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On the Largest Integer that is not a Sum of Distinct Positive nth Powers

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Abstract

It is known that for an arbitrary positive integer n the sequence $S(x^n) = (1^n, 2^n, \ldots)$ is complete, meaning that every sufficiently large integer is a sum of distinct nth powers of positive integers. We prove that every integer

$$m \geq (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n}-1) + 2(b-2))^n - 2a + ab,$$

where $a = n!2^{n^2}$, $b = 2^{n^3}a^{n-1}$, $r = 2^{n^2-n}a$, is a sum of distinct positive nth powers.

1 Introduction

Let $S = (s_1, s_2, \ldots)$ be a sequence of integers. The sequence S is said to be *complete* if every sufficiently large integer can be represented as a sum of distinct elements of S. For a complete sequence S, the largest integer that is not representable as a sum of distinct elements of S is called the *threshold of completeness* of S. We let θ_S denote the threshold of completeness of S.

The threshold of completeness is often very difficult to find even for a simple sequence. For an arbitrary positive integer n, let $S(x^n)$ denote the sequence of nth powers of positive integers, i.e., $S(x^n) = (1^n, 2^n, \ldots)$. The completeness of the sequence was proved in 1948, by Sprague [6]. In 1954, Roth and Szekeres [5] further generalized the result by proving that if $f(x)$ is a polynomial that maps integers into integers, then $S(f) = (f(1), f(2), \ldots)$ is complete if and only if $f(x)$ has a positive leading coefficient and for any prime p there exists an integer m such that p does not divide $f(m)$. In 1964, Graham [2] re-proved the theorem of Roth and Szekeres using alternative elementary techniques.

However, little is known about the threshold of completeness of $S(x^n)$. The value $\theta_{S(x^n)}$ is known only for $n \leq 6$. The values are as follows: $\theta_{S(x)} = 0$, $\theta_{S(x^2)} = 128$ [7], $\theta_{S(x^3)} = 12758$ [2], $\theta_{S(x^4)} = 5134240$ [3], $\theta_{S(x^5)} = 67898771$ [4], $\theta_{S(x^6)} = 11146309947$ [1]. Sprague, Roth and Szekeres, and Graham proved that $S(x^n)$ is complete, but they were not interested in the size of $\theta_{S(x^n)}$. The values $\theta_{S(x^n)}$ for $3 \leq n \leq 6$ were found by methods that require lengthy calculations assisted by computer, and they do not give any idea on the size of $\theta_{S(x^n)}$ for general n.

In this paper, we establish an upper bound of $\theta_{S(x^n)}$ as a function of n. Using the elementary techniques Graham used in his proof, it is possible to obtain an explicit upper bound of the threshold of completeness of $S(x^n) = (1^n, 2^n, 3^n, \ldots)$. Since the case $n = 1$ is trivial, we let n be a positive integer greater than 1. We prove the following theorem:

Theorem 1. *Let $a = n!2^{n^2}$, $b = 2^{n^3}a^{n-1}$ and $r = 2^{n^2-n}a$. Then*

$$\theta_{S(x^n)} < (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n}-1) + 2(b-2))^n - 2a + ab.$$

The theorem yields the result

$$\theta_{S(x^n)} = O((n!)^{n^2-1} \cdot 2^{2n^4+n^3+n^2+(2-\frac{\ln 3}{\ln 2})n}).$$

The upper bound of $\theta_{S(x^n)}$ given by the formula is much greater than 4^{n^4}, while the actual values of $\theta_{S(x^n)}$ for $2 \leq n \leq 6$ are less than 4^{n^2}. So the upper bound obtained in this paper is most likely far from being tight.

2 Preliminary results

Let $S = (s_1, s_2, \ldots)$ be a sequence of integers.

Definition 2. The set $P(S)$ is a set of all sums of the form $\sum_{k=1}^{\infty} \epsilon_k s_k$ where ϵ_k is 0 or 1, all but a finite number of ϵ_k are 0 and at least one of ϵ_k is 1.

