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Alternating Sums Concerning Multiplicative Arithmetic Functions

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Abstract

We deduce asymptotic formulas for the alternating sums $\sum_{n\le x}(-1)^{n-1}f(n)$ and $\sum_{n\le x}(-1)^{n-1}\frac{1}{f(n)}$, where f is one of the following classical multiplicative arithmetic functions: Euler's totient function, the Dedekind function, the sum-of-divisors function, the divisor function, the gcd-sum function. We also consider analogs of these functions, which are associated to unitary and exponential divisors, and other special functions. Some of our results improve the error terms obtained by Bordellès and Cloitre. We formulate certain open problems.

1 Introduction

Alternating sums and series appear in various topics of mathematics and number theory, in particular. For example, it is well-known that for $s \in \mathbb{C}$ with $\Re s > 1$,

$$\eta(s) := \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s} = \left(1-\frac{1}{2^{s-1}}\right)\zeta(s), \tag{1}$$

[1]The present scientific contribution is dedicated to the 650th anniversary of the foundation of the University of Pécs, Hungary.

representing the alternating zeta function or Dirichlet's eta function. Here the left-hand side is convergent for $\Re s > 0$, and this can be used for analytic continuation of the Riemann zeta function for $\Re s > 0$. See, e.g., Tenenbaum [35, Sect. II.3.2].

Bordellès and Cloitre [4] established asymptotic formulas with error terms for alternating sums

$$\sum_{n \le x} (-1)^{n-1} f(n), \tag{2}$$

where $f(n) = 1/g(n)$ and g belongs to a class of multiplicative arithmetic functions, including Euler's totient function φ, the sum-of-divisors function σ and the Dedekind function ψ. It seems that there are no other results in the literature for alternating sums of type (2).

Using a different approach, also based on the convolution method, we show that for many classical multiplicative arithmetic functions f, estimates with sharp error terms for the alternating sum (2) can easily be deduced by using known results for

$$\sum_{n \le x} f(n). \tag{3}$$

For other given multiplicative functions f, a difficulty arises, namely to estimate the coefficients of the reciprocal of a formal power series, more exactly the reciprocal of the Bell series of f for $p = 2$. If the coefficients of the original power series are positive and log-convex, then a result of Kaluza [16] can be used. The obtained error terms for (2) are usually the same, or slightly larger than for (3).

In this way we improve some of the error terms obtained in [4]. We also deduce estimates for other classical multiplicative functions f. As a tool, we use formulas for alternating Dirichlet series

$$D_{\text{altern}}(f, s) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{f(n)}{n^s}, \tag{4}$$

generalizing (1).

In the case of some other functions f, a version of Kendall's renewal theorem (from probability theory) can be applied. Berenhaut, Allen, and Fraser obtained [3] an explicit form of Kendall's theorem (also see [2]), but this cannot be used for the functions we deal with. We prove a new explicit Kendall-type inequality, which can be applied in some cases. As far as we know, there are no other similar applicable results to obtain better error terms in the literature. We formulate several open problems concerning the error terms of the presented asymptotic formulas.

Finally, a generalization of the alternating Dirichlet series (4) and the alternating sum (2) is discussed.

2 General results

2.1 Alternating Dirichlet series

Let

$$D(f,s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{5}$$

denote the Dirichlet series of the function f. If f is multiplicative, then it can be expanded into the Euler product

$$D(f,s) = \prod_{p\in\mathbb{P}} \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^{\nu s}}. \tag{6}$$

If f is completely multiplicative, then

$$\sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^{\nu s}} = \left(1 - \frac{f(p)}{p^s}\right)^{-1} \tag{7}$$

and

$$D(f,s) = \prod_{p\in\mathbb{P}} \left(1 - \frac{f(p)}{p^s}\right)^{-1}. \tag{8}$$

Proposition 1. *If f is a multiplicative function, then*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{f(n)}{n^s} = D(f,s) \left(2 \left(\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^{\nu s}} \right)^{-1} - 1 \right), \tag{9}$$

and if f is completely multiplicative, then

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{f(n)}{n^s} = \left(1 - \frac{f(2)}{2^{s-1}}\right) \prod_{p\in\mathbb{P}} \left(1 - \frac{f(p)}{p^s}\right)^{-1},$$

formally or in case of convergence.

Proof. We have by using (6),

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{f(n)}{n^s} = -\sum_{n=1}^{\infty} \frac{f(n)}{n^s} + 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{f(n)}{n^s} = -D(f,s) + 2 \prod_{\substack{p\in\mathbb{P} \\ p>2}} \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^{\nu s}}$$

$$= -D(f,s) + 2D(f,s) \left(\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^{\nu s}} \right)^{-1} = D(f,s) \left(2 \left(\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^{\nu s}} \right)^{-1} - 1 \right).$$

If f is completely multiplicative, then use identities (7) and (8). □

For special choices of f we obtain formulas for the alternating Dirichlet series (4). For example, let $f = \varphi$ be Euler's totient function. For every prime p,

$$\sum_{\nu=0}^{\infty} \frac{\varphi(p^\nu)}{p^{\nu s}} = \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s-1}}\right)^{-1}. \tag{10}$$

Here the left-hand side of (10) can be computed directly. However, it is more convenient to use the well-known representation of the Dirichlet series of φ (similar considerations are valid for other classical multiplicative function, as well). Namely,

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{s-1}}\right)^{-1}, \tag{11}$$

and using the Euler product,

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_{p \in \mathbb{P}} \sum_{\nu=0}^{\infty} \frac{\varphi(p^\nu)}{p^{\nu s}}. \tag{12}$$

Now a quick look at (11) and (12) gives (10). We deduce from Proposition 1 that

$$D_{\mathrm{altern}}(\varphi, s) = \frac{\zeta(s-1)}{\zeta(s)} \left(2\left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{2^{s-1}}\right) - 1\right), \tag{13}$$

which can be written as (31).

Note that the function $n \mapsto (-1)^{n-1}$ is multiplicative. Therefore, it is possible to give a direct proof of (13) (and of similar formulas, where φ is replaced by another multiplicative function) using Euler products:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\varphi(n)}{n^s} = \left(1 - \sum_{\nu=1}^{\infty} \frac{\varphi(2^\nu)}{2^{\nu s}}\right) \prod_{\substack{p \in \mathbb{P} \\ p > 2}} \left(1 + \sum_{\nu=1}^{\infty} \frac{\varphi(p^\nu)}{p^{\nu s}}\right),$$

but computations are simpler by the previous approach.

2.2 Mean values and alternating sums

Let f be a complex-valued arithmetic function. The (asymptotic) mean value of f is

$$M(f) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n),$$

provided that this limit exists. Let

$$M_{\mathrm{altern}}(f) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} (-1)^{n-1} f(n)$$

denote the mean value of the function $n \mapsto (-1)^{n-1} f(n)$ (if it exists).

Proposition 2. *Assume that f is a multiplicative function and*

$$\sum_{p\in\mathbb{P}} \frac{|f(p)-1|}{p} < \infty, \qquad \sum_{p\in\mathbb{P}} \sum_{\nu=2}^{\infty} \frac{|f(p^\nu)|}{p^\nu} < \infty. \tag{14}$$

Then there exists

$$M(f) = \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu}.$$

Furthermore, if $\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^\nu} \ne 0$, then there exists

$$M_{\text{altern}}(f) = M(f) \left(2 \left(\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^\nu} \right)^{-1} - 1 \right), \tag{15}$$

and if $\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^\nu} = 0$, then $M(f) = 0$ and there exists

$$M_{\text{altern}}(f) = \prod_{\substack{p\in\mathbb{P}\\ p>2}} \left(1-\frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu}. \tag{16}$$

Proof. The result for $M(f)$ is a version of Wintner's theorem for multiplicative functions. See Schwarz and Spilker [24, Cor. 2.3]. It easy to check that assuming (14) for f, the same conditions hold for the multiplicative function $n \mapsto (-1)^{n-1} f(n)$. We deduce that $M_{\text{altern}}(f)$ exists and it is

$$M_{\text{altern}}(f) = \left(1-\frac{1}{2}\right) \left(1 - \sum_{\nu=1}^{\infty} \frac{f(2^\nu)}{2^\nu}\right) \prod_{\substack{p\in\mathbb{P}\\ p>2}} \left(1-\frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu}. \tag{17}$$

Now if $t := \sum_{\nu=1}^{\infty} \frac{f(2^\nu)}{2^\nu} \ne -1$, then

$$M_{\text{altern}}(f) = \frac{1-t}{1+t} \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu},$$

which is (15). If $t = -1$, then (17) gives (16). $\square$

Application 3. Let f be multiplicative such that $f(p) = 1$, $f(p^2) = -6$, $f(p^\nu) = 0$ for every $p \in \mathbb{P}$ and every $\nu \ge 3$. Here $\sum_{\nu=0}^{\infty} \frac{f(2^\nu)}{2^\nu} = 1 + \frac{1}{2} - \frac{6}{4} = 0$. Using Proposition 2 we deduce that $M(f) = 0$ and there exists

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} (-1)^{n-1} f(n) = \prod_{\substack{p\in\mathbb{P}\\ p>2}} \left(1-\frac{1}{p}\right) \left(1+\frac{1}{p}-\frac{6}{p^2}\right) = \prod_{\substack{p\in\mathbb{P}\\ p>2}} \left(1-\frac{7}{p^2}+\frac{6}{p^3}\right) \ne 0.$$

The following result is similar. Let

$$L(f) := \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{f(n)}{n}$$

denote the logarithmic mean value of f and, assuming that f is non-vanishing, let

$$\overline{L}(f) := \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{f(n)},$$

$$\overline{L}_{\text{altern}}(f) := \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} (-1)^{n-1} \frac{1}{f(n)},$$

provided that the limits exist.

Proposition 4. *Assume that f is a non-vanishing multiplicative function and*

$$\sum_{p\in\mathbb{P}} \left| \frac{1}{f(p)} - \frac{1}{p} \right| < \infty, \qquad \sum_{p\in\mathbb{P}} \sum_{\nu=2}^{\infty} \frac{1}{|f(p^\nu)|} < \infty. \tag{18}$$

Then there exists

$$\overline{L}(f) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{1}{f(p^\nu)}.$$

Furthermore, if $\sum_{\nu=0}^{\infty} \frac{1}{f(2^\nu)} \neq 0$, then there exists

$$\overline{L}_{\text{altern}}(f) = \overline{L}(f) \left(2 \left(\sum_{\nu=0}^{\infty} \frac{1}{f(2^\nu)} \right)^{-1} - 1 \right),$$

and if $\sum_{\nu=0}^{\infty} \frac{1}{f(2^\nu)} = 0$, then $\overline{L}(f) = 0$ and there exists

$$\overline{L}_{\text{altern}}(f) = \prod_{\substack{p\in\mathbb{P} \\ p>2}} \left(1 - \frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{1}{f(p^\nu)}.$$

Proof. Apply Proposition 2 for $f(n) := \frac{n}{f(n)}$ and use the following property: If the mean value $M(f)$ exists, then the logarithmic mean value $L(f)$ exists as well, and is equal to $M(f)$. See Hildebrand [14, Thm. 2.13]. $\square$

Application 5. It follows from Proposition 4 that

$$\overline{L}(\varphi) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{\varphi(n)} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{1}{\varphi(p^\nu)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

which is well-known, and

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} (-1)^{n-1} \frac{1}{\varphi(n)} = \overline{L}(\varphi) \left(2 \left(\sum_{\nu=0}^{\infty} \frac{1}{\varphi(2^\nu)} \right)^{-1} - 1 \right) = -\frac{\zeta(2)\zeta(3)}{3\zeta(6)},$$

obtained by Bordellès and Cloitre [4]. Conditions (18) were refined in [4] to deduce asymptotic formulas with error terms for alternating sums of reciprocals of a class of multiplicative arithmetic functions, including Euler's totient function.

