $0.004706\ldots$. Pinch also determined that there are 20128200 additional Carmichael numbers up to 10^{21}. Therefore an upper bound for the sum of reciprocals of all Carmichael numbers $n \leq 10^{21}$ is given by

$$0.004707 + 20128200/1713045574803 < 0.0047188.$$

4.2 The middle range: $10^{21} < n \leq x_0 = \exp(100)$

We will use the following slight modification of Lemma 4, whose proof is nearly identical.

Lemma 14. *Let $a \in \mathbb{Z}$ and $d' = 2\prod_{i=2}^{10} p_i^2$, where p_i denotes the i-th prime number. We have*

$$\sum_{\substack{n \equiv a (d') \\ 10^{21} < n \leq x_0}} \frac{1}{n} < \frac{51.6882395}{d'}.$$

Since Carmichael numbers are odd and squarefree by Lemma 1, they must lie in one of

$$E = \prod_{i=2}^{10} (p_i^2 - 1)$$

residue classes modulo d'.

We may also remove the contribution to the sum from prime numbers. By Lemma 5 we have

$$\sum_{10^{21}<p\leq x_0} \frac{1}{p} > 0.7266133.$$

It follows from Lemma 1 that Carmichael numbers have at least three prime factors, so we may also remove the contribution to the sum from odd, squarefree numbers in the middle range with exactly two prime factors. Let $\pi_2(x) = |\{n = pq \leq x : p < q\}|$ denote the counting function of squarefree numbers with exactly two prime factors. Similarly, let $\pi_2^*(x) = |\{n = pq \leq x : 2 < p < q\}|$ denote the counting function of odd squarefree numbers with exactly two prime factors. By a slight modification of Bayless and Klyve's lower bound [6] on $\pi_2(x)$, we are able to show that for all $x \geq 10^{21}$, we have

$$\pi_2^*(x) \geq \frac{x(\log\log x - 0.300711)}{\log x}.$$

Letting A^* denote the set of odd squarefree numbers with exactly two prime factors, we therefore obtain by partial summation that

$$\sum_{\substack{10^{21}<n\leq x_0\\ n\in A^*}} \frac{1}{n} \geq \frac{\log\log x_0 - 0.300711}{100} - \frac{\log\log 10^{21} - 0.300711}{\log 10^{21}} + I,$$

where

$$\begin{aligned} I &= \int_{10^{21}}^{x_0} \frac{(\log\log t - 0.300711)dt}{t\log t} = \int_{\log 10^{21}}^{100} \frac{(\log w - 0.300711)dw}{w} \\ &= \left[\frac{\log^2 w}{2} - 0.300711\log w\right]_{\log 10^{21}}^{100}. \end{aligned}$$

It follows that

$$\sum_{\substack{10^{21}<n\leq x_0\\ n\in A^*}} \frac{1}{n} > 2.8327533.$$

Therefore, the contribution to the reciprocal sum from the middle range is bounded above by

$$51.6882395E/d' - 0.7266133 - 2.8327533 < 17.5412697.$$

4.3 The large range: $n > x_0$

Finally, we bound the reciprocal sum over Carmichael numbers $n > x_0$. Let $x > x_0$ and recall our definitions $y = \exp(\sqrt{\log x})$ and $y_0 = y(x_0) = \exp(10)$. Write $p = P(n)$. We split the Carmichael numbers $n > x_0$ into two categories.

1. $p \le y$
2. $p > y$.

For $i = 1, 2$, write $\mathscr{B}_i$ for the set of Carmichael numbers $n \le x$ satisfying property i above and put $B_i(x) = |\{n \in \mathscr{B}_i : n \le x\}|$.

By the same argument in the proof of Theorem 12 above, we have

$$B_1(x) = \Psi(x, y) \le \frac{31.928253x}{\exp\left(0.5\sqrt{\log x}\right)}$$

by Lemma 10, and

$$\sum_{\substack{n \in \mathscr{B}_1 \\ n > x_0}} \frac{1}{n} < 10.326283.$$

We next determine an upper bound for $B_2(x)$. Observe that if $p|n$ for a Carmichael number n, then $n \equiv p \pmod{p(p-1)}$. To see this, we clearly have $p|n-p$. Furthermore, since $n - p = (n-1) - (p-1)$ and since $p - 1|n - 1$ by Lemma 1, we have $p - 1|n - p$. It follows that $n \equiv p \pmod{p(p-1)}$, since $\gcd(p, p-1) = 1$.

