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Curious Bounds for Floor Function Sums

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Abstract

Jacobsthal was the first to study a certain kind of sum of floor functions. His work was followed by that of Carlitz, Grimson, and Tverberg. More recently, Onphaeng and Pongsriiam proved some sharp upper and lower bounds for the sums of Jacobsthal and Tverberg. In this paper, we devise concise formulas for these sums, which we then use to prove the upper and lower bounds claimed by Tverberg. Furthermore, we present conjectural lower and upper bounds for these sums.

1 Introduction

In 1957, Jacobsthal [3] defined and studied a function of the form

$$f_m(\{a_1,a_2\},k) = \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_1+k}{m} \right\rfloor - \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor$$

for fixed $m \in \mathbb{Z}^+$ with $a_1, a_2, k \in \mathbb{Z}$. He also defined the functions

$$S_m(\{a_1,a_2\},K) = \sum_{k=0}^{K} f_m(\{a_1,a_2\},k), \qquad 0 \le a_1, a_2, K \le m-1.$$

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It is important to note that we can take advantage of the m-periodicity of f_m, and so we restrict our a_1, a_2, and K values accordingly for the sum. Jacobsthal, and then later Carlitz [1], Grimson [2], and Tverberg [5], proved $S_m(\{a_1, a_2\}, K) \geq 0$. In 2012, Tverberg [5] proposed a generalized notation for these sum functions for any set $A = \{a_1, \ldots, a_n\}$ with $0 \leq a_1, \ldots, a_n, K \leq m-1$ and $n = |A|$, that is,

$$S_m(\{a_1, \ldots, a_n\}, K) = \sum_{k=0}^{K} \sum_{T \subset [1,n]} (-1)^{n-|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor.$$

He also claimed without proof the other upper and lower bounds of S_m for sets $\{a_1, a_2\}$ and $\{a_1, a_2, a_3\}$ (i.e., $n = 2, 3$). In 2017, Onphaeng and Pongsriiam [4] furnished a proof for the upper bounds when n is even and ≥ 4 and the lower bounds when n is odd and ≥ 3. In this paper, we investigate the bounds for S_m for all $n \in \mathbb{Z}^+$ and supply the missing proofs of Tverberg's upper bounds. Furthermore, we conjecture all the bounds for S_m not previously mentioned and summarize the findings in Table 1. Authors who claimed their bounds without proof are denoted with an asterisk (*). Otherwise, a proof is given in their corresponding paper.

n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
1	0	Trivial	$m-1$	Trivial
2	0	Jacobsthal; Carlitz; Grimson; Tverberg; **Theorem 9**	$\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; **Corollary 8**
3	$-2\left\lfloor \frac{m}{2} \right\rfloor$	Tverberg*; Onphaeng, Pongsriiam	$\left\lfloor \frac{m}{3} \right\rfloor$	Tverberg*; **Corollary 14**
4	$-3\left\lfloor \frac{m}{3} \right\rfloor$ (Conjecture)	**Conjecture 17***	$4\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam
odd (≥ 5)	$-2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam	(Conjectures)	**Conjecture 17***
even (≥ 5)	(Conjectures)	**Conjecture 17***	$2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor$	Onphaeng, Pongsriiam

Table 1: Bounds for $S_m(\{a_1, \ldots, a_n\}, K)$

We discuss the case of $n = 1$ in this section as it sets the foundation for the main strategy that we use to prove higher cases. We begin with an explicit definition of Jacobsthal's sum.

Definition 1. For any $m \in \mathbb{Z}$ and $a_1, K \in \mathbb{Z}^+ \cup \{0\}$,

$$S_m(\{a_1\}, K) = \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right).$$

The sum can be written concisely without using the summation symbol, which we show below. Note that the periodicity that existed in the $n \geq 2$ case does not exist here. However, we only prove the following proposition for $0 \leq K \leq m-1$ because that is all that is needed for higher values of n. We let $a \bmod m$ denote the minimal non-negative representative in the $\mathbb{Z}/m$-equivalence class.

Proposition 2. *For* $0 \leq K \leq m - 1$ *and* $a_1 \in \mathbb{Z}^+ \cup \{0\}$,

$$S_m(\{a_1\}, K) = \left\lfloor \frac{a_1}{m} \right\rfloor (K + 1) + \max(0, (a_1 \bmod m) + K - m + 1).$$

Proof. We observe

$$\left\lfloor \frac{a_1 + k}{m} \right\rfloor = \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor,$$

where $\{x\} = x - \lfloor x \rfloor$, the fractional part of x. This notation is distinguished from the usual set notation according to context. Furthermore, since $0 \leq k \leq m - 1$,

$$\left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor = \left\lfloor \frac{a_1 \bmod m}{m} + \frac{k}{m} \right\rfloor.$$