2.3 Method to obtain asymptotic formulas

Assume that f is a nonzero complex-valued multiplicative function. Consider the formal power series

$$S_f(x) := \sum_{\nu=0}^{\infty} a_\nu x^\nu,$$

where $a_\nu = f(2^\nu)$ ($\nu \ge 0$), $a_0 = f(1) = 1$. Note that $S_f(x)$ is the Bell series of the function f for the prime $p = 2$. See, e.g., Apostol [1, Ch. 2]. Let

$$\overline{S}_f(x) := \sum_{\nu=0}^{\infty} b_\nu x^\nu$$

be its formal reciprocal power series. Here the coefficients b_ν are given by $b_0 = 1$ and $\sum_{j=0}^{\nu} a_j b_{\nu-j} = 0$ ($\nu \ge 1$). If both series $S_f(x)$ and $\overline{S}_f(x)$ converge for an $x \in \mathbb{C}$, then $S_f(x)\overline{S}_f(x) = 1$. In particular, if r_f and $\overline{r}_f$ are the radii of convergence of $S_f(x)$, respectively $\overline{S}_f(x)$, then $S_f(x)\overline{S}_f(x) = 1$ for every $x \in \mathbb{C}$ such that $|x| < \min(r_f, \overline{r}_f)$.

It follows from (9) that the convolution identity

$$(-1)^{n-1} f(n) = \sum_{dj=n} h_f(d) f(j) \quad (n \ge 1) \tag{19}$$

holds, where the function h_f is multiplicative, $h_f(p^\nu) = 0$ if $p > 2$, $\nu \ge 1$ and $h_f(2^\nu) = 2b_\nu$ ($\nu \ge 1$), $h_f(1) = 2b_0 - 1 = 1$.

Therefore, by the convolution method,

$$\sum_{n\le x} (-1)^{n-1} f(n) = \sum_{d\le x} h_f(d) \sum_{j\le x/d} f(j), \tag{20}$$

which leads to a good estimate for (2) if an asymptotic formula for $\sum_{n\le x} f(n)$ is known and if the coefficients b_ν of above can be well estimated. Note that, according to (19) and (9),

$$\sum_{n=1}^{\infty} \frac{h_f(n)}{n^s} = \frac{2}{S_f(1/2^s)} - 1, \tag{21}$$

provided that both $S_f(1/2^s)$ and $\overline{S}_f(1/2^s)$ converge. By differentiating,

$$\sum_{n=1}^{\infty} \frac{h_f(n)\log n}{n^s} = -\frac{\log 2}{2^{s-1}} \cdot \frac{S_f'(1/2^s)}{S_f(1/2^s)^2}, \tag{22}$$

assuming that $|1/2^s| < \min(r_f, \overline{r}_f)$. Identities (21) and (22) will be used in applications.

2.4 Two general asymptotic formulas

We prove two general results that will be applied for several special functions in Section 4.

Proposition 6. *Let f be a multiplicative function. Assume that*

(i) there exists a constant C_f such that

$$\sum_{n\le x} f(n) = C_f x^2 + O\left(xR_f(x)\right),$$

where $1 \ll R_f(x) = o(x)$ as $x \to \infty$, and $R_f(x)$ is nondecreasing;

(ii) $S_f(1/4)$ converges;

(iii) the sequence $(b_\nu)_{\nu\ge 0}$ of coefficients of the reciprocal power series $\overline{S}_f(x)$ is bounded.

Then

$$\sum_{n\le x}(-1)^{n-1} f(n) = C_f\left(\frac{2}{S_f(1/4)} - 1\right)x^2 + O\left(xR_f(x)\right).$$

Proof. According to (20),

$$\begin{aligned}\sum_{n\le x}(-1)^{n-1} f(n) &= \sum_{d\le x} h_f(d)\left(C_f\frac{x^2}{d^2} + O\left(\frac{x}{d}R_f(x/d)\right)\right)\\ &= C_f x^2 \sum_{d\le x}\frac{h_f(d)}{d^2} + O\left(xR_f(x)\sum_{d\le x}\frac{|h_f(d)|}{d}\right).\end{aligned}$$

Since the sequence $(b_\nu)_{\nu\ge 0}$ is bounded, the function h_f is bounded. Moreover, the sum

$$\sum_{d\le x}\frac{|h_f(d)|}{d} = \sum_{d=2^\nu\le x}\frac{|h_f(2^\nu)|}{2^\nu} \ll \sum_{2^\nu\le x}\frac{|b_\nu|}{2^\nu}$$

is bounded, as well. Note that $S_f(1/4)$ and $\overline{S}_f(1/4)$ both converge by conditions (ii) and (iii). We deduce, by using (21) for $s=2$, that

$$\begin{aligned}\sum_{n\le x}(-1)^{n-1} f(n) &= C_f x^2\sum_{d=1}^{\infty}\frac{h_f(d)}{d^2} + O\left(x^2\sum_{d>x}\frac{1}{d^2}\right) + O\left(xR_f(x)\right)\\ &= C_f x^2\left(\frac{2}{S_f(1/4)} - 1\right) + O\left(xR_f(x)\right).\end{aligned}$$

□

Proposition 7. *Let f be a nonvanishing multiplicative function. Assume that*

(i) there exist constants D_f and E_f such that

$$\sum_{n \le x} \frac{1}{f(n)} = D_f(\log x + E_f) + O\left(x^{-1} R_{1/f}(x)\right), \tag{23}$$

where $1 \ll R_{1/f}(x) = o(x)$ as $x \to \infty$, and $R_{1/f}(x)$ is nondecreasing;

(ii) the radius of convergence of the series $S_{1/f}(x)$ is $r_{1/f} > 1$;

(iii) the coefficients b_ν of the reciprocal power series $\overline{S}_{1/f}(x)$ satisfy $b_\nu \ll M^\nu$ as $\nu \to \infty$, where $0 < M < 1$ is a real number.

Then

$$\sum_{n \le x} (-1)^{n-1} \frac{1}{f(n)} = D_f \left(\left(\frac{2}{S_{1/f}(1)} - 1 \right) (\log x + E_f) + 2(\log 2) \frac{S'_{1/f}(1)}{S_{1/f}(1)^2} \right) + O\left(T_{1/f}(x)\right), \tag{24}$$

where

$$T_{1/f}(x) = \begin{cases} x^{-1} R_{1/f}(x), & \text{if } 0 < M < \frac{1}{2}; \\ x^{-1} R_{1/f}(x) \log x, & \text{if } M = \frac{1}{2}; \\ x^{\log M / \log 2} \max(\log x, R_{1/f}(x)), & \text{if } \frac{1}{2} < M < 1. \end{cases} \tag{25}$$

Proof. According to (20) we deduce that

$$\begin{aligned} \sum_{n \le x} (-1)^{n-1} \frac{1}{f(n)} &= \sum_{d \le x} h_{1/f}(d) \sum_{j \le x/d} \frac{1}{f(j)} \\ &= \sum_{d \le x} h_{1/f}(d) \left(D_f \left(\log \frac{x}{d} + E_f \right) + O\left((x/d)^{-1} R_{1/f}(x/d) \right) \right) \\ &= D_f(\log x + E_f) \sum_{d \le x} h_{1/f}(d) - D_f \sum_{d \le x} h_{1/f}(d) \log d \\ &\quad + O\left(x^{-1} R_{1/f}(x) \sum_{d \le x} d |h_{1/f}(d)| \right). \end{aligned}$$

That is,

$$\sum_{n \le x} (-1)^{n-1} \frac{1}{f(n)} = D_f(\log x + E_f) \sum_{d=1}^{\infty} h_{1/f}(d) + O\left(\log x \sum_{d > x} |h_{1/f}(d)| \right)$$

$$-D_f \sum_{d=1}^{\infty} h_{1/f}(d) \log d + O\left(\sum_{d > x} |h_{1/f}(d)| \log d \right) + O\left(x^{-1} R_{1/f}(x) \sum_{d \le x} d |h_{1/f}(d)| \right). \tag{26}$$

Note that $\min(r_{1/f}, \overline{r}_{1/f}) > 1$ by conditions (ii) and (iii). By using (21) and (22) for $s = 0$,

$$\sum_{d=1}^{\infty} h_{1/f}(d) = \frac{2}{S_{1/f}(1)} - 1,$$
$$\sum_{d=1}^{\infty} h_{1/f}(d) \log d = -2(\log 2) \frac{S'_{1/f}(1)}{S_{1/f}(1)^2}.$$

Furthermore,

$$\sum_{d>x} |h_{1/f}(d)| = \sum_{d=2^\nu > x} |h_{1/f}(2^\nu)| \ll \sum_{2^\nu > x} |b_\nu| \ll \sum_{2^\nu > x} M^\nu \ll x^{\log M/\log 2},$$
$$\sum_{d>x} |h_{1/f}(d)| \log d \ll \sum_{2^\nu > x} \nu |b_\nu| \ll \sum_{2^\nu > x} \nu M^\nu \ll x^{\log M/\log 2} \log x,$$
$$\sum_{d \le x} d |h_{1/f}(d)| = \sum_{2^\nu \le x} 2^\nu |b_\nu| \ll \sum_{\nu \le \log x/\log 2} (2M)^\nu,$$

where the latter sum is bounded if $0 < M < 1/2$, it is $\ll \log x$ if $M = 1/2$, and is $\ll x^{1+\log M/\log 2}$ if $1/2 < M < 1$.

Inserting these into (26), the proof is complete. $\square$

3 Estimates on coefficients of reciprocal power series

As mentioned in Section 2.3, in order to deduce sharp error terms for alternating sums (2) we need good estimates for the coefficients b_ν of the power series $\overline{S}_f(x)$.

3.1 Theorem of Kaluza

In many (nontrivial) cases the next result can be used.

Lemma 8. *Let $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ be a power series such that $a_\nu > 0$ ($\nu \ge 0$) and the sequence $(a_\nu)_{\nu \ge 0}$ is log-convex, that is $a_\nu^2 \le a_{\nu-1} a_{\nu+1}$ ($\nu \ge 1$). Then for the coefficients b_ν of the (formal) reciprocal power series $\sum_{\nu=0}^{\infty} b_\nu x^\nu$ one has $b_0 = 1/a_0 > 0$ and*

$$-\frac{1}{a_0^2} a_\nu \le b_\nu \le 0 \quad \textit{for all } \nu \ge 1.$$

Proof. The property that $b_\nu \le 0$ for all $\nu \ge 1$ is the theorem of Kaluza [16, Satz 3]. See [6] for a short direct proof of it. Furthermore, we have

$$b_\nu = -\frac{1}{a_0^2} a_\nu - \frac{1}{a_0} \sum_{j=1}^{\nu-1} a_j b_{\nu-j} \ge -\frac{1}{a_0^2} a_\nu \quad (\nu \ge 1).$$

$\square$

For example, consider the sum-of-divisors function σ, where $\sigma(2^\nu) = 2^{\nu+1} - 1$ for every $\nu \ge 0$. The sequence $\left(\frac{1}{2^{\nu+1}-1}\right)_{\nu\ge 0}$ is log-convex. This property allows us to apply Lemma 8 to obtain the estimate of Theorem 23 for the alternating sum $\sum_{n\le x}(-1)^{n-1}\frac{1}{\sigma(n)}$.

3.2 Kendall's renewal theorem

Another related result is Kendall's renewal theorem. Disregarding the probabilistic context, it can be stated as follows. See Berenhaut, Allen, and Fraser [3, Thm. 1.1].