Definition 3. The sequence S is *complete* if $P(S)$ contains every sufficiently large integer.

Definition 4. If S is complete, the *threshold of completeness* θ_S is the largest integer that is not in $P(S)$.

Definition 5. The set $A(S)$ is a set of all sums of the form $\sum_{k=1}^{\infty} \delta_k s_k$ where δ_k is -1, 0 or 1 and all but a finite number of δ_k are 0.

Definition 6. Let k be a positive integer. The sequence S is a $\Sigma(k)$-*sequence* if $s_1 \leq k$, and

$$s_n \leq k + \sum_{j=1}^{n-1} s_j, \quad n \geq 2.$$

For example, if $S = (2, 4, 8, 16, \dots)$ then S is a $\Sigma(2)$-sequence since $2^n = 2 + \sum_{j=1}^{n-1} 2^j$ for all $n \geq 2$.

Definition 7. Let c and k be positive integers. The sequence S is (c, k)-*representable* if $P(S)$ contains k consecutive integers $c + j$, $1 \leq j \leq k$.

For example, if $S = (1, 3, 6, 10, \dots)$ is a sequence of triangle numbers then S is $(8, 3)$-representable since $\{9, 10, 11\} \subset P(S)$.

Definition 8. For a positive integer m, we define $\mathbb{Z}_m(S)$ to be the sequence $(\alpha_1, \alpha_2, \dots)$, where $0 \leq \alpha_i < m$ and $s_i \equiv \alpha_i \pmod{m}$ for all i.

The two following lemmas, slightly modified from Lemma 1 and Lemma 2 in Graham's paper [2], are used to obtain the upper bound.

Lemma 9. *For a positive integer k, let $S = (s_1, s_2, \dots)$ be a strictly increasing $\Sigma(k)$-sequence of positive integers and let $T = (t_1, t_2, \dots)$ be (c, k)-representable. Then $U = (s_1, t_1, s_2, t_2, \dots)$ is complete and $\theta_U \leq c$.*

Proof. It suffices to prove that every positive integer greater than c belongs to $P(U)$. The proof proceeds by induction. Note that all the integers $c + t$, $1 \leq t \leq k$ belong to $P(T)$, and all the integers $c + s_1 + t$, $1 \leq t \leq k$ belong to $P(U)$. If $1 \leq t \leq k$ then

$$c + t \in P(T) \subset P(U),$$

and if $k + 1 \leq t \leq k + s_1$, then $1 \leq k - s_1 + 1 \leq t - s_1 \leq k$ and we have

$$c + t = c + (t - s_1) + s_1 \in P(U).$$

Therefore all the integers

$$c + t, \quad 1 \leq t \leq k + s_1$$

belong to $P(U)$. Now, let $n \geq 2$ and suppose that all the integers

$$c + t, \quad 1 \leq t \leq k + \sum_{j=1}^{n-1} s_j$$

belong to $P(U)$, and that for every such t there is a $P(U)$ representation of $c + t$ such that none of s_m, $m \geq n$ is in the sum. Since all the integers $c + t + s_n$, $1 \leq t \leq k + \sum_{j=1}^{n-1} s_j$ belong to $P(U)$ and $c + 1 + s_n \leq c + 1 + k + \sum_{j=1}^{n-1} s_j$, all the integers

$$c + t, \quad 1 \leq t \leq k + \sum_{j=1}^{n} s_j$$

belong to $P(U)$. Since S is a strictly increasing sequence of positive integers, this completes the induction step and the proof of lemma. □

Lemma 10. *Let $S = (s_1, s_2, \ldots)$ be a strictly increasing sequence of positive integers. If $s_k \leq 2s_{k-1}$ for all $k \geq 2$, then S is a $\Sigma(s_1)$-sequence.*

