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On the Primality of the Generalized Fuss-Catalan Numbers

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Abstract

In this note, we determine all primes among the Fuss-Catalan numbers the generalized Fuss-Catalan numbers, the Lobb numbers, and the ballot numbers.

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1 Introduction

It has long been known (see, for example, Koshy and Salmassi [9]) that the only primes among the Catalan numbers are $C_2 = 2$ and $C_3 = 5$. In this note we determine all the primes among the generalized Fuss-Catalan numbers, the Lobb numbers, and the ballot numbers. The definition of the generalized Fuss-Catalan numbers follows. (The definitions of the Lobb numbers and ballot numbers are given after Corollary 3 and Corollary 4, respectively.)

Let integers $m \geq 2$ and $n, k \geq 1$ be given. The *Catalan numbers* C_n (see A000108 in the *Online Encyclopedia of Integer Sequences* are defined by

$$C_n = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

The *Fuss-Catalan numbers* (see A002293, A002294, A002295, A002296) $F_m(n)$ are the numbers of the form

$$F_m(n) = \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n}. \tag{2}$$

The *generalized Fuss-Catalan numbers* $F_m(n,k)$ are the number of the form

$$F_m(n,k) := \frac{k}{mn+k}\binom{mn+k}{n} = \frac{k}{(m-1)n+k}\binom{mn+k-1}{n}. \tag{3}$$

The generalized Fuss-Catalan numbers are named after N. I. Fuss and E. C. Catalan (see [5, 6, 10, 12, 13]), and are sometimes called the k-fold Fuss-Catalan numbers or the Raney numbers. Note that $F_m(n,1) = F_m(n)$ and $F_2(n) = F_2(n,1) = C_n$.

The Fuss-Catalan numbers have several combinatorial applications; see, for example, [1, 3, 4, 6, 14, 15, 16]. The generating function $F_m(z)$ for the Fuss-Catalan numbers $\{F_m(n,1)\}_{n\geq 0}$, that is,

$$F_m(z) = \sum_{n\geq 0} F_m(n,1)z^n, \tag{4}$$

is called the *generalized binomial series* by Graham, Knuth, and Patashnik [6, p. 200]. Lambert's formula for the Taylor expansion of the powers of $F_m(z)$ (see [6, (5.60)]) is

$$(F_m(z))^k = \sum_{n\geq 0} \frac{k}{mn+k}\binom{mn+k}{n} z^n = \sum_{n\geq 0} F_m(n,k)z^n \tag{5}$$

for all integers $k \geq 1$. An ingenious argument in [6, p. 363] uses (5) to show that

$$F_m(z) = 1 + z(F_m(z))^m. \tag{6}$$

In this note, we are interested in the primality of the generalized Fuss-Catalan numbers, $F_m(n,k)$, defined by (3).

2 Main results

Here is our main result on the primality of the generalized Fuss-Catalan numbers.

Theorem 1. *Let the generalized Fuss-Catalan numbers $F_m(n,k)$ be defined by (3) with $m \geq 2$ and $n, k \geq 1$. Then $F_m(n,k)$ is not prime except in the following cases:*

(a) for $n \geq 3$, the only prime of the form $F_m(n,k)$ is $F_2(3,1) = 5$;

(b) for $n = 2$, $F_p(2,1) = p$, where p is prime, and $F_m(2,2) = 2m+1$ when $m = (p-1)/2$ and p is prime;

(c) for $n = 1$, $F_m(1,k) = k$, where k is prime.

Proof. We separate our proof into three cases as follows: (a) $n \geq 3$; (b) $n = 2$; and (c) $n = 1$.