Lemma 9. *Let $\sum_{\nu=0}^\infty a_\nu x^\nu$ be a power series such that $(a_\nu)_{\nu\ge 0}$ is nonincreasing, $a_0 = 1$, $a_\nu \ge 0$ ($\nu \ge 1$) and $a_\nu \ll q^\nu$ as $\nu \to \infty$, where $0 < q < 1$ is a real number. Then there exists $0 < s < 1$, s real, such that for the coefficients b_ν of the reciprocal power series one has $b_\nu \ll s^\nu$ as $\nu \to \infty$.*

We deduce the next result:

Corollary 10. *Let f be a positive multiplicative function. Assume that*

(i) asymptotic formula (23) is valid with $1 \ll R_{1/f}(x) \ll x^\varepsilon$ as $x \to \infty$, for every $\varepsilon > 0$;

(ii) the sequence $(1/f(2^\nu))_{\nu\ge 0}$ is nonincreasing and $1/f(2^\nu) \ll q^\nu$ as $\nu \to \infty$, where $0 < q < 1$ is a real number.

Then the asymptotic formula (24) holds for $\sum_{n\le x}(-1)^{n-1}\frac{1}{f(n)}$, with error term

$$T_{1/f}(x) = x^{-u}\max(\log x, R_{1/f}(x)) \ll x^{-u_1}$$

for some $u, u_1 > 0$.

Proof. This is a direct consequence of Proposition 7 and Lemma 9, applied for $a_\nu = \frac{1}{f(2^\nu)}$. Note that the radius of convergence of the series $S_{1/f}(x)$ is > 1 by condition (ii). □

In the case of the sum-of-unitary-divisors function σ^* we have $a_\nu = \sigma^*(2^\nu) = 2^\nu + 1$ for every $\nu \ge 1$ and $a_0 = \sigma^*(1) = 1$. The sequence $\left(\frac{1}{a_\nu}\right)_{\nu\ge 0}$ is not log-convex. Lemma 8 cannot be used to estimate the alternating sum $\sum_{n\le x}(-1)^{n-1}\frac{1}{\sigma^*(n)}$. At the same time, Corollary 10, with $t = 2$ furnishes an asymptotic formula. See Section 4.9.

An explicit form of Lemma 9 (Kendall's theorem) was proved in [3, Thm. 1.2]. However, it is restricted to the values $0 < q < 0.32$, and cannot be applied for the above special case, where $q = 1/2$. To find the optimal value of s for pairs (A, q) such that $a_\nu \le Aq^\nu$ ($\nu \ge 1$), not satisfying assumptions of [3, Thm. 1.2] was formulated by Berenhaut, Abernathy, Fan, and Foley [2, Open question 5.4].

We prove a new explicit Kendall-type inequality, based on the following lemma.

Lemma 11. *Let $\sum_{\nu=0}^\infty a_\nu x^\nu$ be a power series such that $a_0 = 1$. Then for the coefficients b_ν of the reciprocal power series $\sum_{\nu=0}^\infty b_\nu x^\nu$ one has $b_0 = 1$ and*

$$b_\nu = \sum_{k=1}^{\nu}(-1)^k \sum_{\substack{j_1,\ldots,j_k\ge 1\\ j_1+\cdots+j_k=\nu}} a_{j_1}\cdots a_{j_k} \tag{27}$$

$$= \sum_{\substack{t_1,\dots,t_\nu \ge 0\\ t_1+2t_2+\cdots+\nu t_\nu=\nu}} (-1)^{t_1+\cdots+t_\nu} \binom{t_1+\cdots+t_\nu}{t_1,\dots,t_\nu} a_1^{t_1}\cdots a_\nu^{t_\nu} \tag{28}$$

for every $\nu \ge 1$, where $\binom{t_1+\cdots+t_\nu}{t_1,\dots,t_\nu}$ are the multinomial coefficients.

Here formula (28) is well known, and it has been recovered several times. See, e.g., [19, Lemma 4]. However, we were not able to find its equivalent version (27) in the literature. For the sake of completeness we present their proofs.

Proof. Using the geometric series formula $(1+x)^{-1} = \sum_{n=0}^\infty (-1)^n x^n$ and the multinomial theorem, we immediately have

$$\left(1+\sum_{\nu=1}^\infty a_\nu x^\nu\right)^{-1} = \sum_{t=0}^\infty (-1)^t \left(\sum_{\nu=1}^\infty a_\nu x^\nu\right)^t$$

$$= \sum_{\nu=0}^\infty x^\nu \sum_{\substack{t_1,\dots,t_\nu \ge 0\\ t_1+2t_2+\cdots+\nu t_\nu=\nu}} (-1)^{t_1+\cdots+t_\nu} \binom{t_1+\cdots+t_\nu}{t_1,\dots,t_\nu} a_1^{t_1}\cdots a_\nu^{t_\nu},$$

giving (28). Furthermore, fix $\nu \ge 1$. By grouping the terms in (28) according to the values $k = t_1+\cdots+t_\nu$, where $1 \le k \le \nu$, we have

$$b_\nu = \sum_{k=1}^\nu (-1)^k \sum_{\substack{t_1,\dots,t_\nu \ge 0\\ t_1+2t_2+\cdots+\nu t_\nu=\nu\\ t_1+\cdots+t_\nu=k}} \binom{t_1+\cdots+t_\nu}{t_1,\dots,t_\nu} a_1^{t_1}\cdots a_\nu^{t_\nu}.$$

Now, identity (27) follows if we show that

$$\sum_{\substack{j_1,\dots,j_k \ge 1\\ j_1+\cdots+j_k=\nu}} a_{j_1}\cdots a_{j_k} = \sum_{\substack{t_1,\dots,t_\nu \ge 0\\ t_1+2t_2+\cdots+\nu t_\nu=\nu\\ t_1+\cdots+t_\nu=k}} \binom{t_1+\cdots+t_\nu}{t_1,\dots,t_\nu} a_1^{t_1}\cdots a_\nu^{t_\nu}. \tag{29}$$

But (29) is immediate by starting with its left-hand side and denoting by $t_1,\dots,t_\nu$ the number of values $j_1,\dots,j_k$ which are equal to $1,\dots,\nu$, respectively. □

Proposition 12. *Assume that $\sum_{\nu=0}^\infty a_\nu x^\nu$ is a power series such that $a_0 = 1$ and $|a_\nu| \le Aq^\nu$ ($\nu \ge 1$) for some absolute constants $A, q > 0$. Then for the coefficients b_ν of the reciprocal power series one has*

$$|b_\nu| \le Aq^\nu (A+1)^{\nu-1} \quad (\nu \ge 1). \tag{30}$$

Proof. By identity (27) and the assumption $|a_\nu| \le Aq^\nu$ $(\nu \ge 1)$ we immediately have

$$|b_\nu| \le \sum_{k=1}^{\nu} A^k \sum_{\substack{j_1,\ldots,j_k \ge 1\\ j_1+\ldots+j_k=\nu}} q^{j_1+\cdots+j_k} = q^\nu \sum_{k=1}^{\nu} A^k \sum_{\substack{j_1,\ldots,j_k \ge 1\\ j_1+\ldots+j_k=\nu}} 1$$

$$= q^\nu \sum_{k=1}^{\nu} A^k \binom{\nu-1}{k-1} = Aq^\nu (A+1)^{\nu-1},$$

as asserted. □

Note that (30) is an explicit Kendall-type inequality provided that $q(A+1) < 1$, in particular if $q \le 1/2$ and $A < 1$.

Corollary 13. *Let f be a positive multiplicative function such that*

(i) asymptotic formula (23) is valid with $1 \ll R_{1/f}(x) = o(x)$ as $x \to \infty$;

(ii) $1/f(2^\nu) \le Aq^\nu$ $(\nu \ge 1)$, where $A, q > 0$ are fixed real constants satisfying $M := q(A+1) < 1$.

Then the asymptotic formula (24) holds for $\sum_{n\le x}(-1)^{n-1}\frac{1}{f(n)}$, with error term (25).

Proof. This follows from Propositions 7 and 12. Note that the radius of convergence of the series $S_{1/f}(x)$ is > 1 by condition (ii). □

We will apply Corollary 13 for the sum-of-bi-unitary-divisors function σ^{**}. See Section 4.13.

4 Results for classical functions

In this section, we investigate alternating sums for classical multiplicative functions. We refer to Apostol [1], Hildebrand [14], and McCarthy [18] for the basic properties of these functions. See Gould and Shonhiwa [12] for a list of Dirichlet series of special arithmetic functions.

4.1 Euler's totient function

First consider Euler's φ function, where $\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right)$ $(n \ge 1)$.

Proposition 14.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\varphi(n)}{n^s} = \frac{2^s-3}{2^s-1}\cdot\frac{\zeta(s-1)}{\zeta(s)} \quad (\Re s > 2). \tag{31}$$

Proof. This was explained in Section 2.1, formula (31) follows at once from (13). □

Theorem 15.

$$\sum_{n\le x}(-1)^{n-1}\varphi(n) = \frac{1}{\pi^2}x^2 + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right). \tag{32}$$

Proof. Apply Proposition 6 for $f = \varphi$. It is known that

$$\sum_{n\le x}\varphi(n) = \frac{3}{\pi^2}x^2 + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right),$$

which is the best error term known to date, due to Walfisz [39, Satz 1, p. 144]. Furthermore,

$$S_\varphi(x) = \sum_{\nu=0}^{\infty}\varphi(2^\nu)x^\nu = 1 + \sum_{\nu=1}^{\infty}2^{\nu-1}x^\nu = \frac{1-x}{1-2x} \quad \left(|x| < \frac{1}{2}\right)$$

(also see (10)). We obtain that the reciprocal power series is

$$\overline{S}_\varphi(x) = \frac{1-2x}{1-x} = 1 - \sum_{\nu=1}^{\infty}x^\nu \quad (|x|<1),$$

for which the coefficients are $b_0 = 1$, $b_\nu = -1$ ($\nu \ge 1$), forming a bounded sequence. The coefficient of the main term in (32) is

$$C_\varphi\left(\frac{2}{S_\varphi(1/4)} - 1\right) = \frac{3}{\pi^2}\cdot\frac{1}{3} = \frac{1}{\pi^2}.$$

□

Remark 16. To find the corresponding constant to be multiplied by $C_\varphi = 3/\pi^2$ observe that by (19), (21) and (31),

$$\frac{2}{S_\varphi(1/4)} - 1 = \left[\frac{2^s-3}{2^s-1}\right]_{s=2} = \frac{1}{3},$$

and similarly for other classical multiplicative functions, if we have the representation of their alternating Dirichlet series.

Theorem 17.

$$\sum_{n\le x}(-1)^{n-1}\frac{1}{\varphi(n)} = -\frac{A}{3}\left(\log x + \gamma - B - \frac{8}{3}\log 2\right) + O\left(x^{-1}(\log x)^{5/3}\right), \tag{33}$$

where γ is Euler's constant and

$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{315\zeta(3)}{2\pi^4}, \qquad B = \sum_{p\in\mathbb{P}}\frac{\log p}{p^2-p+1}. \tag{34}$$

The result (33) improves the error term $O\left(x^{-1}(\log x)^{3}\right)$ obtained by Bordellès and Cloitre [4, Cor. 4, (i)].

Proof. Apply Proposition 7 for $f=\varphi$. The asymptotic formula

$$\sum_{n\le x}\frac{1}{\varphi(n)}=A\left(\log x+\gamma-B\right)+O\left(x^{-1}(\log x)^{2/3}\right)$$

with constants A and B defined by (34) and with the weaker error term $O\left(x^{-1}\log x\right)$ goes back to the work of Landau. See [9, Thm. 1.1]. The error term above was obtained by Sitaramachandrarao [25].