Proof. For $k \geq 2$, we have

$$\begin{aligned} s_k &\leq 2s_{k-1} = s_{k-1} + s_{k-1} \\ &\leq s_{k-1} + 2s_{k-2} = s_{k-1} + s_{k-2} + s_{k-2} \\ &\leq s_{k-1} + s_{k-2} + 2s_{k-3} \leq \cdots \\ &\leq s_1 + \sum_{j=1}^{k-1} s_j. \end{aligned}$$

Therefore, S is a $\Sigma(s_1)$-sequence. □

Lemma 9 shows that if a sequence S can be partitioned into one $\Sigma(k)$-sequence and one (c, k)-representable sequence then S is complete with $\theta_S \leq c$. What we aim to do is to partition $S(x^n)$ into two such sequences for some c and k.

Let $f(x) = x^n$ and let $S(f) = (f(1), f(2), \ldots)$. Let $a = n!2^{n^2}$ and $r = 2^{n^2-n}a$. Partition the elements of the sequence $S(f)$ into four sets B_1, B_2, B_3 and B_4 defined by

$$\begin{aligned} B_1 &= \{f(\alpha a + \beta) :\ 0 \leq \alpha \leq 2^{n^2-n} - 1,\ 1 \leq \beta \leq 2^n\}, \\ B_2 &= \{f(\alpha a + \beta) :\ 0 \leq \alpha \leq 2^{n^2-n} - 1,\ 2^n + 1 \leq \beta \leq a,\ \alpha a + \beta < 2^{n^2-n}a\}, \\ B_3 &= \{f(2^{n^2-n}a), f(2^{n^2-n}a + 2), f(2^{n^2-n}a + 4), \ldots\}, \\ B_4 &= \{f(2^{n^2-n}a + 1), f(2^{n^2-n}a + 3), f(2^{n^2-n}a + 5), \ldots\}, \end{aligned}$$

so that

$$B_1 \cup B_2 = \{f(1), f(2), \ldots, f(r-1)\}$$

and

$$B_3 \cup B_4 = \{f(r), f(r+1), f(r+2), \ldots\}.$$

Let S, T, U and W be the strictly increasing sequences defined by

$$\begin{aligned} S &= (s_1, s_2, \ldots, s_{2n^2}), \quad s_j \in B_1, \\ T &= (t_1, t_2, \ldots), \quad t_j \in B_3, \\ U &= (u_1, u_2, \ldots), \quad u_j \in B_1 \cup B_3, \\ W &= (w_1, w_2, \ldots), \quad w_j \in B_2 \cup B_4. \end{aligned}$$

Then the sequences U and W partition the sequence $S(f)$. First, using Lemma 10, we show that W is a $\Sigma(a)$-sequence.

Lemma 11. *For $a = n!2^{n^2}$ and $r = 2^{n^2-n}a$,*

$$\frac{f(r+1)}{f(r-1)} < \frac{f(a+2^n+1)}{f(a)} < \frac{f(2^n+2)}{f(2^n+1)} \leq 2.$$

Proof. Re-write the inequalities as

$$\left(1+\frac{2}{r-1}\right)^n < \left(1+\frac{2^n+1}{a}\right)^n < \left(1+\frac{1}{2^n+1}\right)^n \leq 2.$$

It is clear that

$$\frac{r-1}{2} > \frac{a}{2^n+1} > 2^n+1,$$

which proves the first two inequalities. The proof of the third inequality

$$\left(1+\frac{1}{2^n+1}\right)^n \leq 2 \iff 1 \leq (2^{\frac{1}{n}}-1)(2^n+1)$$

is also straightforward. □

Corollary 12. *The sequence W is a $\Sigma(a)$-sequence.*

Proof. Note that $w_1 = (2^n+1)^n$. For every $k \geq 2$, $\frac{w_k}{w_{k-1}}$ satisfies one of the following equalities:

$$\frac{w_k}{w_{k-1}} = \frac{f(\alpha+1)}{f(\alpha)}, \quad \text{for} \quad \alpha \geq 2^n+1; \tag{1}$$

$$\frac{w_k}{w_{k-1}} = \frac{f(\beta a+2^n+1)}{f(\beta a)}, \quad \text{for} \quad \beta \geq 1; \tag{2}$$

$$\frac{w_k}{w_{k-1}} = \frac{f(\gamma+2)}{f(\gamma)}, \quad \text{for} \quad \gamma \geq r-1. \tag{3}$$