Therefore, the number of Carmichael numbers $n \le x$ which are divisible by a given prime p is bounded above by $x/(p(p-1))$. We therefore have

$$B_2(x) \le \sum_{p>y} \frac{x}{p(p-1)} = x \sum_{p>y} \frac{1}{p(p-1)}.$$

By partial summation, we have

$$\begin{aligned}
\sum_{p>y} \frac{1}{p(p-1)} &= -\frac{\pi(y)}{y(y-1)} + \int_y^\infty \frac{\pi(t)(2t-1)\ dt}{t^2(t-1)^2} \\
&\le -\frac{\pi(y)}{y^2} + \int_y^\infty \frac{0.112539 \cdot 2t\ dt}{t(t-1)^2} \\
&\le -\frac{1}{y\log y}\left(1 + \frac{1}{\log y}\right) + \int_y^\infty \frac{0.11255 \cdot 2\ dt}{t^2} \\
&= \frac{0.2251}{y} - \frac{1}{y\log y}\left(1 + \frac{1}{\log y}\right).
\end{aligned}$$

Here we used Lemma 6 to bound

$$\frac{t}{\log t}\left(1 + \frac{1}{\log t}\right) \le \pi(t) \le 0.112539t$$

for all $t \ge y_0$. Therefore, for all $x \ge x_0$ we have

$$B_2(x) \le \frac{x}{y}\left(0.2251 - \frac{1}{\log y}\left(1 + \frac{1}{\log y}\right)\right).$$

It follows by another application of partial summation that

$$\sum_{\substack{n\in\mathscr{B}_2\\ n>x_0}} \frac{1}{n} \le \int_{x_0}^{\infty} \frac{1}{t\exp\sqrt{\log t}}\left(0.2251 - \frac{1}{\sqrt{\log t}}\left(1+\frac{1}{\sqrt{\log t}}\right)\right) dt$$

$$= \int_{100}^{\infty} \frac{1}{\exp\sqrt{w}}\left(0.2251 - \frac{1}{\sqrt{w}}\left(1+\frac{1}{\sqrt{w}}\right)\right) dw < 0.000126.$$

Adding the contributions from the small, middle and large ranges, we obtain $0.0047188 + 17.5412697 + 10.326409 < 27.8724$, completing the proof of Theorem 13.

5 Concluding remarks

It will take more work to substantially sharpen the upper bound on the sum of reciprocals of 2-pseudoprimes. In fact, the result is close to optimal for the arguments used, in the following sense.

Reworking the arguments for different choices of the cutoff $x = x_0$ for the large range, the coefficients appearing in the bounds for each case tend to vary relatively slowly. Therefore we may attempt to roughly optimize the bound by minimizing the function

$$f(x) = 0.0152612 + 0.37679(\log x - \log(10^{19} - d + 1)) - I + I_1 + I_2 + I_3,$$

which represents the major contributions from the small, middle and large ranges to the reciprocal sum, where d is as defined in Lemma 4, and

$$I = \log\log x - \log\log 10^{19} - \frac{0.2}{\log^3 x} - \frac{0.2}{\log^3 10^{19}},$$

$$I_1 = \int_{\log x}^{\infty} \frac{31.928253\ dw}{\exp(\sqrt{w/4})},$$

$$I_2 = \int_{\log x}^{\infty} \frac{0.03176\ dw}{\exp\sqrt{w/4}},$$

$$I_3 = -\int_{\log x}^{\infty} \left(\frac{1+1/\sqrt{w}}{\sqrt{w}\cdot\exp(\sqrt{w/16})} + 1.406727\cdot \mathrm{Ei}\left(-\frac{\sqrt{w}}{4}\right)\right) dw.$$

Applying the fundamental theorem of calculus and again substituting $w = \log x$, the derivative is given by

$$f'(x) = \frac{1}{x}\left(0.37679 - J - J_1 - J_2 + J_3\right),$$

where

$$J = \frac{1}{w} + \frac{0.6}{w^4},$$

$$J_1 = \frac{31.928253}{\exp(\sqrt{w/4})},$$

$$J_2 = \frac{0.03176}{\exp(\sqrt{w/4})},$$

and

$$J_3 = \frac{1 + 1/\sqrt{w}}{\sqrt{w} \cdot \exp(\sqrt{w/16})} + 1.406727 \cdot \mathrm{Ei}\left(-\frac{\sqrt{w}}{4}\right).$$

Solving graphically for w, we find that the critical point is approximately $w \approx 79$ so that $x \approx \exp(79)$, and we have $f(\exp(79)) \approx 32$. This indicates that our cutoff of $x_0 = \exp(100)$ for the large range is close to optimal for the method of argument used, especially keeping in mind that lowering the cutoff to $x_0 = \exp(79)$ will in fact raise the constants appearing in the various upper bounds for the large range.