Now

$$S_{1/\varphi}(x)=\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\varphi(2^{\nu})}=1+\sum_{\nu=1}^{\infty}\frac{x^{\nu}}{2^{\nu-1}}=\frac{2+x}{2-x}\quad(|x|<2),$$

$$\overline{S}_{1/\varphi}(x)=\frac{1-x/2}{1+x/2}=1+\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{x^{\nu}}{2^{\nu-1}}\quad(|x|<2);$$

hence $b_{\nu}\ll 2^{-\nu}$ and choose $M=1/2$. Using that $S_{1/\varphi}(1)=3$ and $S'_{1/\varphi}(1)=4$, the proof is complete. □

4.2 Dedekind function

The Dedekind function ψ is given by $\psi(n)=n\prod_{p\mid n}\left(1+\frac{1}{p}\right)$ $(n\ge 1)$.

Proposition 18.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\psi(n)}{n^{s}}=\frac{2^{s}-5}{2^{s}+1}\cdot\frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}\quad(\Re s>2),\tag{35}$$

Proof. It is well-known that

$$\sum_{n=1}^{\infty}\frac{\psi(n)}{n^{s}}=\frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}\quad(\Re s>2),$$

and (35) follows like (31), by using Proposition 1. □

Theorem 19.

$$\sum_{n\le x}(-1)^{n-1}\psi(n)=-\frac{3}{2\pi^{2}}x^{2}+O\left(x(\log x)^{2/3}\right),\tag{36}$$

Proof. Apply Proposition 6 for $f = \psi$. It is known that

$$\sum_{n \le x} \psi(n) = \frac{15}{2\pi^2} x^2 + O\left(x(\log x)^{2/3}\right),$$

the best estimate up to now. See Walfisz [39, Satz 2, p. 100]. Here

$$S_\psi(x) = \sum_{\nu=0}^{\infty} \psi(2^\nu) x^\nu = 1 + 3 \sum_{\nu=1}^{\infty} 2^{\nu-1} x^n = \frac{1+x}{1-2x} \quad \left(|x| < \frac{1}{2}\right).$$

We obtain that the reciprocal power series is

$$\overline{S}_\psi(x) = \frac{1-2x}{1+x} = 1 + 3 \sum_{\nu=1}^{\infty} (-1)^\nu x^\nu \quad (|x| < 1),$$

for which the coefficients are $b_0 = 1$, $b_\nu = 3(-1)^\nu$ $(\nu \ge 1)$, forming a bounded sequence. The coefficient of the main term in (36) is

$$C_\psi \left(\frac{2}{S_\psi(1/4)} - 1 \right) = \frac{15}{2\pi^2} \cdot (-\frac{1}{5}) = -\frac{3}{2\pi^2}.$$

□

Theorem 20.

$$\sum_{n \le x} (-1)^{n-1} \frac{1}{\psi(n)} = \frac{C}{5} \left(\log x + \gamma + D + \frac{24}{5} \log 2 \right) + O\left(x^{-1} (\log x)^{2/3} (\log\log x)^{4/3}\right), \quad (37)$$

where γ is Euler's constant and

$$C = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p(p+1)}\right), \qquad D = \sum_{p \in \mathbb{P}} \frac{\log p}{p^2 + p - 1}. \quad (38)$$

The result (37) improves the error term $O\left(x^{-1}(\log x)^2\right)$ obtained by Bordellès and Cloitre [4, Cor. 4, (iii)]. The constant $C \doteq 0.704442$ is sometimes called the carefree constant, and its digits form the sequence A065463 in Sloane's Online Encyclopedia of Integer Sequences (OEIS) [31]. Also see Finch [11, Sect. 2.5.1].

Proof. Apply Proposition 7 for $f = \psi$. The asymptotic formula

$$\sum_{n \le x} \frac{1}{\psi(n)} = C\left(\log x + \gamma + D\right) + O\left(x^{-1} (\log x)^{2/3} (\log\log x)^{4/3}\right),$$

where C and D are the constants given by (38), is due to Sita Ramaiah and Suryanarayana [29, Cor. 4.2].

Furthermore,

$$S_{1/\psi}(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\psi(2^\nu)} = 1 + \frac{2}{3} \sum_{\nu=1}^{\infty} \frac{x^\nu}{2^\nu} = \frac{6-x}{3(2-x)} \quad (|x| < 2),$$

$$\overline{S}_{1/\psi}(x) = \frac{1-x/2}{1-x/6} = 1 - 2\sum_{\nu=1}^{\infty} \frac{x^\nu}{6^\nu} \quad (|x| < 6),$$

which shows that

$$b_\nu = -\frac{2}{6^\nu} \quad (\nu \geq 1).$$

Hence $b_\nu \ll 6^{-\nu}$ and choose $M = 1/6$. Using that $S_{1/\psi}(1) = \frac{5}{3}$ and $S'_{1/\psi}(1) = \frac{4}{3}$, we deduce (37). □

4.3 Sum-of-divisors function

Consider the function $\sigma(n) = \sum_{d \mid n} d$ $(n \geq 1)$.

Proposition 21.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sigma(n)}{n^s} = \left(1 - \frac{6}{2^s} + \frac{4}{2^{2s}}\right) \zeta(s)\zeta(s-1) \quad (\Re s > 2). \tag{39}$$

Note that $(-1)^{n-1}\sigma(n)$ is sequence A143348 in the OEIS [31], where identity (39) is given.

Proof. Use the familiar formula

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1) \quad (\Re s > 2)$$

and Proposition 1. □

Theorem 22.

$$\sum_{n \leq x} (-1)^{n-1} \sigma(n) = -\frac{\pi^2}{48} x^2 + O\left(x (\log x)^{2/3}\right). \tag{40}$$

Proof. Apply Proposition 6 for $f = \sigma$. It is known that

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O\left(x(\log x)^{2/3}\right),$$

the best up to now. See Walfisz [39, Satz 4, p. 99]. Here

$$S_\sigma(x) = \sum_{\nu=0}^{\infty} \sigma(2^\nu) x^\nu = \sum_{\nu=0}^{\infty} (2^{\nu+1} - 1) x^n = \frac{1}{(1-x)(1-2x)} \quad \left(|x| < \frac{1}{2}\right).$$

Hence

$$\overline{S}_\sigma(x) = (1-x)(1-2x) = 1 - 3x + 2x^2,$$

for which the coefficients are $b_0 = 1$, $b_1 = -3$, $b_2 = 2$, $b_\nu = 0$ ($\nu \geq 3$). The coefficient of the main term in (40) is from (39),

$$\frac{\pi^2}{12}\left[1 - \frac{6}{2^s} + \frac{4}{2^{2s}}\right]_{s=2} = -\frac{\pi^2}{48}.$$

□

The following asymptotic formula is due to Sita Ramaiah and Suryanarayana [29, Cor. 4.1]:

$$\sum_{n\leq x} \frac{1}{\sigma(n)} = E\left(\log x + \gamma + F\right) + O\left(x^{-1}(\log x)^{2/3}(\log\log x)^{4/3}\right), \tag{41}$$

where γ is Euler's constant,

$$E = \prod_{p\in\mathbb{P}} \alpha(p), \qquad F = \sum_{p\in\mathbb{P}} \frac{(p-1)^2\beta(p)\log p}{p\alpha(p)},$$

$$\alpha(p) = \left(1 - \frac{1}{p}\right)\sum_{\nu=0}^{\infty}\frac{1}{\sigma(p^\nu)} = 1 - \frac{(p-1)^2}{p}\sum_{j=1}^{\infty}\frac{1}{(p^j-1)(p^{j+1}-1)},$$

$$\beta(p) = \sum_{j=1}^{\infty}\frac{j}{(p^j-1)(p^{j+1}-1)}.$$

We prove the next result:

Theorem 23.

$$\sum_{n\leq x}(-1)^{n-1}\frac{1}{\sigma(n)} = E\left(\left(\frac{2}{K}-1\right)(\log x + \gamma + F) + 2(\log 2)\frac{K'}{K^2}\right) \tag{42}$$

$$+O\left(x^{-1}(\log x)^{5/3}(\log\log x)^{4/3}\right),$$

where

$$K = \sum_{j=0}^{\infty}\frac{1}{2^{j+1}-1}, \qquad K' = \sum_{j=1}^{\infty}\frac{j}{2^{j+1}-1}. \tag{43}$$

The result (42) improves the error term $O\left(x^{-1}(\log x)^4\right)$ obtained by Bordellès and Cloitre [4, Cor. 4, (v)]. Here $K \doteq 1.606695$ is the Erdős-Borwein constant, known to be irrational. See sequence A065442 in the OEIS [31].

Proof. Apply Proposition 7 for $f = \sigma$, using formula (41). Now

$$S_{1/\sigma}(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\sigma(2^\nu)} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{2^{\nu+1}-1}$$

and $S_{1/\sigma}(1) = K$. Note that $S'_{1/\psi}(1) = K'$, given above.

The coefficients b_ν of the reciprocal power series are $b_0 = 1$, $b_1 = -\frac{1}{3}$, $b_2 = -\frac{2}{63}$, $b_3 = -\frac{8}{945}$, etc. Observe that the sequence $\left(\frac{1}{2^{\nu+1}-1}\right)_{\nu\geq 0}$ is log-convex. Therefore, according to Lemma 8,

$$-\frac{1}{2^{\nu+1}-1} \leq b_\nu \leq 0 \quad (\nu \geq 1),$$

which shows that $b_\nu \ll 2^{-\nu}$ and we can choose $M = 1/2$. □

4.4 Divisor function

Now consider another classical function, the divisor function $\tau(n) = \sum_{d|n} 1$ $(n \geq 1)$. Using the familiar formula

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta^2(s) \quad (\Re s > 1),$$

and Proposition 1 we deduce

Proposition 24.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\tau(n)}{n^s} = \left(1 - \frac{4}{2^s} + \frac{2}{2^{2s}}\right) \zeta^2(s) \quad (\Re s > 1).$$

By similar considerations we also have

Proposition 25.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\tau(n) n^s} = \left(\frac{1}{2^{s-1}} \left(\log\left(1 - \frac{1}{2^s}\right)\right)^{-1} + 1\right) \prod_{p\in\mathbb{P}} p^s \log\left(1 - \frac{1}{p^s}\right) \quad (\Re s > 1).$$

Theorem 26.

$$\sum_{n\leq x} (-1)^{n-1} \tau(n) = -\frac{1}{2} x \log x + \left(\frac{1}{2} - \gamma + \log 2\right) x + O\left(x^{\theta+\varepsilon}\right),$$

where θ is the best exponent in Dirichlet's divisor problem.

Proof. Proposition 6 cannot be applied. Using that

$$\sum_{n\leq x} \tau(n) = x \log x + (2\gamma - 1)x + O\left(x^{\theta+\varepsilon}\right)$$

the result follows by similar arguments. □

Note that the actual best result for θ is $\theta = 131/416 \doteq 0.314903$, due to Huxley [15].