Also, for every $\alpha \geq 2^n+1$, $\beta \geq 1$ and $\gamma \geq r-1$ we have

$$\frac{f(\alpha+1)}{f(\alpha)} \leq \frac{f(2^n+2)}{f(2^n+1)},$$

$$\frac{f(\beta a+2^n+1)}{f(\beta a)} \leq \frac{f(a+2^n+1)}{f(a)},$$

$$\frac{f(\gamma+2)}{f(\gamma)} \leq \frac{f(r+1)}{f(r-1)}.$$

Thus, by Lemma 11, $\frac{w_k}{w_{k-1}} \leq 2$ for $k \geq 2$, and therefore by Lemma 10, W is a $\Sigma((2^n+1)^n)$-sequence. To complete the proof, it remains to prove that $(2^n+1)^n < a$ for all $n > 1$. The inequality is true for $n = 2$ and $n = 3$, and for $n > 3$ we have

$$(2^n+1)^n < (2^n+2^n)^n = 2^n 2^{n^2} < n!2^{n^2} = a.$$

Therefore, W is a $\Sigma(a)$-sequence. □

Now, we prove that U is (d, a)-representable for some positive integer d. By Lemma 9, the value d is the upper bound of $\theta_{S(x^n)}$. Note that the sequences S and T partition U. Lemma 13 shows that $P(S)$ contains a complete residue system modulo a, and Lemma 14 and 15 together show that $P(T)$ contains arbitrarily long arithmetic progression of integers with common difference a. The properties of S and T are used in Lemma 16 to prove that $P(U)$ contains a consecutive integers.

Lemma 13. *The set $P(S)$ contains a complete residue system modulo a.*

Proof. It suffices to prove that $\{1, 2, \ldots, a\} \subset P(\mathbb{Z}_a(S))$. Let $S_1, S_2, \ldots, S_{2^n}$ be the sequences defined by

$$S_j = (j^n, j^n, \ldots, j^n), \quad 1 \leq j \leq 2^n$$

where $|S_j| = 2^{n^2-n}$ for all j. Since for each $0 \leq \alpha \leq 2^{n^2-n} - 1$, $1 \leq \beta \leq 2^n$ we have

$$f(\alpha a + \beta) \equiv \beta^n \pmod{a},$$

and S is the sequence of such $f(\alpha a + \beta)$ in increasing order, the sequences $S_1, S_2, \ldots, S_{2^n}$ partition the sequence $\mathbb{Z}_a(S)$. Note that

$$P(S_1) = \{1, 2, \ldots, 2^{n^2-n}\}, \quad P(S_2) = \{2^n, 2 \cdot 2^n, 3 \cdot 2^n, \ldots, 2^{n^2-n} \cdot 2^n\}.$$

Since for every integer $1 \leq m \leq 2^{n^2-n}(1 + 2^n)$ there exist $0 \leq \alpha \leq 2^{n^2-n}$, $1 \leq \beta \leq 2^n$ such that

$$m = \alpha 2^n + \beta,$$

we have

$$P(S_1 \cup S_2) = \{1, 2, 3, \ldots, 2^{n^2-n}(1 + 2^n)\}.$$

Likewise, for every $j \geq 3$, the inequality

$$j^n < 2^n(j-1)^n < 2^{n^2-n}(1 + 2^n + \cdots + (j-1)^n)$$

holds, and therefore for every $1 \leq m \leq 2^{n^2-n}(1 + 2^n + \cdots + j^n)$ there exists $0 \leq \alpha \leq 2^{n^2-n}$, $1 \leq \beta \leq 2^{n^2-n}(1 + 2^n + \cdots (j-1)^n)$ such that $m = \alpha j^n + \beta$. Therefore

$$P(\mathbb{Z}_a(S)) = P(S_1 \cup S_2 \cup \cdots \cup S_{2^n}) = \{1, 2, 3, \ldots, 2^{n^2-n}(1 + 2^n + 3^n + \cdots + 2^{n^2})\}.$$