The result is derived as follows:

$$\begin{aligned}
&\sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right) \\
&\qquad = \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right) \\
&\qquad = \left\lfloor \frac{a_1}{m} \right\rfloor (K + 1) + \sum_{k=0}^{K} \left\lfloor \frac{a_1 \bmod m}{m} + \frac{k}{m} \right\rfloor
\end{aligned}$$

$$= \left\lfloor \frac{a_1}{m} \right\rfloor (K+1) + \sum_{k=0}^{m-(a_1 \bmod m)-1} \left\lfloor \frac{(a_1 \bmod m)+k}{m} \right\rfloor$$

$$+ \sum_{k=m-(a_1 \bmod m)}^{K} \left\lfloor \frac{(a_1 \bmod m)+k}{m} \right\rfloor$$

$$= \left\lfloor \frac{a_1}{m} \right\rfloor (K+1) + \sum_{k=0}^{m-(a_1 \bmod m)-1} 0 + \sum_{k=m-(a_1 \bmod m)}^{K} 1$$

$$= \left\lfloor \frac{a_1}{m} \right\rfloor (K+1) + \max(0, (a_1 \bmod m) + K - m + 1).$$

□

The bounds for $S_m(\{a_1\}, K)$ are then easily attained from Proposition 2.

Corollary 3. *For $0 \leq a_1, K \leq m-1$,*

$$0 \leq S_m(\{a_1\}, K) \leq m-1.$$

In particular, the maximum occurs precisely when $a_1 = K = m-1$.

Proof. The result follows from the fact that

$$S_m(\{a_1\}, K) = \max(0, a_1 + K - m + 1).$$

□

In the following sections, we use similar methods to provide bounds when $n > 1$ for the sums $S_m(\{a_1, \ldots, a_n\}, K)$.

2 Lower and upper bounds for $n = 2$

Carlitz [1], Grimson [2], Jacobsthal [3], and Tverberg [5] all proved the lower bound for $n = 2$, while Tverberg was first to mention the upper bound for this case. In this section, we prove the upper bound by introducing a new form for the sum just as before. We also use this form to give a new proof for its lower bound. We do this by first writing out Jacobsthal's sum explicitly, and then making a generalization of Proposition 2.

Definition 4. For any $m \in \mathbb{Z}^+$ and any $a_1, a_2, K \in \mathbb{Z}^+ \cup \{0\}$,

$$S_m(\{a_1, a_2\}, K) = \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_1+k}{m} \right\rfloor - \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \right).$$

We show that the sum can be written concisely without using the summation symbol, similar to Proposition 2.

Proposition 5. *For* $0 \leq K \leq m-1$, *and any* $a_1, a_2 \in \mathbb{Z}^+ \cup \{0\}$,

$$\begin{aligned} S_m(\{a_1,a_2\},K) = &\left(\left\lfloor \frac{a_1+a_2}{m} \right\rfloor - \left\lfloor \frac{a_1}{m} \right\rfloor - \left\lfloor \frac{a_2}{m} \right\rfloor\right)(K+1) \\ &+ \max(0, ((a_1+a_2) \bmod m) + K - m + 1) \\ &- \max(0, (a_1 \bmod m) + K - m + 1) \\ &- \max(0, (a_2 \bmod m) + K - m + 1). \end{aligned}$$

Proof. Rewriting the two-variable sum in Definition 4 as a series of one-variable sums,

$$S_m(\{a_1,a_2\},K) = S_m(\{a_1+a_2\},K) - S_m(\{a_1\},K) - S_m(\{a_2\},K),$$

allows us to apply Proposition 2 to each sum to get our result. □

A symmetry exists in Proposition 5. We outline the pattern in the lemma below. This, along with a partial result in Theorem 7, gives us the desired upper and lower bounds in Corollary 8 and Theorem 9, respectively.

Lemma 6. *(Mirrored Sums) For* $0 \leq a_1, a_2 \leq m-1$ *and* $0 \leq K \leq m-2$,

$$S_m(\{a_1,a_2\},K) = S_m(\{m-a_1, m-a_2\}, m-2-K).$$

Proof. It is enough to show the claim for $0 \leq a_1 + a_2 \leq m$. Otherwise, we have that $(m-a_1)+(m-a_2) < m$, in which case we can use a similar argument by substituting a_1 with $m-a_1$ and a_2 with $m-a_2$. For the case $a_1 = a_2 = 0$, the result trivially holds by Definition 4. For the case $0 < a_1 + a_2 < m$, Proposition 5 simplifies to

$$\begin{aligned} S_m(\{a_1,a_2\},K) = &\max(0, a_1+a_2+K-m+1) \\ &- \max(0, a_1+K-m+1) - \max(0, a_2+K-m+1). \end{aligned} \tag{1}$$