Now we consider the following result, which goes back to the work of Ramanujan [21, Eq. (7)]. See Wilson [40, Sect. 3] for its proof:

$$\sum_{n\le x} \frac{1}{\tau(n)} = x \sum_{j=1}^{N} \frac{A_j}{(\log x)^{j-1/2}} + O\left(\frac{x}{(\log x)^{N+1/2}}\right),$$

valid for every real $x \ge 2$ and every fixed integer $N \ge 1$ where A_j $(1 \le j \le N)$ are computable constants,

$$A_1 = \frac{1}{\sqrt{\pi}} \prod_{p\in\mathbb{P}} \left(\sqrt{p^2-p}\ \log\left(\frac{p}{p-1}\right)\right).$$

We prove

Theorem 27.

$$\sum_{n\le x} (-1)^{n-1} \frac{1}{\tau(n)} = x \sum_{t=1}^{N} \frac{B_t}{(\log x)^{t-1/2}} + O\left(\frac{x}{(\log x)^{N+1/2}}\right),$$

valid for every real $x \ge 2$ and every fixed integer $N \ge 1$ where B_t $(1 \le t \le N)$ are computable constants. In particular,

$$B_1 = A_1 \left(\frac{1}{\log 2} - 1\right).$$

Proof. Now

$$S_{1/\tau}(x) = \sum_{\nu=0}^{\infty} \frac{1}{\tau(2^\nu)} x^\nu = \sum_{\nu=0}^{\infty} \frac{1}{\nu+1} x^\nu = -\frac{\log(1-x)}{x} \qquad (|x|<1)$$

and the reciprocal power series is

$$\overline{S}_{1/\tau}(x) = -\frac{x}{\log(1-x)} = \sum_{\nu=0}^{\infty} b_\nu x^\nu,$$

where $b_0 = 1$, $b_1 = -1/2$, $b_2 = -1/12$, $b_3 = -1/24$, etc. Note that the sequence $\left(\frac{1}{\nu+1}\right)_{\nu\ge 0}$ is log-convex. According to Lemma 8 (this example was considered by Kaluza [16]),

$$-\frac{1}{\nu+1} \le b_\nu \le 0 \qquad (\nu \ge 1).$$

This shows, using (20), that

$$(-1)^{n-1} \frac{1}{\tau(n)} = \sum_{dj=n} h_{1/\tau}(d) \frac{1}{\tau(j)} \qquad (n \ge 1),$$

where the function $h_{1/\tau}$ is multiplicative, $h_{1/\tau}(2^\nu) \ll \frac{1}{\nu}$ as $\nu \to \infty$ and $h_{1/\tau}(p^\nu) = 0$ for every prime $p > 2$ and $\nu \geq 1$.

Hence

$$T(x) := \sum_{n \leq x} (-1)^{n-1} \frac{1}{\tau(n)} = \sum_{d \leq x/2} h_{1/\tau}(d) \sum_{j \leq x/d} \frac{1}{\tau(j)} + \sum_{x/2 < d \leq x} h_{1/\tau}(d)$$

$$= \sum_{d \leq x/2} h_{1/\tau}(d) \left(\frac{x}{d} \sum_{j=1}^{N} \frac{A_j}{(\log(x/d))^{j-1/2}} + O\left(\frac{x/d}{(\log(x/d))^{N+1/2}} \right) \right) + \sum_{x/2 < d \leq x} h_{1/\tau}(d)$$

$$= x \sum_{j=1}^{N} \frac{A_j}{(\log x)^{j-1/2}} \sum_{d \leq x/2} \frac{h_{1/\tau}(d)}{d(1 - \frac{\log d}{\log x})^{j-1/2}} + O\left(\frac{x}{(\log x)^{N+1/2}} \sum_{d \leq x/2} \frac{|h_{1/\tau}(d)|}{d(1 - \frac{\log d}{\log x})^{N+1/2}} \right)$$
$$+ \sum_{x/2 < d \leq x} h_{1/\tau}(d).$$

Here the last term is small:

$$\sum_{x/2 < d \leq x} h_{1/\tau}(d) \ll \sum_{d = 2^\nu \leq x} |h_{1/\tau}(2^\nu)| \ll \sum_{\nu \leq \log x / \log 2} \frac{1}{\nu} \ll \log \log x.$$

Using the power series expansion

$$(1+x)^t = \sum_{j=0}^{\infty} \binom{t}{j} x^j \qquad (x, t \in \mathbb{R}, |x| < 1),$$

we deduce

$$\sum_{d \leq x/2} \frac{|h_{1/\tau}(d)|}{d(1 - \frac{\log d}{\log x})^{N+1/2}} = \sum_{d \leq x/2} \frac{|h_{1/\tau}(d)|}{d} \left(1 + O\left(\frac{\log d}{\log x} \right) \right)$$
$$= \sum_{d = 2^\nu \leq x/2} \frac{|h_{1/\tau}(2^\nu)|}{2^\nu} + O\left(\frac{1}{\log x} \sum_{d = 2^\nu \leq x/2} \frac{|h_{1/\tau}(2^\nu)|}{2^\nu} \log 2^\nu \right)$$
$$\ll \sum_{2^\nu \leq x/2} \frac{1}{\nu 2^\nu} + \frac{1}{\log x} \sum_{2^\nu \leq x/2} \frac{1}{2^\nu} \ll 1.$$

Therefore, the remainder term of above is

$$O\left(\frac{x}{(\log x)^{N+1/2}} \right).$$

Furthermore,

$$\begin{aligned}
\sum_{d\le x/2} \frac{h_{1/\tau}(d)}{d(1-\frac{\log d}{\log x})^{j-1/2}} &= \sum_{d\le x/2} \frac{h_{1/\tau}(d)}{d} \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\left(\frac{\log d}{\log x}\right)^{\ell} \\
&= \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\frac{1}{(\log x)^{\ell}} \sum_{d\le x/2}\frac{h_{1/\tau}(d)}{d}(\log d)^{\ell} \\
&= \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\frac{1}{(\log x)^{\ell}}\left(K_{\ell}+O\left(\frac{(\log x)^{\ell-1}}{x}\right)\right),
\end{aligned}$$

where for every $\ell \ge 0$ the series

$$K_{\ell} := \sum_{d=1}^{\infty} \frac{h_{1/\tau}(d)}{d}(\log d)^{\ell} = \sum_{\substack{d=2^{\nu}\\ \nu\ge 0}} \frac{h_{1/\tau}(2^{\nu})}{2^{\nu}}(\log 2^{\nu})^{\ell}$$

is absolutely convergent, since $|h_{1/\tau}(2^{\nu})| \ll \frac{1}{\nu}$, and

$$\sum_{d>x/2} \frac{|h_{1/\tau}(d)|}{d}(\log d)^{\ell} = \sum_{d=2^{\nu}>x/2} \frac{|h_{1/\tau}(2^{\nu})|}{2^{\nu}}(\log 2^{\nu})^{\ell} \ll \sum_{\nu>\log x/\log 2} \frac{\nu^{\ell-1}}{2^{\nu}} \ll \frac{(\log x)^{\ell-1}}{x}.$$

We deduce that

$$\begin{aligned}
T(x) = x\sum_{j=1}^{N} \frac{A_j}{(\log x)^{j-1/2}} \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-j+1/2}{\ell}\frac{1}{(\log x)^{\ell}}\left(K_{\ell}+O\left(\frac{(\log x)^{\ell-1}}{x}\right)\right) \\
+O\left(\frac{x}{(\log x)^{N+1/2}}\right) \\
= x\sum_{t=1}^{N} \frac{1}{(\log x)^{t-1/2}} \sum_{j=1}^{N}(-1)^{t-j}\binom{-j+1/2}{t-j}A_j K_{t-j} + O\left(\frac{x}{(\log x)^{N+1/2}}\right).
\end{aligned}$$

The proof is complete by denoting

$$B_t = \sum_{j=1}^{N}(-1)^{t-j}\binom{-j+1/2}{t-j}A_j K_{t-j},$$

where $B_1 = A_1 K_0 = A_1(\frac{1}{\log 2}-1)$ by (21) (applied for $s=1$). $\square$

Note that a similar asymptotic formula can be deduced for the alternating sum

$$\sum_{n\le x}(-1)^{n-1}\frac{1}{\tau_k(n)},$$

where $\tau_k(n)$ is the Piltz divisor function, based on the result for $\sum_{n\le x}\frac{1}{\tau_k(n)}$, due to De Koninck and Ivić [9, Thm. 1.2].

4.5 Gcd-sum function

Let $P(n) = \sum_{k=1}^{n} \gcd(k,n)$ be the gcd-sum function. Known results include the following: P is multiplicative, $P(p^\nu) = p^{\nu-1}(p(\nu+1)-\nu)$ $(\nu \geq 1)$,

$$\sum_{n=1}^{\infty} \frac{P(n)}{n^s} = \frac{\zeta^2(s-1)}{\zeta(s)} \quad (\Re s > 2),$$

$$\sum_{n\leq x} P(n) = \frac{3}{\pi^2} x^2 \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O\left(x^{1+\theta+\varepsilon}\right),$$

where θ is the best exponent in Dirichlet's divisor problem. See the survey of the author [37]. We have

Proposition 28.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{P(n)}{n^s} = \left(2\left(1 - \frac{1}{2^{s-1}}\right)^2 \left(1 - \frac{1}{2^s}\right)^{-1} - 1 \right) \frac{\zeta^2(s-1)}{\zeta(s)} \quad (\Re s > 2).$$

Theorem 29.

$$\sum_{n\leq x} (-1)^{n-1} P(n) = -\frac{1}{\pi^2} x^2 \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} - \frac{10}{3}\log 2 \right) + O\left(x^{1+\theta+\varepsilon}\right),$$

where θ is the best exponent in Dirichlet's divisor problem.

Proof. Similar to the proofs of above. Here $h_P(2) = -6$, $h_P(2^\nu) = 2$ $(\nu \geq 2)$. □

The next formula was proved by Chen and Zhai [7, Thm. 4], sharpening a result of the author [37, Thm. 6]:

$$\sum_{n\leq x} \frac{1}{P(n)} = \sum_{j=0}^{N} \frac{K_j}{(\log x)^{j-1/2}} + O\left(\frac{1}{(\log x)^{N+1/2}} \right),$$

valid for every real $x \geq 2$ and every fixed integer $N \geq 1$ where K_j $(1 \leq j \leq N)$ are computable constants,

$$K_0 = \frac{2}{\sqrt{\pi}} \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p}\right)^{1/2} \sum_{\nu=0}^{\infty} \frac{1}{P(p^\nu)}.$$

We have

Theorem 30.

$$\sum_{n\leq x} (-1)^{n-1} \frac{1}{P(n)} = \sum_{t=0}^{N} \frac{D_t}{(\log x)^{t-1/2}} + O\left(\frac{1}{(\log x)^{N+1/2}} \right),$$

valid for every real $x \geq 2$ and every fixed integer $N \geq 1$ where D_t $(0 \leq t \leq N)$ are computable constants. In particular,

$$D_0 = K_0 \left(\frac{1}{2(\log 2 - 1)} - 1 \right).$$

Proof. Similar to the proof of Theorem 27. Here $P(2^\nu) = (\nu+2)2^{\nu-1}$ $(\nu \geq 1)$. The crucial fact is that the sequence $\left(\frac{1}{(\nu+2)2^{\nu-1}}\right)_{\nu \geq 0}$ is log-convex, and therefore Lemma 8 can be used to deduce that $h_{1/P}(2^\nu) \ll \frac{1}{\nu 2^\nu}$. □

4.6 Squarefree kernel

Now we move to the function $\kappa(n) = \prod_{p|n} p$ $(n \geq 1)$, the squarefree kernel of n (radical of n). It is known that

$$\sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} = \zeta(s) \prod_{p \in \mathbb{P}} \left(1 + \frac{p-1}{p^s}\right) \quad (\Re s > 2)$$

and for $x \geq 3$,

$$\sum_{n \leq x} \kappa(n) = \frac{C}{2} x^2 + O\left(R_\kappa(x)\right),$$

where C is the constant defined by (38), $R_\kappa(x) = x^{3/2}\delta(x)$ unconditionally and $R_\kappa(x) = x^{7/5}\omega(x)$ assuming the Riemann hypothesis (RH), with

$$\delta(x) = \exp\left(-c_1(\log x)^{3/5}(\log\log x)^{-1/5}\right), \tag{44}$$

$$\omega(x) = \exp\left(c_2(\log x)(\log\log x)^{-1}\right), \tag{45}$$

c_1, c_2 being positive constants. These estimates are due (for a more general function) to Suryanarayana and Subrahmanyam [34, Cor. 4.3.5 and 4.4.5].