Furthermore, adjusting the coefficient in our definition of $y = e^{\sqrt{c \log x}}$ and reconsidering the argument, our choice of $c = 1$ used above gave better bounds than larger or smaller values of c that we also considered. However, it may be possible to further optimize the choice of coefficients, including those used in the proof of Lemma 10 bounding the count of smooth numbers.

Another possible way to improve this bound would be to find a way to effectively utilize more information about the 2-pseudoprimes. For instance, more specific residue classes in the middle range could be ruled out using an inclusion-exclusion argument.

Perhaps the bound on the sum of reciprocals of Carmichael numbers could be further optimized by utilizing more information about them, as well as optimizing the cutoff x_0. Another possibility for improving the bound for the sum of reciprocals of Carmichael numbers in the middle range is to use an inclusion-exclusion argument to remove not only the odd squarefree numbers with exactly two prime factors, but also the ones which fall in certain specific residue classes that can be ruled out.

Furthermore, Damgård et al [8, Thm. 5] proved that for all $x \geq 1$, the number $N(x)$ of Carmichael numbers up to x with exactly three prime factors satisfies the upper bound

$$N(x) \leq 0.25\sqrt{x}(\log x)^{11/4}.$$

Obtaining an explicit lower bound on the count of odd, squarefree numbers up to x having exactly three prime factors would therefore allow one to remove the contribution from all such numbers in the middle range and replace it using this tighter bound.

6 Acknowledgments

The authors would like to thank the referees for suggestions improving both the writing and the mathematics of this paper.

References

[1] W. R. Alford, A. Granville, and C. Pomerance, There are infinitely many Carmichael numbers. *Ann. of Math.* **139** (1994), 703–722.

[2] T. Apostol, *Introduction to Analytic Number Theory*, Springer Science & Business Media, 2013.

[3] J. Bayless and P. Kinlaw, On consecutive coincidences of Euler's function. *Int. J. Number Theory* **12** (2015), 1011–1026.

[4] J. Bayless and D. Klyve, On the sum of reciprocals of amicable numbers. *Integers* **11** (2011), 315–322.

[5] J. Bayless and D. Klyve, Reciprocal sums as a knowledge metric: theory, computation, and perfect numbers. *Amer. Math. Monthly* **120** (2013), 822–831.

[6] J. Bayless and D. Klyve, Sums over primitive sets with a fixed number of prime factors. *Submitted.*

[7] R. Crandall and C. Pomerance, *Prime Numbers: A Computational Perspective*, Lecture Notes in Statistics, Vol. 182, Springer Science & Business Media, 2006.

[8] I. Damgård, P. Landrock, and C. Pomerance, Average case error estimates for the strong probable prime test. *Math. Comp.* **61** (1993), 177–194.

[9] J. M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, Amer. Math. Soc., 2012.

[10] P. Dusart, Explicit estimates of some functions over primes. *Ramanujan J.* (2016), 1–25.

[11] P. Erdős, On pseudoprimes and Carmichael numbers. *Publ. Math. Debrecen* **4** (1956), 201–206.

[12] J. Feitsma, Pseudoprimes, 2013, http://www.janfeitsma.nl/math/psp2/index.

[13] G. Harman, Watt's mean value theorem and Carmichael numbers. *Int. J. Number Theory* **4** (2008), 241–248.

[14] H. Nguyen, The reciprocal sum of the amicable numbers. Dartmouth College undergraduate honors thesis, Hanover, NH, 2014.

[15] R. Pinch, The Carmichael numbers up to 10^{21}, *Proceedings of the Conference on Algorithmic Number Theory, Turku Centre for Computer Science* **7** (2007), 129–131.

[16] R. Pinch, Tables relating to Carmichael numbers, 2005, http://www.chalcedon.demon.co.uk/rgep/cartable.html.

[17] D. Platt and T. Trudgian, On the first sign change of $\theta(x) - x$. *Math. Comp.* **85** (2016), 1539–1547.

[18] C. Pomerance, On the distribution of pseudoprimes. *Math. Comp.* **37** (1981), 587–593.

[19] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers. *Illinois J. Math.* **6** (1962), 64–94.

[20] L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II, *Math. Comp.* **6** (1976), 243–269.

2010 *Mathematics Subject Classification*: Primary 11N25; Secondary 11Y99.
Keywords: Carmichael number, pseudoprime, reciprocal sum, smooth number.

(Concerned with sequences A001567 and A002997.)

Received April 7 2017; revised versions received May 31 2017; June 7 2017. Published in *Journal of Integer Sequences*, June 25 2017.

Return to Journal of Integer Sequences home page.