Furthermore, we note that $m < 2m - (a_1+a_2) < 2m$, which gives

$$\begin{aligned} S_m(\{m-a_1, m-a_2\}, m-2-K) = &m-1-K + \max(0, m-(a_1+a_2)-K-1) \\ &- \max(0, m-a_1-K-1) - \max(0, m-a_2-K-1). \end{aligned} \tag{2}$$

Now consider the following equations that use the fact that

$$\max(0,x) - \max(0,-x) = x, \text{ for all } x \in \mathbb{R}.$$

$$\begin{aligned}&\max(0, a_1 + a_2 + K - m + 1) - \max(0, -a_1 - a_2 - K + m - 1)\\&\qquad = a_1 + a_2 + K - m + 1,\end{aligned} \tag{3}$$

$$\begin{aligned}&\max(0, m - a_1 - K - 1) - \max(0, a_1 + K - m + 1)\\&\qquad = m - a_1 - K - 1,\end{aligned} \tag{4}$$

$$\begin{aligned}&\max(0, m - a_1 - K - 1) - \max(0, a_2 + K - m + 1)\\&\qquad = m - a_2 - K - 1.\end{aligned} \tag{5}$$

Then, (3)+(4)+(5) confirms (1) = (2). Finally, we consider $a_1 + a_2 = m$. Here, Proposition 5 gives

$$\begin{aligned}&S_m(\{a_1, a_2\}, K)\\&\quad = K + 1 - \max(0, a_1 + K - m + 1) - \max(0, a_2 + K - m + 1),\end{aligned} \tag{6}$$

$$\begin{aligned}&S_m(\{m - a_1, m - a_2\}, m - 2 - K)\\&\quad = m - (K + 1) - \max(0, m - a_1 - K - 1) - \max(0, m - a_2 - K - 1).\end{aligned} \tag{7}$$

In this case, (4)+(5) confirms (6) = (7). This concludes the proof. □

We show the upper bound for half the range of K using differences.

Theorem 7. *For $0 \le a_1, a_2 \le m - 1$ and $0 \le K \le \lfloor \frac{m}{2} \rfloor - 1$,*

$$S_m(\{a_1, a_2\}, K) \le \left\lfloor \frac{m}{2} \right\rfloor .$$

Proof. We show the stronger result, that for $0 \le K \le \lfloor \frac{m}{2} \rfloor - 1$,

$$S_m(\{a_1, a_2\}, K) \le K + 1.$$

For the case $K = 0$,

$$\begin{aligned}S_m(\{a_1, a_2\}, 0) &= \left\lfloor \frac{a_1 + a_2}{m} \right\rfloor + \max(0, ((a_1 + a_2) \bmod m) - m + 1)\\&\quad - \max(0, a_1 - m + 1) - \max(0, a_2 - m + 1)\\&= \left\lfloor \frac{a_1 + a_2}{m} \right\rfloor + 0 - 0 - 0\\&\le 1.\end{aligned}$$

For the case $1 \le K \le \lfloor \frac{m}{2} \rfloor - 1$, it is enough to show that

$$\Delta_m := S_m(\{a_1, a_2\}, K) - S_m(\{a_1 + 1, a_2\}, K - 1) \le 1.$$

By using Proposition 5 we explicitly write out the two sums

$$\begin{aligned}
S_m(\{a_1, a_2\}, K) = &\left\lfloor \frac{a_1 + a_2}{m} \right\rfloor (K+1) \\
&+ \max(0, ((a_1 + a_2) \bmod m) + K - m + 1) \\
&- \max(0, a_1 + K - m + 1) \\
&- \max(0, a_2 + K - m + 1), \\
S_m(\{a_1 + 1, a_2\}, K - 1) = &\left\lfloor \frac{a_1 + a_2 + 1}{m} \right\rfloor K \\
&+ \max(0, ((a_1 + a_2 + 1) \bmod m) + K - m) \\
&- \max(0, a_1 + K - m + 1) \\
&- \max(0, a_2 + K - m).
\end{aligned}$$

We determine Δ_m according to the possible values of $a_1 + a_2$ and $a_2 + K - m + 1$.

Case 1: $(a_1 + a_2 < m$ **and** $a_2 + K - m + 1 \leq 0)$ If $a_1 + a_2 < m - 1$, then both sums are the same.

$$\begin{aligned}
S_m(\{a_1, a_2\}, K) &= 0 + \max(0, a_1 + a_2 + K - m + 1) \\
&\quad - \max(0, a_1 + K - m + 1) - 0, \\
S_m(\{a_1 + 1, a_2\}, K - 1) &= 0 + \max(0, (a_1 + 1) + a_2 + K - m) \\
&\quad - \max(0, (a_1 + 1) + K - m) - 0.
\end{aligned}$$

If $a_1 + a_2 = m - 1$, then both sums evaluate to K. Therefore, we get $\Delta_m = 0$.