We have

Proposition 31.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\kappa(n)}{n^s} = \frac{2^s - 3}{2^s + 1} \zeta(s) \prod_{p \in \mathbb{P}} \left(1 + \frac{p-1}{p^s}\right) \quad (\Re s > 2).$$

Theorem 32.

$$\sum_{n \leq x} (-1)^{n-1} \kappa(n) = \frac{C}{10} x^2 + O\left(R_\kappa(x)\right),$$

where C is given by (38) and $R_\kappa(x)$ is defined above.

Proof. Here, to deduce the unconditional result, Proposition 6 cannot be applied, since the function $\delta(x)$ is not increasing. However, $x^{\varepsilon}\delta(x)$ is increasing for any $\varepsilon > 0$ and we obtain by (20),

$$\begin{aligned}\sum_{n\le x}(-1)^{n-1}\kappa(n) &= \sum_{d\le x} h_\kappa(d)\left(\frac{C}{2}\left(\frac{x}{d}\right)^2 + O\left(\left(\frac{x}{d}\right)^{3/2}\delta(x/d)\right)\right) \\ &= \frac{Cx^2}{2}\sum_{d\le x}\frac{h_\kappa(d)}{d^2} + O\left(x^{\varepsilon}\delta(x)x^{3/2-\varepsilon}\sum_{d\le x}\frac{|h_\kappa(d)|}{d^{3/2-\varepsilon}}\right).\end{aligned}$$

Note that $h_\kappa(2^\nu) = 4(-1)^\nu$ ($\nu \ge 1$). Hence the function h_κ is bounded and the result is obtained by the usual arguments.

Assuming RH, Proposition 6 can directly be applied, since $\omega(x)$ is increasing. □

It is known that

$$K(x) := \sum_{n\le x}\frac{1}{\kappa(n)} = \exp\left((1+o(1))\left(\frac{8\log x}{\log\log x}\right)^{1/2}\right) \quad (x\to\infty),$$

due to de Bruijn [5], confirming a conjecture of Erdős. In fact,

$$K(x) \sim \frac{1}{2}e^{\gamma}F(\log x)(\log\log x) \quad (x\to\infty),$$

where γ is Euler's constant and

$$F(t) := \frac{6}{\pi^2}\sum_{m=1}^{\infty}\frac{\min(1,e^t/m)}{\prod_{p|m}(p+1)} \quad (t\ge 0),$$

which follows from a more precise asymptotic formula with error term, recently established by Robert and Tenenbaum [23, Thm. 4.3]. We point out that according to [23, Eq. (2.12)], there exists a sequence of polynomials $(Q_j)_{j\ge1}$ with $\deg Q_j \le j$ ($j\ge 1$) such that for any $N\ge 1$,

$$F(t) = \exp\left(\left(\frac{8t}{\log t}\right)^{1/2}\left(1+\sum_{j=1}^{N}\frac{Q_j(\log\log t)}{(\log t)^j} + O\left(\left(\frac{\log\log t}{\log t}\right)^{N+1}\right)\right)\right) \quad (t\ge 3).$$

Note that

$$S_{1/\kappa}(x) = \sum_{\nu=0}^{\infty}\frac{1}{\kappa(2^\nu)}x^\nu = \frac{2-x}{2(1-x)} \quad (|x|<1),$$

$$\overline{S}_{1/\kappa}(x) = \frac{2(1-x)}{2-x} = 1-\sum_{\nu=1}^{\infty}\frac{x^\nu}{2^\nu} \quad (|x|<2),$$

therefore $h_{1/\kappa}(2^\nu) = -\frac{1}{2^{\nu-1}}$ $(\nu \geq 1)$ and $\sum_{n=1}^\infty h_{1/\kappa}(n) = -1$. It follows that

$$K_{\text{altern}}(x) := \sum_{n \leq x} (-1)^{n-1} \frac{1}{\kappa(n)} = K(x) - 2 \sum_{2 \leq 2^\nu \leq x} \frac{1}{2^\nu} K\left(\frac{x}{2^\nu}\right). \tag{46}$$

Identity (46) and the deep analytic results of Robert and Tenenbaum [23] lead to the following:

Theorem 33. (Tenenbaum [36]) *One has*

$$K_{\text{altern}}(x) \sim -K(x) \quad (x \to \infty) \tag{47}$$

and a genuine asymptotic formula with effective remainder term may be derived for $K_{\text{altern}}(x)$.

4.7 Squarefree numbers

Now consider the squarefree numbers for which the characteristic function is μ^2, where μ is the Möbius function. It is well-known that

$$\sum_{n=1}^\infty \frac{\mu^2(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \quad (\Re s > 1)$$

and

$$\sum_{n \leq x} \mu^2(n) = \frac{6}{\pi^2} x + O\left(R_{\mu^2}(x)\right),$$

where $R_{\mu^2}(x) = x^{1/2}\delta(x)$, with $\delta(x)$ defined by (44), unconditionally, due to Walfisz [39, Satz 1, p. 192], and $R_{\mu^2}(x) = x^{11/35+\varepsilon}$ $(\varepsilon > 0)$ assuming RH, due very recently to Liu [17].

Proposition 34.

$$\sum_{n=1}^\infty (-1)^{n-1} \frac{\mu^2(n)}{n^s} = \frac{2^s - 1}{2^s + 1} \cdot \frac{\zeta(s)}{\zeta(2s)} \quad (\Re s > 1).$$

Theorem 35.

$$\sum_{n \leq x} (-1)^{n-1} \mu^2(n) = \frac{2}{\pi^2} x + O\left(R_{\mu^2}(x)\right).$$

Proof. Similar to the proof of Theorem 32. Note that here $h_{\mu^2}(2^\nu) = 2(-1)^\nu$ $(\nu \geq 1)$. Hence the function h_κ is bounded. □

4.8 Number of abelian groups of a given order

Let $a(n)$ denote, as usual, the number of abelian groups of order n. This is another classical multiplicative function, investigated by several authors. It is known that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{k=1}^{\infty} \zeta(ks) \quad (\Re s > 1),$$

$$\sum_{n\le x} a(n) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + O\left(x^{1/4+\varepsilon}\right), \tag{48}$$

where

$$C_j = \prod_{\substack{k=1\\ k\ne j}}^{\infty} \zeta(k/j) \quad (j = 1,2,3),$$

this best error term to date due to Robert and Sargos [22].

We have

Proposition 36.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{a(n)}{n^s} = \left(2 \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^{ks}}\right) - 1 \right) \prod_{k=1}^{\infty} \zeta(ks) \quad (\Re s > 1).$$

Theorem 37.

$$\sum_{n\le x} (-1)^{n-1} a(n) = C_1 K_1 x + C_2 K_2 x^{1/2} + C_3 K_3 x^{1/3} + O\left(x^{1/4+\varepsilon}\right),$$

where

$$K_j = 2 \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^{k/j}}\right) - 1 \quad (j = 1,2,3).$$

Note that $K_1 \doteq -0.422423$, where the digits of $\prod_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right) \doteq 0.288788$ form the sequence A048651 in the OEIS [31].

Proof. We use the method described in Section 2.3. According to (20),

$$\sum_{n\le x} (-1)^{n-1} a(n) = \sum_{d\le x} h_a(d) \sum_{j\le x/d} a(j).$$

Remark that by Euler's pentagonal number theorem,

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{2^{ks}}\right) = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\frac{1}{2^{(3j-1)js/2}} + \frac{1}{2^{(3j+1)js/2}} \right) \quad (|2^s| > 1),$$

which shows that $h_a(2^\nu) \in \{-2, 0, 2\}$ for every $\nu \ge 1$ and $h_a(p^\nu) = 0$ for every prime $p > 2$ and every $\nu \ge 1$. Hence the function h_a is bounded. Now using (48), the proof can be carried out by the usual arguments. □

It is known that

$$\sum_{n\le x} \frac{1}{a(n)} = Dx + O\left(x^{1/2}(\log x)^{-1/2}\right),$$

where

$$D = \prod_{p\in\mathbb{P}} \left(1+\sum_{k=2}^{\infty}\left(\frac{1}{P(k)}-\frac{1}{P(k-1)}\right)\frac{1}{p^k}\right) \doteq 0.752015$$

(sequence A084911 in the OEIS [31]), due to De Koninck and Ivić [9, Thm. 1.3]. Here $P(k)$ denotes the number of unrestricted partitions of k (not to be confused with the gcd-sum function from Section 4.5, denoted also by P). See Nowak [20] for a more precise asymptotic formula.

It follows by (15) that the limit

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x}(-1)^{n-1}\frac{1}{a(n)} = D\left(2\left(1+\sum_{\nu=1}^{\infty}\frac{1}{P(\nu)2^{\nu}}\right)^{-1}-1\right)$$

exists.

To establish an asymptotic formula for

$$\sum_{n\le x}(-1)^{n-1}\frac{1}{a(n)} \tag{49}$$

we need to estimate the coefficients of the reciprocal of the power series $S_{1/a}(x) = 1 + \sum_{\nu=1}^{\infty}\frac{1}{P(\nu)}x^{\nu}$. Here Lemma 8 cannot be used, since the sequence $(a_\nu)_{\nu\ge 0}$ with $a_0 = 1$ and $a_\nu = \frac{1}{P(\nu)}$ ($\nu \ge 1$) is not log-convex. However, observe that DeSalvo and Pak [10, Thm. 1.1] recently proved that the sequence $(P(n))$ is log-concave for $n > 25$, that is, $(1/P(n))$ is log-convex for $n > 25$.

Open Problem 38. Estimate the alternating sum (49).

4.9 Sum-of-unitary-divisors function

Recall that d is said to be a unitary divisor of n if $d \mid n$ and $\gcd(d, n/d) = 1$. Let $\sigma^*(n)$ denote, as usual, the sum of unitary divisors of n. The function σ^* is multiplicative and $\sigma^*(p^\nu) = p^\nu + 1$ ($\nu \ge 1$). One has

$$\sum_{n=1}^{\infty}\frac{\sigma^*(n)}{n^s} = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s-1)} \quad (\Re s > 2).$$

Furthermore,

$$\sum_{n\le x}\sigma^*(n) = \frac{\pi^2}{12\zeta(3)}x^2 + O\left(x(\log x)^{5/3}\right),$$

established by Sitaramachandrarao and Suryanarayana [27, Eq. (1.4)].

Proposition 39.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sigma^*(n)}{n^s} = \left(1-\frac{6}{2^s}+\frac{6}{2^{2s}}\right)\left(1-\frac{2}{2^{2s}}\right)^{-1}\frac{\zeta(s)\zeta(s-1)}{\zeta(2s-1)} \quad (\Re s > 2). \tag{50}$$

Theorem 40.

$$\sum_{n\le x}(-1)^{n-1}\sigma^*(n) = -\frac{\pi^2}{84\zeta(3)}x^2 + O\left(x(\log x)^{5/3}\right).$$

Proof. Apply Proposition 6. The Dirichlet series representation (50) can be used, the function h_{σ^*} is bounded. □

It is known that

$$\sum_{n\le x}\frac{1}{\sigma^*(n)} = B^*\log x + D^* + O\left(x^{-1}(\log x)^{5/3}(\log\log x)^{4/3}\right),$$

obtained by Sita Ramaiah and Suryanarayana [30, p. 1352], where B^* and D^* are explicit constants. Here, according to Proposition 4,

$$B^* = \prod_{p\in\mathbb{P}}\left(1-\frac{1}{p}\right)\left(1+\sum_{\nu=1}^{\infty}\frac{1}{p^\nu+1}\right).$$

It follows from the same Proposition 4 that the limit

$$E^* := \lim_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}(-1)^{n-1}\frac{1}{\sigma^*(n)} = B^*\left(2\left(1+\sum_{\nu=1}^{\infty}\frac{1}{2^\nu+1}\right)^{-1}-1\right)$$

exists.

