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The Arithmetic Jacobian Matrix and Determinant

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Abstract

Let $a_1, \ldots, a_m$ be such real numbers that can be expressed as a finite product of prime powers with rational exponents. Using arithmetic partial derivatives, we define the arithmetic Jacobian matrix $\mathbf{J_a}$ of the vector $\mathbf{a} = (a_1, \ldots, a_m)$ analogously to the Jacobian matrix $\mathbf{J_f}$ of a vector function $\mathbf{f}$. We introduce the concept of multiplicative independence of $\{a_1, \ldots, a_m\}$ and show that $\mathbf{J_a}$ plays in it a similar role as $\mathbf{J_f}$ does in functional independence. We also present a kind of arithmetic implicit function

theorem and show that $\mathbf{J_a}$ applies to it somewhat analogously as $\mathbf{J_f}$ applies to the ordinary implicit function theorem.

1 Introduction

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{P}$ stand for the set of real numbers, rational numbers, integers, nonnegative integers, and primes, respectively.

If $a \in \mathbb{R}$, there may be a sequence of rational numbers $(\nu_p(a))_{p\in\mathbb{P}}$ with only finitely many nonzero terms satisfying

$$a = (\operatorname{sgn} a) \prod_{p\in\mathbb{P}} p^{\nu_p(a)}, \tag{1}$$

where sgn is the sign function. We let $\mathbb{R}'$ and $\mathbb{R}'_+$ denote the set of all such real numbers and the subset consisting of its positive elements, respectively. Formula (1) is also valid for $a = 0$, as we define $\nu_p(0) = 0$ for all $p \in \mathbb{P}$. If $\nu_p(a) \neq 0$, we say that p *divides* a and denote $p \mid a$. Otherwise, we denote $p \nmid a$.

Proposition 1. *Let $a \in \mathbb{R}'$ and $V_a = \{\nu_p(a) \mid p \in \mathbb{P}\}$. Then*

(a) *$a \in \mathbb{Z}$ if and only if $V_a \subset \mathbb{N}$;*

(b) *$a \in \mathbb{Q}$ if and only if $V_a \subset \mathbb{Z}$.*

Proof. Simple and omitted. □

Proposition 2. *If $a \in \mathbb{R}'$, then the sequence $(\nu_p(a))_{p\in\mathbb{P}}$ is unique.*

Proof. This is well known if $a \in \mathbb{Q}$. Otherwise, see [8, Lemma 1]. □

We define the *arithmetic derivative* of $a \in \mathbb{R}'$ by

$$a' = a \sum_{p\in\mathbb{P}} \frac{\nu_p(a)}{p} = \sum_{p\in\mathbb{P}} a'_p,$$

where

$$a'_p = \frac{\nu_p(a)}{p} a \tag{2}$$

is the *arithmetic partial derivative* of a with respect to p. For the background and history of these concepts, see, e.g., [1, 8, 4, 3]. These references mainly concern the arithmetic derivative in $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$, but most of the results can be extended to $\mathbb{R}'$ in an obvious way, see [8, Section 9].

Let $\mathbf{f} = (f_1, \ldots, f_m) : E \to \mathbb{R}^m$ be a continuously differentiable function, where $E \subseteq \mathbb{R}^n$ is a connected open set. Its *Jacobian matrix* at $\mathbf{t} = (t_1, \ldots, t_n) \in E$ is defined by

$$\mathbf{J_f}(\mathbf{t}) = \begin{pmatrix} (f_1)'_{t_1}(\mathbf{t}) & (f_1)'_{t_2}(\mathbf{t}) & \ldots & (f_1)'_{t_n}(\mathbf{t}) \\ (f_2)'_{t_1}(\mathbf{t}) & (f_2)'_{t_2}(\mathbf{t}) & \ldots & (f_2)'_{t_n}(\mathbf{t}) \\ & & \vdots & \\ (f_m)'_{t_1}(\mathbf{t}) & (f_m)'_{t_2}(\mathbf{t}) & \ldots & (f_m)'_{t_n}(\mathbf{t}) \end{pmatrix},$$

where $(f_i)'_{t_j} = \partial f_i/\partial t_j$. If $m = n$, then $\det \mathbf{J_f}(\mathbf{x})$ is the *Jacobian determinant* (or, more briefly, the *Jacobian*) of $\mathbf{f}$.