Case 2: $(a_1 + a_2 < m$ **and** $a_2 + K - m + 1 > 0)$ Again, we assume $a_1 + a_2 < m - 1$.

$$\begin{aligned}
S_m(\{a_1, a_2\}, K) &= 0 + \max(0, a_1 + a_2 + K - m + 1) \\
&\quad - \max(0, a_1 + K - m + 1) - (a_2 + K - m + 1), \\
S_m(\{a_1 + 1, a_2\}, K - 1) &= 0 + \max(0, (a_1 + 1) + a_2 + K - m + 1) \\
&\quad - \max(0, (a_1 + 1) + K - m) - (a_2 + K - m).
\end{aligned}$$

Therefore, $\Delta_m = S_m(\{a_1, a_2\}, K) - S_m(\{a_1 + 1, a_2\}, K - 1) = -1$. A similar argument holds for $a_1 + a_2 = m - 1$.

Case 3: $(a_1 + a_2 \geq m$ **and** $a_2 + K - m + 1 \leq 0)$

$$\begin{aligned}
S_m(\{a_1, a_2\}, K) &= (K + 1) + \max(0, (a_1 + a_2 - m) + K - m + 1) \\
&\quad - \max(0, a_1 + K - m + 1) - 0, \\
S_m(\{a_1 + 1, a_2\}, K - 1) &= K + \max(0, (a_1 + 1 + a_2 - m) + K - m) \\
&\quad - \max(0, (a_1 + 1) + K - m) - 0.
\end{aligned}$$

Therefore, $\Delta_m = S_m(\{a_1, a_2\}, K) - S_m(\{a_1 + 1, a_2\}, K - 1) = +1$.

Case 4: ($a_1 + a_2 \geq m$ **and** $a_2 + K - m + 1 > 0$) $\Delta_m = 0$ using similar reasoning to Case 2 and Case 3.

We summarize the four cases for $0 < K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$ in the table below.

Case	$a_1 + a_2$	$a_2 + K - m + 1$	Δ_m
1	$< m$	≤ 0	0
2	$< m$	> 0	-1
3	$\geq m$	≤ 0	$+1$
4	$\geq m$	> 0	0

Table 2: Summary of Δ_m values.

This shows that $-1 \leq \Delta_m \leq 1$. Thus, we have shown that

$$S_m(\{a_1, a_2\}, K) \leq S_m(\{a_1, a_2\}, K - 1) + 1 \leq K + 1 \leq \left\lfloor \frac{m}{2} \right\rfloor,$$

for all $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$. □

We apply Lemma 6 and Theorem 7 to show the upper bound for S_m.

Corollary 8. *For* $0 \leq a_1, a_2, K \leq m - 1$,

$$S_m(\{a_1, a_2\}, K) \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. For $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$, Theorem 7 gives the result. For $\left\lfloor \frac{m}{2} \right\rfloor \leq K \leq m - 2$, $S_m(\{a_1, a_2\}, K) = S_m(\{m - a_1, m - a_2\}, m - 2 - K)$ by Lemma 6. From there, we apply Theorem 7 to the right hand side and get the result. Finally, for $K = m - 1$, it is easily seen that $S_m(\{a_1, a_2\}, K) = 0$. This completes the proof. □

The lower bound is now easy to show using Δ_m.

Theorem 9. *For* $0 \leq a_1, a_2, K \leq m - 1$,

$$0 \leq S_m(\{a_1, a_2\}, K).$$

Proof. Without loss of generality, we assume that $0 \leq a_2 \leq a_1 \leq m - 1$. Consider $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$. Thus, the conditions of Case 2 (i.e., $a_1 + a_2 < m$ and $a_2 + K - m + 1 > 0$) cannot be met because if $a_1 + a_2 < m$, then $a_2 \leq \left\lfloor \frac{m}{2} \right\rfloor$. So, $a_2 + K - m + 1 \leq 0$. This shows that $\Delta_m \neq -1$, which means $\Delta_m = 0$ or 1. This, along with $S_m(\{a_1, a_2\}, 0) \geq 0$, gives us the lower bound as desired. Next, we consider $\left\lfloor \frac{m}{2} \right\rfloor \leq K \leq m - 2$, and apply Lemma 6 to complete the argument. Lastly, $S_m(\{a_1, a_2\}, m - 1) = 0$. We now have the complete result. □

3 Upper bound for $n = 3$

Onphaeng and Pongsriiam [4] already proved the lower bound for this case. In this section, we follow the style of the previous sections by rewriting the sum, observing its symmetry, and using a difference to prove the upper bound. This time, we use Tverberg's formulation to write out the sum explicitly.

Definition 10. For any $m \in \mathbb{Z}^+$ and $a_1, a_2, a_3, K \in \mathbb{Z}^+ \cup \{0\}$,

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) = \sum_{k=0}^{K} \Bigg(& \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor \\ & - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor \\ & + \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \Bigg). \end{aligned}$$

Like before, we show that the sum can be written concisely without using the summation symbol.