Moreover, by Corollary 10 we deduce that

$$\sum_{n\le x}(-1)^{n-1}\frac{1}{\sigma^*(n)} = E^*\log x + F^* + O\left(x^{-u}(\log x)^{5/3}(\log\log x)^{4/3}\right), \tag{51}$$

with an explicit constant F^* and some $u>0$. Bordellès and Cloitre [4, Cor. 4, (vi)] established that the error term of (51) is $O\left(x^{-1}(\log x)^4\right)$.

To use our method, we need a better estimate for the coefficients b_ν of the reciprocal of the power series

$$S_{1/\sigma^*}(x) = 1+\sum_{\nu=1}^{\infty}\frac{x^\nu}{2^\nu+1}.$$

Here $b_0=1$, $b_1=-\frac{1}{3}$, $b_2=-\frac{4}{45}$, $b_3=-\frac{2}{135}$, $b_4=\frac{32}{34425}$, etc.

Open Problem 41. We conjecture that $b_\nu \ll 1/2^\nu$ as $\nu\to\infty$. If this is true, then it follows from Proposition 7 that the error term in (51) can be improved into $O\left(x^{-1}(\log x)^{8/3}(\log\log x)^{4/3}\right)$.

We pose as an open problem to prove this.

4.10 Unitary Euler function

Let φ^* be the unitary analog of Euler's φ function. The function φ^* is multiplicative and $\varphi(p^\nu) = p^\nu - 1$ for every prime power p^ν ($\nu \ge 1$). One has

$$\sum_{n=1}^{\infty} \frac{\varphi^*(n)}{n^s} = \zeta(s)\zeta(s-1) \prod_p \left(1 - \frac{2}{p^s} + \frac{1}{p^{2s-1}}\right) \quad (\Re s > 2).$$

Furthermore,

$$\sum_{n \le x} \varphi^*(n) = \frac{C}{2} x^2 + O\left(x(\log x)^{5/3}(\log\log x)^{4/3}\right),$$

where C is defined by (38). See Sitaramachandrarao and Suryanarayana [27, Eq. (1.5)].

Proposition 42.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\varphi^*(n)}{n^s} = \left(1 - \frac{1}{2^{s-2}} + \frac{1}{2^{2s-1}}\right)\left(1 - \frac{1}{2^{s-1}} + \frac{1}{2^{2s-1}}\right)^{-1} \sum_{n=1}^{\infty} \frac{\varphi^*(n)}{n^s} \quad (\Re s > 2), \tag{52}$$

Theorem 43.

$$\sum_{n \le x} (-1)^{n-1} \varphi^*(n) = \frac{C}{10} x^2 + O\left(x(\log x)^{5/3}(\log\log x)^{4/3}\right),$$

where C is defined by (38).

Proof. Apply Proposition 6. The Dirichlet series representation (52) can be used. The function h_{φ^*} is bounded. □

It is known that

$$\sum_{n \le x} \frac{1}{\varphi^*(n)} = L^* \log x + M^* + O\left(x^{-1}(\log x)^{5/3}\right),$$

due to Sita Ramaiah and Subbarao [28, Thm. 3.1], improving the error term of Sita Ramaiah and Suryanarayana [30, Thm. 3.2], where L^* and M^* are explicit constants. Here, according to Proposition 4,

$$L^* = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\nu=1}^{\infty} \frac{1}{p^\nu - 1}\right).$$

It follows from the same Proposition 4 that the limit

$$T^* := \lim_{x \to \infty} \frac{1}{\log x} \sum_{n \le x} (-1)^{n-1} \frac{1}{\varphi^*(n)} = L^* \left(\frac{2}{1+K} - 1\right)$$

exists, where K is the Erdős-Borwein constant defined by (43). Moreover, by Corollary 10 (take $q = 1/2$) we deduce that

$$\sum_{n \le x} (-1)^{n-1} \frac{1}{\varphi^*(n)} = T^* \log x + U^* + O\left(x^{-u} (\log x)^{5/3}\right). \tag{53}$$

with an explicit constant U^* and some $u > 0$.

Note that this example was not considered by Bordellès and Cloitre [4]. To use our method one needs to consider the power series

$$S_{1/\varphi^*}(x) = 1 + \sum_{\nu=1}^{\infty} \frac{x^\nu}{2^\nu - 1},$$

where the sequence $a_0 = 1$, $a_\nu = \frac{1}{2^\nu - 1}$ is log-convex but only for $\nu \ge 1$, that is $a_\nu^2 \le a_{\nu-1} a_{\nu+1}$ holds for $\nu \ge 2$ and is false for $\nu = 1$. Hence Lemma 8 cannot be used. In fact, the coefficients b_ν of the reciprocal power series are $b_0 = 1$, $b_1 = -1$, $b_2 = \frac{2}{3}$, $b_3 = -\frac{10}{21}$, $b_4 = \frac{104}{315}$, etc. (not all of $b_1, b_2, \ldots$ are negative).

Open Problem 44. We conjecture that $b_\nu \ll 1/2^\nu$ as $\nu \to \infty$. If this is true, then it follows from Proposition 7 that the error term in (53) can be improved into $O\left(x^{-1} (\log x)^{8/3}\right)$.

We pose as an open problem to prove this.

4.11 Unitary squarefree kernel

Let $\kappa^*(n)$ denote the greatest squarefree unitary divisor of n. The function κ^* is multiplicative, $\kappa^*(p) = p$ and $\kappa^*(p^\nu) = 1$ for every prime p and $\nu \ge 2$. One has

$$\sum_{n \le x} \kappa^*(n) = \frac{1}{2} \widetilde{C} x^2 + O(R_{\kappa^*}(x)), \tag{54}$$

where

$$\widetilde{C} = \prod_{p \in \mathbb{P}} \left(1 - \frac{p^2 + p - 1}{p^3 (p+1)}\right), \tag{55}$$

$R_{\kappa^*}(x) = x^{3/2} \delta(x)$, with $\delta(x)$ defined by (44), unconditionally, and $R_{\kappa^*}(x) = x^{7/5} \omega(x)$, with $\omega(x)$ defined by (45), assuming RH, due to Sita Ramaiah and Suryanarayana [26, Thm. 5.7, 5.8].

Theorem 45. *With the notation above,*

$$\sum_{n \le x} (-1)^{n-1} \kappa^*(n) = \frac{5}{38} \widetilde{C} x^2 + O(R_{\kappa^*}(x)). \tag{56}$$

Proof. We have

$$S_{\kappa^*}(x) = \frac{x^2 - x - 1}{x - 1} \quad (|x| < 1)$$

and the proof is quite similar to the proof of Theorem 32. □

An asymptotic formula for the reciprocal of $\kappa^*(n)$ is simpler to obtain than for the reciprocal of the squarefree kernel $\kappa(n)$, discussed in Section 4.6. It is a result of Suryanarayana and Subrahmanyam [33, Cor. 3.4.1] that

$$\sum_{n \le x} \frac{1}{\kappa^*(n)} = \frac{A\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{B\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}), \tag{57}$$

where

$$A = \prod_{p \in \mathbb{P}} \left(1 + \frac{\sqrt{p} - 1}{p(p - \sqrt{p} + 1)}\right), \quad B = \prod_{p \in \mathbb{P}} \left(1 + \frac{p^{1/3} - 1}{p(p^{2/3} - p^{1/3} + 1)}\right),$$

We deduce the next result.

Theorem 46.

$$\sum_{n \le x} (-1)^{n-1} \frac{1}{\kappa^*(n)} = \frac{A^*\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{B^*\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}), \tag{58}$$

where

$$A^* = \frac{A(9 - 12\sqrt{2})}{23}, \quad B^* = \frac{B(2^{5/3} - 3 \cdot 2^{1/3} - 1)}{2^{5/3} - 2^{1/3} + 1}. \tag{59}$$

Proof. We have

$$S_{1/\kappa^*}(x) = \frac{x^2 - x + 2}{2(1 - x)} \quad (|x| < 1),$$

hence

$$\overline{S}_{1/\kappa^*}(x) = \frac{2(1 - x)}{x^2 - x + 2} = \sum_{\nu=0}^{\infty} b_\nu x^\nu \quad (|x| < \sqrt{2}),$$

where

$$b_\nu = \frac{1}{4^{\nu+1}\sqrt{7}} \operatorname{Re}\left((\sqrt{7} + i)(1 - i\sqrt{7})^{\nu+1} + (\sqrt{7} - i)(1 + i\sqrt{7})^{\nu+1}\right) \quad (\nu \ge 0).$$

Therefore,

$$|b_\nu| \le \frac{4}{\sqrt{7}} \cdot \frac{1}{2^{\nu/2}} \quad (\nu \ge 1)$$

and using our method this implies (58). □

4.12 Powerful part of a number

It is possible to deduce similar formulas for many other special multiplicative functions. We consider the following further example. Every positive integer n can be uniquely written as $n = ab$, where $\gcd(a, b) = 1$, a is squarefree and b is squareful. Here b is called the *powerful part* of n and is denoted by $\operatorname{pow}(n)$. See Cloutier, De Koninck, Doyon [8]. Note that

$$\operatorname{pow}(n) = \frac{n}{\kappa^*(n)} \quad (n \geq 1), \tag{60}$$

where $\kappa^*(n)$ is the unitary squarefree kernel of n, discussed in Section 4.11.

By partial summation we deduce from (57) that

$$\sum_{n \leq x} \operatorname{pow}(n) = \frac{1}{3} c_1 x^{3/2} + \frac{1}{4} c_2 x^{4/3} + O(x^{6/5}), \tag{61}$$

where

$$c_1 = \prod_{p \in \mathbb{P}} \left(1 + \frac{2}{p^{3/2}} - \frac{1}{p^{5/2}}\right),$$
$$c_2 = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^{2/3}} + \frac{2}{p^{4/3}} - \frac{1}{p^{7/3}}\right).$$

We remark that (61) is better than [8, Eq. (1)], where the error term is $O(x^{4/3})$.

Theorem 47.

$$\sum_{n \leq x} (-1)^{n-1} \operatorname{pow}(n) = \frac{A^* \zeta(3/2)}{3\zeta(3)} x^{3/2} + \frac{B^* \zeta(4/3)}{4\zeta(2)} x^{4/3} + O(x^{6/5}),$$

where the constants A^* and B^* are defined by (59).

Proof. Use formulas (60), (58) and partial summation. Alternatively, formula (61) and the method of the present paper can be applied. □

By partial summation again, we deduce from (60) and (54) that

$$\sum_{n \leq x} \frac{1}{\operatorname{pow}(n)} = \widetilde{C} x + O(R_{1/\operatorname{pow}}(x)) \tag{62}$$

where $\widetilde{C}$ is defined by (55), $R_{1/\operatorname{pow}}(x) = x^{1/2}\delta(x)$, with $\delta(x)$ defined by (44), unconditionally, and $R_{1/\operatorname{pow}}(x) = x^{2/5}\omega(x)$, with $\omega(x)$ defined by (45), assuming RH. Note that this error term is better than $O(x^{1/2})$, indicated in [8, Eq. (3)].

Theorem 48.

$$\sum_{n \leq x} (-1)^{n-1} \frac{1}{\operatorname{pow}(n)} = \frac{5}{19} \widetilde{C} x + O(R_{1/\operatorname{pow}}(x)),$$

with the notation above.