Let $a_1, \ldots, a_m \in \mathbb{R}'_+$ (actually, we could study $\mathbb{R}'$ instead of $\mathbb{R}'_+$, which, however, would not benefit us in any significant way), and denote

$$P = \{p_1, \ldots, p_n\} = \{p \in \mathbb{P} \mid \exists a_i : p \mid a_i\} \tag{3}$$

and

$$\alpha_{ij} = \nu_{p_j}(a_i), \quad i = 1, \ldots, m,\ j = 1, \ldots, n. \tag{4}$$

Then

$$a_i = \prod_{p \in \mathbb{P}} p^{\nu_p(a_i)} = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \cdots p_n^{\alpha_{in}}, \quad i = 1, \ldots, m. \tag{5}$$

Further, let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}, \qquad \boldsymbol{\alpha}_i = \begin{pmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \vdots \\ \alpha_{in} \end{pmatrix}, \quad i = 1, \ldots, m, \tag{6}$$

and

$$\mathbf{A_a} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ & \vdots & & \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \vdots \\ \boldsymbol{\alpha}_m^T \end{pmatrix}. \tag{7}$$

We define the *arithmetic Jacobian matrix* of $\mathbf{a}$ by

$$\mathbf{J_a} = \begin{pmatrix} (a_1)'_{p_1} & (a_1)'_{p_2} & \ldots & (a_1)'_{p_n} \\ (a_2)'_{p_1} & (a_2)'_{p_2} & \ldots & (a_2)'_{p_n} \\ & \vdots & & \\ (a_m)'_{p_1} & (a_m)'_{p_2} & \ldots & (a_m)'_{p_n} \end{pmatrix}$$

and, if $m = n$, the *arithmetic Jacobian determinant* (or, more briefly, the *arithmetic Jacobian*) of $\mathbf{a}$ by

$$\det \mathbf{J_a} = \begin{vmatrix} (a_1)'_{p_1} & (a_1)'_{p_2} & \ldots & (a_1)'_{p_m} \\ (a_2)'_{p_1} & (a_2)'_{p_2} & \ldots & (a_2)'_{p_m} \\ & \vdots & & \\ (a_m)'_{p_1} & (a_m)'_{p_2} & \ldots & (a_m)'_{p_m} \end{vmatrix}.$$

Let $\mathbf{f}$ be as above. The functions $f_1, \ldots, f_m$ are functionally independent (i.e., there is no function $\phi : \mathbb{R}^m \to \mathbb{R}$ such that $\nabla\phi(\mathbf{f}(\mathbf{t})) \neq \mathbf{0}$ and $\phi(f_1(\mathbf{t}), \ldots, f_m(\mathbf{t})) = 0$ for all $\mathbf{t} \in E$) if and only if $m \leq n$ and $\operatorname{rank} \mathbf{J_f}(\mathbf{t}) = m$ for all $\mathbf{t} \in E$. (See, e.g., [5].) In Section 2, we will define the concept of multiplicative independence of the numbers $a_1, \dots, a_m$ and study the role of $\mathbf{J}_{\mathbf{a}}$ there.

We outline the implicit function theorem [7, Theorem 9.28]. Assuming $m < n$, write $\mathbf{t} = (\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^{n-m}$. Let $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^{n-m}$ satisfy $(\mathbf{a}, \mathbf{b}) \in E$. Define the function $\boldsymbol{\phi}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \mathbf{b})$, where $\mathbf{x}$ is "close to" $\mathbf{a}$. If it satisfies $\det \mathbf{J}_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}) \neq 0$ and if $\mathbf{y}$ is "close to" $\mathbf{b}$, then the equation $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ is uniquely "solvable" with respect to $\mathbf{x}$. We will in Section 3 present a theorem where the arithmetic Jacobian matrix and determinant play a somewhat similar role. Section 4 is devoted to some concluding remarks.