It remains to prove that

$$a = n! 2^{n^2} \leq 2^{n^2-n}(1 + 2^n + 3^n + \cdots + 2^{n^2}).$$

Since

$$\left(\frac{1 + 2^n + \cdots + 2^{n^2}}{2^n} \right)^{\frac{1}{n}} \geq \frac{1 + 2 + \cdots + 2^n}{2^n},$$

we have

$$2^{n^2-n}(1+2^n+\cdots+2^{n^2}) \geq (1+2+\cdots+2^n)^n = \left(\frac{2^n(2^n+1)}{2}\right)^n.$$

Since $2^n+1 > 2j$ for every positive integer $j \leq n$, we have

$$\begin{aligned}\frac{2^{n^2-n}(1+2^n+\cdots+2^{n^2})}{n!2^{n^2}} &\geq \left(\frac{2^n(2^n+1)}{2}\right)^n \cdot \frac{1}{n!2^{n^2}} \\ &= \frac{(2^n+1)^n}{n!2^n} \\ &= \prod_{j=1}^{n} \frac{2^n+1}{2j} \\ &> 1.\end{aligned}$$

Therefore, $a = n!2^{n^2} < 2^{n^2-n}(1+2^n+\cdots+2^{n^2})$ and it completes the proof. □

Lemma 14. *For every positive integer m,*

$$a \in A\Big(\big(f(m), f(m+2), f(m+4), \ldots, f(m+\frac{2}{3}(2^{2n}-1))\big)\Big).$$

Proof. Define $\Delta_k : \mathbb{Q}[x] \to \mathbb{Q}[x]$ by:

$$\begin{aligned}\Delta_1(g(x)) &= g(4x+2) - g(4x), \\ \Delta_k(g(x)) &= \Delta_1(\Delta_{k-1}(g(x))), \quad 2 \leq k \leq n,\end{aligned}$$

so that for $1 \leq k \leq n$, $\Delta_k(f(x))$ is a polynomial of degree $n-k$. For example,

$$\Delta_2(f(x)) = \Delta_1(f(4x+2) - f(4x))$$

$$= \Big(f(16x+10) + f(16x)\Big) - \Big(f(16x+8) + f(16x+2)\Big)$$

and

$$\begin{aligned}\Delta_3((f(x)) &= \Delta_1(\Delta_2(f(x))) \\ &= \Big(f(64x+42) + f(64x+32) + f(64x+8) + f(64x+2)\Big) \\ &\quad - \Big(f(64x+40) + f(64x+34) + f(64x+10) + f(64x)\Big).\end{aligned}$$

It is easy to check that there are 2^{k-1} positive terms and 2^{k-1} negative terms in $\Delta_k(f(x))$, and all of the terms are distinct. Therefore, for each $1 \leq k \leq n$, there exist 2^k distinct integers $\alpha_k(1) > \alpha_k(2) > \cdots > \alpha_k(2^{k-1})$, $\beta_k(1) > \beta_k(2) > \cdots > \beta_k(2^{k-1})$ with $\alpha_k(1) > \beta_k(1)$ such that

$$\Delta_k(f(x)) = \sum_{i=1}^{2^{k-1}} f\big(2^{2k}x + \alpha_k(i)\big) - \sum_{i=1}^{2^{k-1}} f\big(2^{2k}x + \beta_k(i)\big).$$