Proposition 11. *For $0 \le K \le m - 1$ and $a_1, a_2, a_3 \in \mathbb{Z}^+ \cup \{0\}$,*

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) = \Bigg(& \left\lfloor \frac{a_1 + a_2 + a_3}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3}{m} \right\rfloor \\ & - \left\lfloor \frac{a_1 + a_3}{m} \right\rfloor + \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{a_2}{m} \right\rfloor + \left\lfloor \frac{a_3}{m} \right\rfloor \Bigg) (K + 1) \\ & + \max(0, ((a_1 + a_2 + a_3) \bmod m) + K - m + 1) \\ & - \max(0, ((a_1 + a_2) \bmod m) + K - m + 1) \\ & - \max(0, ((a_2 + a_3) \bmod m) + K - m + 1) \\ & - \max(0, ((a_1 + a_3) \bmod m) + K - m + 1) \\ & + \max(0, (a_1 \bmod m) + K - m + 1) \\ & + \max(0, (a_2 \bmod m) + K - m + 1) \\ & + \max(0, (a_3 \bmod m) + K - m + 1). \end{aligned}$$

Proof. We can rewrite our three-variable sum in Definition 10 in terms of two-variable sums, that is,

$$S_m(\{a_1, a_2, a_3\}, K) = S_m(\{a_1, a_2 + a_3\}, K) - S_m(\{a_1, a_2\}, K) - S_m(\{a_1, a_3\}, K). \tag{8}$$

We then apply Proposition 5 to each sum to get our result.

□

A symmetry exists for $S_m(\{a_1, a_2, a_3\}, K)$, similar to Lemma 6.

Lemma 12. *(Mirrored Sums) For* $0 \le a_1, a_2, a_3 \le m-1$ *and* $0 \le K \le m-2$,

$$S_m(\{a_1, a_2, a_3\}, K) = S_m(\{m-a_1, m-a_2, m-a_3\}, m-2-K).$$

Proof. As above, we rewrite the three-variable sum as a series of two-variable sums and then reason as follows:

$$\begin{aligned}
&S_m(\{a_1, a_2, a_3\}, K)\\
&\quad = S_m(\{a_1, a_2+a_3\}, K) - S_m(\{a_1, a_2\}, K) - S_m(\{a_1, a_3\}, K)\\
&\quad = S_m(\{a_1, (a_2+a_3) \bmod m\}, K) - S_m(\{a_1, a_2\}, K) - S_m(\{a_1, a_3\}, K)\\
&\quad = S_m(\{m-a_1, m-(a_2+a_3) \bmod m\}, m-2-K)\\
&\quad - S_m(\{m-a_1, m-a_2\}, m-2-K) - S_m(\{m-a_1, m-a_3\}, m-2-K)\\
&\quad = S_m(\{m-a_1, (m-a_2)+(m-a_3)\}, m-2-K)\\
&\quad - S_m(\{m-a_1, m-a_2\}, m-2-K) - S_m(\{m-a_1, m-a_3\}, m-2-K)\\
&\quad = S_m(\{m-a_1, m-a_2, m-a_3\}, m-2-K).
\end{aligned}$$

The first and fifth equalities come from (8). We take advantage of the a_i-periodicity of the sums in the second and fourth equalities. Lastly, we apply Lemma 6 in the third equality. □

Next, we show the upper bound for half the range of K using Δ_m.

Theorem 13. *For* $0 \le a_1, a_2, a_3 \le m-1$ *and* $0 \le K \le \left\lfloor \frac{m}{2} \right\rfloor - 1$,

$$S_m(\{a_1, a_2, a_3\}, K) \le \left\lfloor \frac{m}{3} \right\rfloor.$$

Proof. Without loss of generality, we assume that $0 \le a_3 \le a_2 \le a_1 \le m-1$. We break down the proof into two cases of K, that is, we would like to show

$$S_m(\{a_1, a_2, a_3\}, K) \le \begin{cases} K+1, & \text{if } 0 \le K \le \left\lfloor \frac{m}{3} \right\rfloor - 1 \text{ (}\textbf{Case A}\text{)};\\ \left\lfloor \frac{m}{3} \right\rfloor, & \text{if } \left\lfloor \frac{m}{3} \right\rfloor \le K \le \left\lfloor \frac{m}{2} \right\rfloor - 1 \text{ (}\textbf{Case B}\text{)}. \end{cases}$$