2 Multiplicative independence

Let $\operatorname{seq}\mathbb{Q}$ be the set of infinite sequences of rational numbers with finitely many nonzero terms. Consider the mapping

$$\boldsymbol{\nu} : \mathbb{R}'_+ \to \operatorname{seq}\mathbb{Q} : \boldsymbol{\nu}(a) = (\nu_p(a))_{p \in \mathbb{P}}.$$

Defining in $\operatorname{seq}\mathbb{Q}$ addition and scalar multiplication in the ordinary way, this set becomes a vector space over $\mathbb{Q}$. On the other hand, defining in $\mathbb{R}'_+$ addition as the ordinary multiplication and scalar multiplication as the ordinary exponentation, also $\mathbb{R}'_+$ becomes a vector space over $\mathbb{Q}$. Then $\boldsymbol{\nu}$ is an isomorphism, and we can identify $\mathbb{R}'_+$ with $\operatorname{seq}\mathbb{Q}$. Because linear independence is a well-defined concept in $\operatorname{seq}\mathbb{Q}$, we may so define linear independence in $\mathbb{R}'_+$. However, we find the term "multiplicative independence" more appropriate. (We quote this term from Pong [6], who studied this concept in an Abelian group.) So we say that a set

$$S = \{a_1, \dots, a_m\} \subset \mathbb{R}'_+ \tag{8}$$

is (and the numbers $a_1, \dots, a_m$ are) *multiplicatively independent* if the set $\{\boldsymbol{\nu}(a_1), \dots, \boldsymbol{\nu}(a_m)\}$ is linearly independent. Otherwise, S is (and $a_1, \dots, a_m$ are) *multiplicatively dependent*.

Proposition 3. *Let S be as in (8) and $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m$ as in (6). The following conditions are equivalent:*

(a) *The set S is multiplicatively independent.*

(b) *The only rational numbers $\lambda_1, \dots, \lambda_m$ satisfying $a_1^{\lambda_1} \cdots a_m^{\lambda_m} = 1$ are $\lambda_1 = \cdots = \lambda_m = 0$.*

(c) *The set $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m\}$ is linearly independent in the vector space $\mathbb{Q}^n$, where n is as in (3).*

Proof. Simple and omitted. $\square$

We now present such properties of the arithmetic Jacobian determinant that have relevance to multiplicative independence.

Proposition 4. *Let $\mathbf{a}$ be as in (6). If $n = m$ in (3), then*

$$\det \mathbf{J}_{\mathbf{a}} = \frac{a_1 a_2 \cdots a_m}{p_1 p_2 \cdots p_m} \det \mathbf{A}_{\mathbf{a}}.$$

Proof. By (2),

$$\det \mathbf{J_a} = \begin{vmatrix} \frac{\alpha_{11}}{p_1}a_1 & \frac{\alpha_{12}}{p_2}a_1 & \dots & \frac{\alpha_{1m}}{p_m}a_1 \\ \frac{\alpha_{21}}{p_1}a_2 & \frac{\alpha_{22}}{p_2}a_2 & \dots & \frac{\alpha_{2m}}{p_m}a_2 \\ & \vdots & & \\ \frac{\alpha_{m1}}{p_1}a_m & \frac{\alpha_{m2}}{p_2}a_m & \dots & \frac{\alpha_{mm}}{p_m}a_m \end{vmatrix}.$$

Take the factor a_1 from the first row, a_2 from the second one, etc. Further, take the factor $1/p_1$ from the remaining first column, $1/p_2$ from the second one, etc. We obtain

$$\det \mathbf{J_a} = a_1 a_2 \cdots a_m \begin{vmatrix} \frac{\alpha_{11}}{p_1} & \frac{\alpha_{12}}{p_2} & \dots & \frac{\alpha_{1m}}{p_m} \\ \frac{\alpha_{21}}{p_1} & \frac{\alpha_{22}}{p_2} & \dots & \frac{\alpha_{2m}}{p_m} \\ & \vdots & & \\ \frac{\alpha_{m1}}{p_1} & \frac{\alpha_{m2}}{p_2} & \dots & \frac{\alpha_{mm}}{p_m} \end{vmatrix} = \frac{a_1 a_2 \cdots a_m}{p_1 p_2 \cdots p_m} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ & \vdots & & \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mm} \end{vmatrix}.$$

□

Corollary 5. *If* $\mathbf{a}$ *is as in* (6)*, then* $\operatorname{rank} \mathbf{J_a} = \operatorname{rank} \mathbf{A_a}$.