Since $\alpha_1(1) = 2$ and $\alpha_k(1) = 4\alpha_{k-1}(1) + 2$ for $k \geq 2$, we have

$$\alpha_k(1) = \frac{2}{3}(2^{2k} - 1).$$

Also, we have $\{\alpha_k(2^{k-1}), \beta_k(2^{k-1})\} = \{0, 2\}$. Therefore

$$\Delta_k(f(x)) \in A\Big(\big(f(2^{2k}x), f(2^{2k}x + 2), \ldots, f(2^{2k}x + \frac{2}{3}(2^{2k} - 1))\big)\Big).$$

On the other hand, since

$$\begin{aligned}\Delta_1(f(x)) &= f(4x + 2) - f(4x) \\ &= (4x + 2)^n - (4x)^n \\ &= n2^{2n-1}x^{n-1} + \text{terms of lower degree,}\end{aligned}$$

we have

$$\begin{aligned}\Delta_n(f(x)) &= n(n-1)(n-2)\cdots 1 \cdot 2^{2n-1}2^{2n-3}2^{2n-5}\cdots 2^1 \\ &= n!2^{n^2} \\ &= a.\end{aligned}$$

Therefore,

$$a \in A\Big(\big(f(2^{2n}x), f(2^{2n}x + 2), \ldots, f(2^{2n}x + \frac{2}{3}(2^{2n} - 1))\big)\Big).$$

Since the $\Delta_n(f(x))$ is a polynomial of degree 0, the value $a = \Delta_n(f(x))$ is independent of x. Therefore, we can replace $2^{2n}x$ with an arbitrary positive integer m and we have

$$a \in A\Big(\big(f(m), f(m + 2), f(m + 4), \ldots, f(m + \frac{2}{3}(2^{2n} - 1))\big)\Big).$$ □

Lemma 15. *For every positive integer t, there exists a positive integer c such that all the integers*

$$c + ja, \quad 1 \leq j \leq t$$

belong to $P(T)$ and

$$c < (t-1)2^{n-1}(r + \frac{2}{3}(t-1)(2^{2n} - 1) + 2(t-2))^n - a.$$

Proof. Let $\alpha = \frac{2}{3}(2^{2n} - 1)$, and let $T_1, T_2, \ldots, T_{t-1}$ be the sequences defined by

$$\begin{aligned}T_1 &= \Big(f(r), f(r+2), f(r+4), \ldots, f(r+\alpha)\Big), \\ T_2 &= \Big(f(r+\alpha+2), f(r+\alpha+4), \ldots, f(r+2\alpha+2)\Big), \\ T_3 &= \Big(f(r+2\alpha+4), f(r+2\alpha+6), \ldots, f(r+3\alpha+4)\Big), \ \ldots \\ T_{t-1} &= \Big(f(r+(t-2)\alpha+2(t-2)), \ldots, f(r+(t-1)\alpha+2(t-2))\Big).\end{aligned}$$

By Lemma 14, $a \in A(T_j)$ for every $1 \le j \le t-1$, and there exists

$$A_j, B_j \in P(T_j)$$

such that $A_j - B_j = a$, both A_j and B_j consist of 2^{n-1} terms, and all 2^n terms of A_j and B_j are distinct. Let

$$\begin{aligned}
C_1 &= B_1 + B_2 + B_3 + \cdots + B_{t-1}, \\
C_2 &= A_1 + B_2 + B_3 + \cdots + B_{t-1}, \\
C_3 &= A_1 + A_2 + B_3 + \cdots + B_{t-1}, \ \ldots \\
C_j &= \sum_{i=1}^{j-1} A_i + \sum_{i=j}^{t-1} B_i, \ \ldots \\
C_t &= A_1 + A_2 + A_3 + \cdots + A_{t-1}.
\end{aligned}$$

Then each C_j belongs to $P(T)$, and $(C_1, C_2, \ldots, C_t)$ is an arithmetic progression of t integers with common difference a. Thus, they are exactly the integers $c + ja$, $1 \le j \le t$ with $c = C_1 - a = B_1 + B_2 + \cdots + B_{t-1} - a$. Since each B_j, $1 \le j \le t-1$ is a sum of 2^{n-1} terms in T, and all of the terms are less than or equal to

$$f(r + (t-1)\alpha + 2(t-2)) = (r + \frac{2}{3}(t-1)(2^{2n}-1) + 2(t-2))^n,$$

we have

$$c = C_1 - a < (t-1)2^{n-1}(r + \frac{2}{3}(t-1)(2^{2n}-1) + 2(t-2))^n - a. \qquad \square$$