First, we show that for $n \geq 3$, $F_m(n,k)$ is not prime except for $F_2(3,1) = 5$. Indeed,

$$\begin{aligned} F_m(n,k) &= \frac{k}{(m-1)n+k}\binom{mn+k-1}{n} \\ &= \frac{k}{(m-1)n+k}\frac{mn+k-1}{n}\frac{mn+k-2}{n-1}\cdots\frac{mn+k-(n-1)}{2}\frac{mn+k-n}{1} \\ &= k\frac{mn+k-1}{n}\frac{mn+k-2}{n-1}\cdots\frac{mn+k-(n-2)}{3}\frac{mn+k-(n-1)}{2}. \end{aligned} \tag{7}$$

Since $m \geq 2$, $n \geq 3$, and $k \geq 1$, we have

$$\frac{mn+k-1}{n}, \frac{mn+k-2}{n-1}, \ldots, \frac{mn+k-(n-3)}{4} > 1$$

and

$$\frac{mn+k-(n-2)}{3} \geq 2.$$

Noting $m \geq 2$, and combining the above estimates, we have

$$F_m(n,k) > k((m-1)n+k+1) \geq mn+k-1$$

for $k \geq 2$. Thus,

$$F_m(n,k) > mn+k-1 \tag{8}$$

for $m \geq 2$, $n \geq 3$ and $k \geq 2$. Note that every prime factor of $F_m(n,k)$ is less than or equal to $mn+k-1$ because each factor of $F_m(n,k)$ must be a factor of the numerator in the definition of $F_m(n,k)$ (see (7)). So, if $F_m(n,k)$ were a prime, we would have $F_m(n,k) \leq mn+k-1$, which contradicts to (8). Hence, $F_m(n,k)$ is a composite number.

We have finished the discussion for the sub-case of (a) where $n \geq 3$ and $k \geq 2$, and we now consider the sub-case of $n \geq 3$ and $k = 1$, i.e., the case of

$$F_m(n,1) = \frac{1}{(m-1)n+1}\binom{mn}{n}. \tag{9}$$

If $n = 3$, then (9) gives

$$F_m(3,1) = \frac{1}{3m-2}\binom{3m}{3} = \frac{m(3m-1)}{2}. \tag{10}$$

If $m = 2\ell+1$, $\ell \geq 1$, then the above equation (10) implies that $F_{2\ell+1}(3,1) = (2\ell+1)(3\ell+1)$ is not a prime number. If $m = 2\ell$, $\ell \geq 1$, then (10) implies that $F_{2\ell}(3,1) = \ell(6\ell-1)$ is not prime unless $\ell = 1$. Thus, $F_2(3,1) = 5$ is the only prime number for $m \geq 2$ and $n = 3$. The final phase of case (a) that we have not yet considered is $n \geq 4$ and $k = 1$. Similarly to (7), we may write (9) as

$$\begin{aligned}
&F_m(n,1) = \frac{1}{(m-1)n+1}\binom{mn}{n} \\
= \;& \frac{1}{(m-1)n+1}\frac{mn}{n}\frac{mn-1}{n-1}\cdots\frac{mn-(n-3)}{3}\frac{mn-(n-2)}{2}\frac{mn-(n-1)}{1} \\
= \;& \frac{mn}{n}\frac{mn-1}{n-1}\cdots\frac{mn-(n-4)}{4}\frac{mn-(n-3)}{3}\frac{mn-(n-2)}{2} \\
> \;& \frac{mn-(n-4)}{4}\frac{mn-(n-3)}{3}\frac{mn-(n-2)}{2} \\
\geq \;& 2\frac{7}{3}\frac{mn-(n-2)}{2} \\
> \;& 2(mn-(n-2)) > mn,
\end{aligned}$$

where we used $n \geq 4$ and $m \geq 2$ so that $mn-(n-4)/4 \geq 2$ and $(mn-(n-3)/3 \geq 7/3$, and the last inequality is due to $m \geq 2$. Thus, $F_m(n,1)$ is not prime when $n \geq 4$ by similar arguments as those stated right after (8). This finishes the proof of case (a).

In case (b), under the assumption $n = 2$, we have

$$F_m(2,k) = \frac{k}{2(m-1)+k}\binom{2m+k-1}{2} = \frac{k(2m+k-1)}{2}\ .$$

Thus, if $k = 1$ and $m = p$, a prime number, then $F_m(2,k) = p$ is prime. If $k = 2\ell$ ($\ell \geq 1$), then

$$F_m(2,2\ell) = \ell(2m+2\ell-1)$$

is not prime unless $\ell = 1$ and $2m+1$ is prime, or equivalently, $\ell = 1$ and $m = (p-1)/2$, where p is prime. If $k = 2\ell+1$, $\ell > 1$, then

$$F_m(2,k) = (2\ell+1)(m+1)$$

is not prime.