As above, we define a difference of sums,

$$\square_m := S_m(\{a_1, a_2, a_3\}, K) - S_m(\{a_1+1, a_2, a_3\}, K-1),$$

and note that $\square_m$ can be converted to Δ_m via (8) as follows:

$$\begin{aligned}
\square_m &= S_m(\{a_1, a_2, a_3\}, K) - S_m(\{a_1+1, a_2, a_3\}, K-1)\\
&= (S_m(\{a_1, a_2+a_3\}, K) - S_m(\{a_1+1, a_2+a_3\}, K-1))\\
&- (S_m(\{a_1, a_2\}, K) - S_m(\{a_1+1, a_2\}, K-1))\\
&- (S_m(\{a_1, a_3\}, K) - S_m(\{a_1+1, a_3\}, K-1))\\
&= \Delta_m(\{a_1, a_2+a_3\}, K) - \Delta_m(\{a_1, a_2\}, K) - \Delta_m(\{a_1, a_3\}, K).
\end{aligned}$$

Case A: From the proof of Theorem 9, we know that

$$\Delta_m(\{a_1, a_2\}, K) \geq 0 \text{ and } \Delta_m(\{a_1, a_3\}, K) \geq 0.$$

This, coupled with the fact that $\Delta_m(\{a_1, a_2 + a_3\}, K) \leq 1$ (by Table 2), gives $\square_m \leq 1$. Furthermore, with $S_m(\{a_1, a_2, a_3\}, 0) \leq 1$, we get $S_m \leq K + 1$ as in the proof of Theorem 7.

Case B: If $\square_m \leq 0$, then we can use Case A (i.e., $S_m \leq K + 1 \leq \lfloor \frac{m}{3} \rfloor$) to show

$$S_m(\{a_1, a_2, a_3\}, K) \leq S_m(\{a_1 + 1, a_2, a_3\}, K - 1) \leq \left\lfloor \frac{m}{3} \right\rfloor.$$

If $\square_m = 1$, we show $S_m \leq \lfloor \frac{m}{3} \rfloor$ directly. Observe that the only way to obtain $\square_m = 1$ is when

$$\begin{aligned} \Delta_m(\{a_1, a_2 + a_3\}, K) &= +1, \\ \Delta_m(\{a_1, a_2\}, K) &= 0, \\ \Delta_m(\{a_1, a_3\}, K) &= 0. \end{aligned}$$

This arrangement is achieved when our assumed a_1, a_2, a_3, K also satisfies all conditions from Table 2. So, in Table 3, we organize each row according to the restrictions that must be applied. The 'type' refers to the logical operator on the conditions in the same row in order to achieve that particular Δ_m value. Keeping these conditions in mind, we bound $S_m(\{a_1, a_2, a_3\}, K)$ according to whether $a_2 + a_3 < m$ or not.

Δ_m	Condition **a**		Type	Condition **b**	
+1	(1a)	$a_1 + (a_2 + a_3) \bmod m \geq m$	AND	(1b)	$((a_2 + a_3) \bmod m) + K - m + 1 \leq 0$
0	(2a)	$a_1 + a_2 \geq m$	XOR	(2b)	$a_2 + K - m + 1 \leq 0$
0	(3a)	$a_1 + a_3 \geq m$	XOR	(3b)	$a_3 + K - m + 1 \leq 0$

Table 3: Conditions on Δ_m values for Case B

Case B1: $a_2 + a_3 < m$. Conditions (1a) and (1b) simplify to

$$m \leq a_1 + a_2 + a_3 < 2m \text{ AND } a_2 + a_3 + K - m + 1 \leq 0,$$

which therefore satisfies (2b) and (3b), implying that it also satisfies $\sim$(2a) and $\sim$(3a) by the XOR condition. Thus, we get

$$\begin{aligned} a_1 + a_2 < m &\text{ AND } a_2 + K - m + 1 \leq 0, \\ a_1 + a_3 < m &\text{ AND } a_3 + K - m + 1 \leq 0. \end{aligned}$$

Applying these conditions to Proposition 11 results in the following formula:

$$\begin{aligned}S_m(\{a_1,a_2,a_3\},K) = K+1-\max(0,a_1+a_2+K-m+1)\\ -\max(0,a_1+a_3+K-m+1)+\max(0,a_1+K-m+1),\end{aligned}$$

which can be broken down into the following four subcases.

Case B1.1: $(a_1+K-m+1>0)$

$$S_m(\{a_1,a_2,a_3\},K) = -a_1-a_2-a_3+m \le -m+m = 0.$$

Case B1.2: $(a_1+K-m+1 \le 0$ **and** $a_1+a_3+K-m+1>0)$

$$\begin{aligned}S_m(\{a_1,a_2,a_3\},K) &= K+1-(a_1+a_2+K-m+1)-(a_1+a_3+K-m+1)\\ &= (m-a_1-a_2-a_3)-(a_1+K)+(m-1)\\ &\le 0-\left\lfloor\frac{2m}{3}\right\rfloor+(m-1)\\ &\le \left\lfloor\frac{m}{3}\right\rfloor,\end{aligned}$$

because $a_1+a_2+a_3 \ge m$ (1a) and $a_1 \ge a_2 \ge a_3$ gives $a_1 \ge \lfloor\frac{m}{3}\rfloor$. This, along with $K \ge \lfloor\frac{m}{3}\rfloor$, gives us the second to last line.