Finally, we show that $P(U)$ contains a consecutive integers $k_1 + t_1, k_2 + t_2, \ldots, k_a + t_a$, where $\{k_1, k_2, \ldots, k_a\}$ is a complete residue system of a in $P(S)$ and $t_1, t_2, \ldots, t_a$ are taken from the arithmetic progression in $P(T)$.

Lemma 16. *Let $b = 2^{n^3} a^{n-1}$. The sequence U is (d, a)-representable for a positive integer d such that*

$$d < (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n}-1) + 2(b-2))^n - 2a + ab.$$

Proof. By Lemma 15, $P(T)$ contains an arithmetic progression of b integers,

$$c + ja, \quad 1 \le j \le b$$

with

$$c < (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n}-1) + 2(b-2))^n - a.$$

By Lemma 13, there exist positive integers $1 = k_1 < k_2 < \cdots < k_a$ in $P(S)$ such that $\{k_1, k_2, \ldots, k_a\}$ is a complete residue system modulo a. For $1 \le j \le a$, let

$$n_j = \left\lfloor \frac{k_a - k_j}{a} \right\rfloor + 1.$$

Then for each $1 \le j \le a$,

$$\frac{k_a - k_j}{a} < n_j \le \frac{k_a - k_j}{a} + 1 \iff k_a < n_j a + k_j \le k_a + a.$$

Also, if $i \ne j$ then $n_i a + k_i \not\equiv n_j a + k_j \pmod{a}$. Therefore

$$\{c + n_1 a + k_1, c + n_2 a + k_2, \ldots, c + n_a a + k_a\}$$

is the set of a consecutive integers

$$\{c + k_a + 1, c + k_a + 2, \ldots, c + k_a + a\}.$$

It remains to prove that each $c + n_j a + k_j$ is in $P(U)$. Let $\Sigma(S)$ denote the sum of every element of S. Since $|S| = 2^{n^2}$, and

$$s_j \le f((2^{n^2-n} - 1)a + 2^n) = (r - a + 2^n)^n < r^n - (a - 2^n)^n < r^n - n!$$

for each $s_j \in S$, we have

$$\Sigma(S) < 2^{n^2}(r^n - n!) = 2^{n^2} r^n - a.$$

Therefore, for each $1 \le j \le a$ we have

$$1 \le n_j < \frac{k_a}{a} + 1 \le \frac{1}{a}\Sigma(S) + 1 < \frac{1}{a} 2^{n^2} r^n = 2^{n^3} a^{n-1} = b$$

and thus all of $c + n_j a$ belong to $P(T)$. Since all of k_j belong to $P(S)$, all of $c + n_j a + k_j$ belong to $P(U)$. Therefore, U is $(c + k_a, a)$-representable. Let

$$d = c + k_a.$$

Since $k_a < \Sigma(S) < 2^{n^2} r^n - a = ab - a$,

$$d = c + k_a < (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n} - 1) + 2(b-2))^n - 2a + ab. \qquad \square$$

Now we have everything we need to prove the theorem.

3 Proof of the theorem

Recall that U and W are disjoint subsequences of $S(f)$. By Corollary 12, W is a $\Sigma(a)$-sequence and by Lemma 16, U is (d, a)-representable with

$$d < (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n} - 1) + 2(b-2))^n - 2a + ab.$$

Therefore by Lemma 9, $S(x^n) = S(f)$ is complete and

$$\theta_{S(x^n)} \le d < (b-1)2^{n-1}(r + \frac{2}{3}(b-1)(2^{2n} - 1) + 2(b-2))^n - 2a + ab. \qquad \square$$

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