In case (c), we assume $n = 1$; then

$$F_m(1, k) = k$$

is prime exactly when k is prime. This completes the proof of the theorem. □

Corollary 2. *Let $F_m(n, 1)$ be the Fuss-Catalan numbers defined by (2). Then none of these are prime except for $F_p(2, 1) = p$, where p is prime, and $F_2(3, 1) = 5$.*

For the Catalan numbers $C_n = F_2(n, 1)$, Koshy and Salmassi [9] use a different method to prove the following special case of Theorem 1.

Corollary 3. *The Catalan numbers C_n are not prime except $C_2 = 2$ and $C_3 = 5$.*

Lobb [11] defines the Lobb numbers A039599

$$L_{n,m} := \frac{2m+1}{n+m+1}\binom{2n}{n+m}$$

for $n \geq m \geq 0$, which have the combinatorial interpretation as follows: $L_{n,m}$ is the number of sequences of length $2n$ with $n + m$ of the terms equal to 1 and $n - m$ of the terms equal to -1, such that no partial sum is negative. Bobrowski et al. [2] extend Lobb numbers to the number of sequences with $(k-1)n + m$ terms equal to 1 and $n - m$ terms equal to $1 - k$. The extended Lobb numbers are denoted by $L^k_{m,n}$ and are defined by

$$L^k_{n,m} := \frac{km+1}{(k-1)n+m+1}\binom{kn}{(k-1)n+m}. \tag{11}$$

Generalized Lobb numbers include many number sequences as special cases. For example, when $k = 2$, the numbers $L^2_{n,m}$ are the classical Lobb numbers. When $m = 0$, the numbers

$$L^k_{n,0} = \frac{1}{(k-1)n+1}\binom{kn}{n} = F_k(n)$$

are the Fuss-Catalan numbers. When $k = 2$ and $m = 0$, the numbers

$$L^2_{n,0} = \frac{1}{n+1}\binom{2n}{n} = C_n$$

are the classical Catalan numbers. When $k = 1$, the numbers

$$L^1_{n,m} = \binom{n}{m}$$

are the binomial coefficients. Other special cases can be seen in [7].

Equations (3) and (10) imply that

$$L_{n,m}^{k} = F_k(n - m, km + 1). \tag{12}$$

Hence, we have the inverse relationship

$$F_m(n, k) = L_{n+(k-1)/m,(k-1)/m}^{m}, \tag{13}$$

which can be used to transfer the results of Theorem 1 for the generalized Fuss-Catalan numbers to corresponding results for the generalized Lobb numbers. This gives us the following corollary to Theorem 1.

Corollary 4. *For integers $m \geq 2$ and $n, k \geq 1$, the only extended Lobb numbers which are prime are $L_{1+(k-1)/m,(k-1)/m}^{m} = k$, where k is prime, $L_{2,0}^{p} = p$, where p is prime, $L_{3,0}^{2} = 5$, and $L_{2+1/m,1/m}^{m} = 2m + 1$, where $m = (p - 1)/2$ and p is prime.*

The Lobb numbers $L_{n,m}^{2}$ are also related to the ballot numbers A002026 (see, for example, [6])

$$B(a, b) = \frac{a - b}{a + b}\binom{a + b}{a} = \frac{a - b}{a + b}\binom{a + b}{b}. \tag{14}$$

Namely, when $L_{n,m}^{k}$ and $B(a, b)$ are defined by (13) and (14), respectively, we have

$$L_{n,m}^{2} = B(n + m + 1, n - m), \tag{15}$$

or equivalently,

$$B(n, m) = L_{\frac{n+m-1}{2},\frac{n-m-1}{2}}^{2}. \tag{16}$$

Hence, $L_{n,m}^{2}$ is a ballot number. From Corollary 4, we immediately have

Corollary 5. *Let the ballot numbers $B(a, b)$ be defined by (11). Then the only prime numbers of the form $B(a, b)$ are $B(k + 1, 1) = k$, where k is prime, $B(3, 2) = 2$, and $B(4, 2) = B(4, 3) = 5$.*

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