Proof. Given $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$ with equal number of elements, let $\mathbf{J_a}(I, J)$ and $\mathbf{A_a}(I, J)$ denote the submatrix of $\mathbf{J_a}$ and respectively $\mathbf{A_a}$ with row indices in I and column indices in J. By Proposition 4, these matrices are either both nonsingular or both singular. Since rank is the largest dimension of a nonsingular square submatrix, the claim therefore follows. □

Theorem 6. *Let* S*,* $\mathbf{a}$*, and* P *be as in* (8), (6), *and* (3), *respectively. The set* S *is multiplicatively independent if and only if* $\operatorname{rank} \mathbf{J_a} = m$.

Proof. Apply Proposition 3 and Corollary 5. (Note that $\operatorname{rank} \mathbf{J_a} = m$ implies $m \le n$.) □

Proposition 7. *Let* $\mathbf{a}$ *and* P *be as in* (6) *and* (3)*, respectively, let* $0 \ne x_1, \dots, x_m \in \mathbb{Q}$*, and denote* $\mathbf{a}^{\mathbf{x}} = (a_1^{x_1}, \dots, a_m^{x_m})$*. Then*

$$\operatorname{rank} \mathbf{J}_{\mathbf{a}^{\mathbf{x}}} = \operatorname{rank} \mathbf{J_a}.$$

Proof. In general, we have

$$(a^x)'_p = \frac{\nu_p(a^x)}{p} a^x = \frac{x\nu_p(a)}{p} a^x = x a^{x-1} \frac{\nu_p(a)}{p} a = x a^{x-1} a'_p$$

for all $a \in \mathbb{R}'_+$, $x \in \mathbb{Q}$, $p \in \mathbb{P}$. Consequently,

$$\mathbf{J}_{\mathbf{a}^{\mathbf{x}}} = \begin{pmatrix} (a_1^{x_1})'_{p_1} & (a_1^{x_1})'_{p_2} & \dots & (a_1^{x_1})'_{p_n} \\ (a_2^{x_2})'_{p_1} & (a_2^{x_2})'_{p_2} & \dots & (a_2^{x_2})'_{p_n} \\ & \vdots & & \\ (a_m^{x_m})'_{p_1} & (a_m^{x_m})'_{p_2} & \dots & (a_m^{x_m})'_{p_n} \end{pmatrix} =$$

$$\begin{pmatrix} x_1 a_1^{x_1-1}(a_1)'_{p_1} & x_1 a_1^{x_1-1}(a_1)'_{p_2} & \dots & x_1 a_1^{x_1-1}(a_1)'_{p_n} \\ x_2 a_2^{x_2-1}(a_2)'_{p_1} & x_2 a_2^{x_2-1}(a_2)'_{p_2} & \dots & x_2 a_2^{x_2-1}(a_2)'_{p_n} \\ & \vdots & & \\ x_m a_m^{x_m-1}(a_m)'_{p_1} & x_m a_m^{x_m-1}(a_m)'_{p_2} & \dots & x_m a_m^{x_m-1}(a_m)'_{p_n} \end{pmatrix} = \mathbf{D}\mathbf{J_a},$$

where

$$\mathbf{D} = \operatorname{diag}(x_1 a_1^{x_1-1}, \dots, x_m a_m^{x_m-1}).$$

Since $\mathbf{D}$ is invertible, the claim follows. $\square$

3 An arithmetic implicit function theorem

In this section, we establish a kind of arithmetic implicit function theorem where the arithmetic Jacobian matrix and determinant apply. The problem is that arithmetic differentiation operates on numbers, not on functions; so, we must include variables in this attempt.

Let $a_1, \dots, a_m, b_1, \dots, b_r \in \mathbb{R}'_+$. We consider the equation

$$a_1^{x_1} \cdots a_m^{x_m} = b_1^{y_1} \cdots b_r^{y_r}, \tag{9}$$

where $x_1, \dots, x_m \in \mathbb{Q}$ are variables and $y_1, \dots, y_r \in \mathbb{Q} \setminus \{0\}$ are constants. Factorizing $b_1, \dots, b_r$ as in (1), we can reduce (9) to

$$a_1^{x_1} \cdots a_m^{x_m} = q_1^{z_1} \cdots q_s^{z_s}, \tag{10}$$

where $q_1, \dots, q_s \in \mathbb{P}$ are distinct and $z_1, \dots, z_s \in \mathbb{Q} \setminus \{0\}$ are constants. We define P and α_{ij} by (3) and (4), respectively; then (5) holds. We also denote