Proof. Apply formulas (60), (56) and partial summation. Alternatively, formula (62) and the method of the present paper can be used. □

4.13 Sum-of-bi-unitary-divisors function

Let $\sigma^{**}(n)$ denote, as usual, the sum of bi-unitary divisors of n. Recall that a divisor d of n is a bi-unitary divisor if the greatest common unitary divisor of d and n/d is 1. The function σ^{**} is multiplicative and for any prime power p^ν ($\nu \geq 1$),

$$\sigma^{**}(p^\nu) = \begin{cases} \sigma(p^\nu), & \text{if } \nu \text{ is odd;} \\ \sigma(p^\nu) - p^{\nu/2}, & \text{if } \nu \text{ is even.} \end{cases}$$

It is the result of Suryanarayana and Subbarao [32, Cor. 3.4.3] that

$$\sum_{n \leq x} \sigma^{**}(n) = \frac{1}{2} C^{**} x^2 + O(x(\log x)^3),$$

where

$$C^{**} = \zeta(2)\zeta(3) \prod_{p \in \mathbb{P}} \left(1 - \frac{2}{p^3} + \frac{1}{p^4} + \frac{1}{p^5} - \frac{1}{p^6}\right).$$

Theorem 49. *We have*

$$\sum_{n \leq x} (-1)^{n-1} \sigma^{**}(n) = -\frac{11}{106} C^{**} x^2 + O(x(\log x)^3).$$

Proof. Similar to the proof of (40), by applying Proposition 6 for $f = \sigma^{**}$. □

Sitaramaiah and Subbarao [28, Thm. 3.2] established that

$$\sum_{n \leq x} \frac{1}{\sigma^{**}(n)} = A^{**} \log x + B^{**} + O(x^{-1}(\log x)^{14/3}(\log\log x)^{4/3}),$$

where A^{**}, B^{**} are certain explicit constants.

Theorem 50. *We have*

$$\sum_{n \leq x} (-1)^{n-1} \frac{1}{\sigma^{**}(n)} = A_1^{**} \log x + B_1^{**} + O(x^c(\log x)^{14/3}(\log\log x)^{4/3}), \tag{63}$$

where A_1^{**}, B_1^{**} are explicit constants and $c = (\log 9/10)/(\log 2) \doteq -0.152003$.

Proof. Now Lemma 8 (theorem of Kaluza) cannot be used, since the sequence $\left(\frac{1}{\sigma^{**}(2^\nu)}\right)_{\nu \geq 0}$ is not log-convex. But it is easy to check that

$$\frac{1}{\sigma^{**}(2^\nu)} \leq \frac{4}{5} \cdot \frac{1}{2^\nu} \quad (\nu \geq 1),$$

hence Corollary 13 can be applied with $A = 4/5$, $q = 1/2$, where $M = q(A+1) = 9/10 < 1$. □

Open Problem 51. Improve the error term of (63).

4.14 Alternating sum-of-divisors function

Consider the function $\beta(n) = \sum_{d|n} d\,\lambda(n/d)$ $(n \geq 1)$, where λ is the Liouville function. The function β is multiplicative and $\beta(p^\nu) = p^\nu - p^{\nu-1} + p^{\nu-2} - \cdots + (-1)^\nu$ for every prime power p^ν $(\nu \geq 1)$. See the survey paper of the author [38].

Proposition 52.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\beta(n)}{n^s} = \left(1 - \frac{2}{2^s} - \frac{4}{2^{2s}}\right) \frac{\zeta(s-1)\zeta(2s)}{\zeta(s)} \quad (\Re s > 2). \tag{64}$$

Proof. Use Proposition 1 and the representation

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s)}{\zeta(s)} \quad (\Re s > 2).$$

□

Theorem 53.

$$\sum_{n \leq x} (-1)^{n-1} \beta(n) = \frac{\pi^2}{120} x^2 + O\left(x(\log x)^{2/3} (\log\log x)^{4/3}\right). \tag{65}$$

Proof. Apply Proposition 6 for $f = \beta$. It is known that

$$\sum_{n \leq x} \beta(n) = \frac{\pi^2}{30} x^2 + O\left(x(\log x)^{2/3} (\log\log x)^{4/3}\right),$$

see [38, Eq. (15)], which is a consequence of the result of Walfisz [39, Satz 4, p. 144] for Euler's φ function. The coefficient of the main term in (65) is from (64),

$$\frac{\pi^2}{30} \left[1 - \frac{2}{2^s} - \frac{4}{2^{2s}}\right]_{s=2} = \frac{\pi^2}{120}.$$

□

It is known ([38, Eq. (17)]) that for every $\varepsilon > 0$,

$$\sum_{n \leq x} \frac{1}{\beta(n)} = K_1 \log x + K_2 + O(x^{-1+\varepsilon}), \tag{66}$$

where K_1 and K_2 are constants. Since $(\beta(2^\nu))_{\nu \geq 0}$ is nondecreasing and $\beta(2^\nu) \geq 2^{\nu-1}$ $(\nu \geq 1)$, Corollary 10 can be applied (take $q = 1/2$). We deduce that

$$\sum_{n \leq x} (-1)^{n-1} \frac{1}{\beta(n)} = K_3 \log x + K_4 + O(x^{-u}) \tag{67}$$

with some constants K_3, K_4 and some $u > 0$.

Open Problem 54. Improve the error terms of (66) and (67).

4.15 Exponential divisor function

The exponential divisor function $\tau^{(e)}$ is multiplicative and $\tau^{(e)}(p^\nu) = \tau(\nu)$ for every prime power p^ν ($\nu \geq 1$), where τ is the classical divisor function. There are constants A_1 and A_2 such that

$$\sum_{n\leq x} \tau^{(e)}(n) = A_1 x + A_2 x^{1/2} + O(R_{\tau^{(e)}}(x)),$$

where

$$A_1 = \prod_{p\in\mathbb{P}} \left(1 + \sum_{\nu=2}^{\infty} \frac{\tau(\nu) - \tau(\nu-1)}{p^\nu}\right)$$

is the mean value of $\tau^{(e)}$ and $R_{\tau^{(e)}}(x) = x^{2/9}\log x$, as shown by Wu [41, Thm. 1]. This error term is strongly related to estimates on the divisor function $d(1,2;n) = \sum_{ab^2=n} 1$. It can be sharpened into $O(x^{1057/4785+\varepsilon})$ by using [13, Thm. 1]. Also see [41, Remark, p. 135].

It follows from Proposition 2 that the limit

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} (-1)^{n-1} \tau^{(e)}(n) = A_1 \left(\frac{2}{1+K} - 1\right)$$

exists, where $K = \sum_{\nu=1}^{\infty} \frac{\tau(\nu)}{2^\nu} = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu - 1}$ is the Erdős-Borwein constant, already quoted above.

Open Problem 55. Investigate the alternating sums

$$\sum_{n\leq x} (-1)^{n-1} \tau^{(e)}(n), \qquad \sum_{n\leq x} (-1)^{n-1} \frac{1}{\tau^{(e)}(n)}.$$

5 Generalized alternating sums

It is possible to investigate the following generalization of the alternating sums discussed above. Let Q be an arbitrary subset of the set of primes $\mathbb{P}$, let

$$t_Q(n) := \begin{cases} 1, & \text{if } q \nmid n \text{ for every } q \in Q; \\ -1, & \text{otherwise,} \end{cases}$$

and

$$D_Q(f,s) := \sum_{n=1}^{\infty} t_Q(n) \frac{f(n)}{n^s}. \tag{68}$$

If $Q = \{2\}$, then $t_{\{2\}}(n) = (-1)^{n-1}$ and (68) reduces to the alternating Dirichlet series (4). If $Q = \{2,3\}$, just to illustrate another special case, then we have

$$D_{\{2,3\}}(f,s) = \frac{f(1)}{1^s} - \frac{f(2)}{2^s} - \frac{f(3)}{3^s} - \frac{f(4)}{4^s} + \frac{f(5)}{5^s} - \frac{f(6)}{6^s} + \frac{f(7)}{7^s} + \cdots,$$

while the choice $Q=\emptyset$ gives the classical Dirichlet series (5).

Note that the function $n \mapsto t_Q(n)$ is multiplicative if and only if $Q=\{q\}$ having one element. Proof: If $Q=\{q\}$, then the function $t_{\{q\}}(n)$ is multiplicative. On the other hand, if there are distinct $q_1,q_2\in Q$, then $t_Q(q_1q_2)=-1\neq 1=(-1)(-1)=t_Q(q_1)t_Q(q_2)$. However, the function

$$c_Q(n):=\begin{cases}1, & \text{if } q\nmid n \text{ for every } q\in Q;\\ 0, & \text{otherwise};\end{cases}$$

is multiplicative for every $Q\subseteq\mathbb{P}$.

Proposition 56. *Let Q be an arbitrary subset of $\mathbb{P}$. If f is a multiplicative function, then*

$$D_Q(f,s)=D(f,s)\left(2\prod_{q\in Q}\left(\sum_{\nu=0}^{\infty}\frac{f(q^\nu)}{q^{\nu s}}\right)^{-1}-1\right),$$

and if f is completely multiplicative, then

$$D_Q(f,s)=\prod_{p\in\mathbb{P}}\left(1-\frac{f(p)}{p^s}\right)^{-1}\left(2\prod_{q\in Q}\left(1-\frac{f(q)}{q^s}\right)-1\right),$$

formally or in case of convergence.

Proof. We have

$$D_Q(f,s)=-\sum_{n=1}^{\infty}\frac{f(n)}{n^s}+2\sum_{n=1}^{\infty}c_Q(n)\frac{f(n)}{n^s}=-D(f,s)+2\prod_{p\notin Q}\sum_{\nu=0}^{\infty}\frac{f(p^\nu)}{p^{\nu s}}$$
$$=-D(f,s)+2D(f,s)\prod_{q\in Q}\left(\sum_{\nu=0}^{\infty}\frac{f(q^\nu)}{q^{\nu s}}\right)^{-1}=D(f,s)\left(2\prod_{q\in Q}\left(\sum_{\nu=0}^{\infty}\frac{f(q^\nu)}{q^{\nu s}}\right)^{-1}-1\right).$$

□

If $Q=\{2\}$, then Proposition 56 reduces to Proposition 1.

Some of the discussed asymptotic formulas can also be generalized to certain subsets $Q\subseteq\mathbb{P}$. For example, we have the next result.

Theorem 57. *Let Q be an arbitrary finite subset of $\mathbb{P}$. Then*

$$\sum_{n\le x}t_Q(n)\sigma(n)=\frac{\pi^2}{12}\left(2\prod_{p\in Q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)-1\right)x^2+O\left(x(\log x)^{2/3}\right). \tag{69}$$

Proof. We have

$$\sum_{n\le x} t_Q(n)\sigma(n) = -\sum_{n\le x}\sigma(n) + 2\sum_{n\le x} c_Q(n)\sigma(n),$$

where $c_Q(n)\sigma(n)$ is multiplicative and

$$\sum_{n=1}^{\infty} \frac{c_Q(n)\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)\prod_{p\in Q}\left(1-\frac{p+1}{p^s}+\frac{p}{p^{2s}}\right).$$

It turns out that

$$\sum_{n\le x} c_Q(n)\sigma(n) = \sum_{d\le x} h_Q(d)\sum_{j\le x/d}\sigma(j),$$

where the function h_Q is multiplicative and for every prime power p^ν ($\nu \ge 1$),

$$h_Q(p^\nu) = \begin{cases} -(p+1), & \text{if } p\in Q,\ \nu=1;\\ p, & \text{if } p\in Q,\ \nu=2;\\ 0, & \text{otherwise.}\end{cases}$$

Now the proof runs similar to the proof of (40). □

In the case $Q=\{2\}$ formula (69) reduces to (40).

Open Problem 58. Deduce asymptotic formulas for

$$\sum_{n\le x} t_Q(n)\sigma(n)$$

and for similar sums if Q is an arbitrary fixed subset of the primes.

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