Case B1.3: $(a_1+a_3+K-m+1 \le 0$ **and** $a_1+a_2+K-m+1>0)$

$$S_m(\{a_1,a_2,a_3\},K) = m-a_1-a_2 \le \left\lfloor\frac{m}{3}\right\rfloor,$$

because $a_1+a_2 \ge \lfloor\frac{2m}{3}\rfloor$.

Case B1.4: $(a_1+a_2+K-m+1 \le 0)$ This case cannot happen because $a_1+a_2+a_3 \ge m$ gives that $a_1+a_2 \ge \lfloor\frac{2m}{3}\rfloor$ which means $a_1+a_2+K+1 \ge m$, contradicting the condition.

We summarize these four subcases in Table 4.

Subcase	$a_1+a_2+K-m+1$	$a_1+a_3+K-m+1$	$a_1+K-m+1$	S_m
1	>0	>0	>0	≤ 0
2	>0	>0	≤ 0	$\le \lfloor\frac{m}{3}\rfloor$
3	>0	≤ 0	≤ 0	$\le \lfloor\frac{m}{3}\rfloor$
4	≤ 0	≤ 0	≤ 0	N/A

Table 4: Subcases for B1

Case B2: $m \le a_2+a_3 < 2m$.

Conditions (1a) and (1b) simplify to

$$a_1 + a_2 + a_3 \geq 2m \text{ AND } a_2 + a_3 + K - 2m + 1 \leq 0,$$

which therefore satisfies (2a) and (3a), implying that it also satisfies $\sim$(2b) and $\sim$(3b) by the XOR condition. Thus, we get

$$\begin{aligned} a_1 + a_2 \geq m \text{ AND } a_2 + K - m + 1 > 0, \\ a_1 + a_3 \geq m \text{ AND } a_3 + K - m + 1 > 0. \end{aligned}$$

Applying these conditions to Proposition 11 results in the following formula:

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) = &-(K+1) - \max(0, a_1 + a_2 + K - 2m + 1) \\ &- \max(0, a_1 + a_3 + K - 2m + 1) + (a_1 + K - m + 1) \\ &+ (a_2 + K - m + 1) + (a_3 + K - m + 1), \end{aligned}$$

which can be broken down into the following three subcases.

B2.1: $(a_1 + a_2 + K - 2m + 1 > 0$ **and** $a_1 + a_3 + K - 2m + 1 > 0)$

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) &= -(a_1 + a_2 + K - 2m + 1) - (a_1 + a_3 + K - 2m + 1) \\ &\quad + (a_1 + a_2 + K - 2m + 1) + (a_3 + K - m + 1) \\ &= m - a_1 \\ &\leq \left\lfloor \frac{m}{3} \right\rfloor, \end{aligned}$$

because $a_1 + a_2 + a_3 \geq m$ and $a_1 \geq a_2 \geq a_3$ gives $a_1 \geq \lfloor \frac{2m}{3} \rfloor$.

B2.2: $(a_1 + a_2 + K - 2m + 1 > 0$ **and** $a_1 + a_3 + K - 2m + 1 \leq 0)$

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) &= (m - a_1) + (a_1 + a_3 + K - 2m + 1) \\ &\leq m - a_1 \\ &\leq \left\lfloor \frac{m}{3} \right\rfloor. \end{aligned}$$

B2.3: $(a_1 + a_2 + K - 2m + 1 \leq 0)$

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) &= (m - a_1) + (a_1 + a_2 + K - 2m + 1) \\ &\quad + (a_1 + a_3 + K - 2m + 1) \\ &\leq (m - a_1) \\ &\leq \left\lfloor \frac{m}{3} \right\rfloor. \end{aligned}$$

We summarize the three cases in Table 5.

Subcase	$a_1 + a_2 + K - 2m + 1$	$a_1 + a_3 + K - 2m + 1$	S_m
1	> 0	> 0	$\leq \left\lfloor \frac{m}{3} \right\rfloor$
2	> 0	≤ 0	$\leq \left\lfloor \frac{m}{3} \right\rfloor$
3	≤ 0	≤ 0	$\leq \left\lfloor \frac{m}{3} \right\rfloor$

Table 5: Subcases for B2

The results from Tables 4 and 5 complete the proof of Case B. Hence, we have proven the theorem.

□

Lemma 12 and Theorem 13 give the main result, which we state as a corollary.