$$Q = \{q_1, \dots, q_s\}.$$

Theorem 8. *If (10) has a solution, then*

$$Q \subseteq P. \tag{11}$$

If (11) holds and

$$\operatorname{rank} \mathbf{J}_{\mathbf{a}} = n, \tag{12}$$

where $\mathbf{a}$ *is as in (6), then (10) has a solution.*

Proof. First, assume that (10) has a solution. Let $q_i \in Q$. Since $z_i \neq 0$, q_i divides the left-hand side of (10); so, $q_i = p_j$ for some j, and (11) follows.

Second, assume that (11) and (12) hold. By reordering the indices of $p_1, \dots, p_n$, we can write $Q = \{p_1, \dots, p_s\}$. Let $\mathbf{a}$ and $\mathbf{A}_{\mathbf{a}}$ be as in (6) and (7). Then

$$a_i^{x_i} = \prod_{j=1}^{n} p_j^{\alpha_{ij} x_i}, \quad i = 1, \dots, m,$$

and (10) reads

$$\prod_{i=1}^{m} \prod_{j=1}^{n} p_j^{\alpha_{ij} x_i} = \prod_{j=1}^{s} p_j^{z_j},$$

i.e.,

$$\prod_{j=1}^{n} p_j^{\sum_{i=1}^{m} \alpha_{ij} x_i} = \prod_{j=1}^{s} p_j^{z_j}. \tag{13}$$

By Proposition 2, this is equivalent to

$$\sum_{i=1}^{m} \alpha_{ij} x_i = z_j, \quad j = 1, \ldots, s,$$
$$\sum_{i=1}^{m} \alpha_{ij} x_i = 0, \quad j = s+1, \ldots, n.$$

In matrix form, this reads

$$\mathbf{A}_{\mathbf{a}}^T \mathbf{x} = \mathbf{z}, \tag{14}$$

where $z_{s+1} = \cdots = z_n = 0$ and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

By Corollary 5 and (12),

$$\operatorname{rank} \mathbf{A}_{\mathbf{a}}^T = \operatorname{rank} \mathbf{A}_{\mathbf{a}} = \operatorname{rank} \mathbf{J}_{\mathbf{a}} = n. \tag{15}$$

Therefore, $\mathbf{A}_{\mathbf{a}}^T \mathbb{Q}^m = \mathbb{Q}^n$, which implies that (14) has a solution. (Note that $m \geq n$ by (15).) $\square$

Corollary 9. *Assume that $m = n$ in (4). Then (10) has a unique solution if and only if (11) holds and* $\det \mathbf{J}_{\mathbf{a}} \neq 0$.

Proof. The claim follows from (15). $\square$

4 Concluding remarks

We defined the arithmetic Jacobian matrix and multiplicative independence. We saw that the arithmetic Jacobian matrix relates to multiplicative independence in the same way as the ordinary Jacobian matrix relates to functional independence. We also noticed that the arithmetic Jacobian matrix and determinant play a role in establishing a certain kind of implicit function theorem somewhat similarly as the ordinary Jacobian matrix and determinant do in the ordinary implicit function theorem. For this purpose, we needed to introduce extra variables.

Another functional interpretation of arithmetic differentiation may be obtained as follows: If $a \in \mathbb{R}'_+$ and $p \in \mathbb{P}$, then there are unique $\alpha \in \mathbb{Q}$ and $\tilde{a} \in \mathbb{R}'_+$ such that $a = \tilde{a}p^\alpha$ and $p \nmid \tilde{a}$; in fact, $\alpha = \nu_p(a)$. Further, $a'_p = \alpha\tilde{a}p^{\alpha-1}$. So, in studying arithmetic partial derivatives of first order, the primes behave, in a certain sense, like variables, and the rational numbers like functions.

Since the isomorphism $\boldsymbol{\nu}$ brings the vector space structure to $\mathbb{R}'_+$, we can define all linear algebra concepts there. In particular, we have already studied the "arithmetic inner product", see [2]. (In that paper, we considered only $\mathbb{Q}_+$ but every argument applies also for $\mathbb{R}'_+$.)

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