Corollary 14. *For* $0 \leq a_1, a_2, a_3, K \leq m - 1$,

$$S_m(\{a_1, a_2, a_3\}, K) \leq \left\lfloor \frac{m}{3} \right\rfloor.$$

Proof. For $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$, Theorem 13 gives us our result. For $\left\lfloor \frac{m}{2} \right\rfloor \leq K \leq m - 2$, $S_m(\{a_1, a_2, a_3\}, K) = S_m(\{m-a_1, m-a_2, m-a_3\}, m-2-K)$ by Lemma 12. From there, we apply Theorem 13 to the right hand side and get the result. Finally, $S_m(\{a_1, a_2, a_3\}, m-1) = 0$. This completes the proof. □

4 (Not so sharp) lower bound for $n = 4$

Given that $0 \leq a_1, a_2, a_3, a_4, K \leq m - 1$, the pattern of the maximum and minimum values of $S_m(\{a_1, a_2, a_3, a_4\}, K)$ is less clear, as evidenced by some results of the computer program:

Maximum Values of Sums

$$\begin{aligned}(S_1, S_2, \ldots) = (&0, 4, 3, 8, 7, 12, 11, 16, 15, 20, 19, 24\\ &23, 28, 27, 32, 31, 36, 35, 40, 39, 44, \ldots).\end{aligned}$$

Minimum Values of Sums

$$\begin{aligned}(S_1, S_2, \ldots) = (&0, 0, -3, -2, -3, -6, -5, -6, -9,\\ &-8, -9, -12, -11, -12, -15, -14,\\ &-15, -18, -17, -18, -21, -20, \ldots).\end{aligned}$$

Onphaeng and Pongsriiam [4] showed the upper bound

$$S_m(\{a_1, a_2, a_3, a_4\}, K) \leq 4 \left\lfloor \frac{m}{2} \right\rfloor.$$

We conjecture the lower bound

$$-3\left\lfloor \frac{m}{3} \right\rfloor \leq S_m(\{a_1, a_2, a_3, a_4\}, K).$$

In an attempt to prove this lower bound, we found that writing a difference of sums (like Δ_m or $\square_m$) is not an efficient way to approach the problem. Accordingly, we use another method to obtain the following partial result.

Theorem 15. *For* $0 \leq a_1, a_2, a_3, a_4, K \leq m-1$,

$$-2\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{3} \right\rfloor \leq S_m(\{a_1, a_2, a_3, a_4\}, K).$$

Proof. We can combine the following bounds from $n = 2, 3$, namely,

$$\begin{aligned}
0 &\leq S_m(\{a_1 + a_2 + a_3, a_4\}, K), \\
-\left\lfloor \frac{m}{2} \right\rfloor &\leq -S_m(\{a_1 + a_2, a_4\}, K), \\
-\left\lfloor \frac{m}{2} \right\rfloor &\leq -S_m(\{a_1 + a_3, a_4\}, K), \\
-\left\lfloor \frac{m}{3} \right\rfloor &\leq -S_m(\{a_2, a_3, a_4\}, K), \\
0 &\leq S_m(\{a_1, a_4\}, K),
\end{aligned}$$

along with the identity

$$\begin{aligned}
S_m(\{a_1, a_2, a_3, a_4\}, K) = S_m(\{a_1 + a_2 + a_3, a_4\}, K) - S_m(\{a_1 + a_2, a_4\}, K) \\
- S_m(\{a_1 + a_3, a_4\}, K) - S_m(\{a_2, a_3, a_4\}, K) + S_m(\{a_1, a_4\}, K),
\end{aligned}$$

to obtain the claimed result.

□

5 Conjectures

In order to complete the analysis on this type of floor function problem, we want to show all the upper bounds and lower bounds for any number of variables, n. Onphaeng and Pongsriiam [4] were able to show the upper bound when n is even and the lower bound when n is odd.

Theorem 16 (Onphaeng, Pongsriiam)**.** *When n is even and m is even,*

$$S_m \leq 2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor.$$

When n is odd and m is even,

$$-2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor \leq S_m.$$

The bounds on both cases are obtained exactly at

$$A = \{m/2, m/2, \ldots, m/2\}, \quad K = m/2 - 1.$$

We conjecture the missing bounds, namely, the lower bounds when n is even and the upper bounds when n is odd. To make these conjectures, we wrote a Maple program to calculate the values of $S_m(A,K)$ for specific m, A, and K. For each n, the sets A and K that give extreme values of S_m form interesting patterns, which depend on m. Once we determine such values in A and K, we can quickly compute the extreme values of $S_m(A,K)$ for each n and then use the resulting data to form a holonomic ansatz. The resulting recurrence becomes a ninth order recurrence with polynomial coefficients of degree at most 2. We summarize our findings in the following conjecture. For interested readers, this Maple code can be found on Thanatipanonda's website (www.thotsaporn.com).

Conjecture 17. Let $A := \{a_1, a_2, \ldots, a_n\}$ and define max/min $S_m(A,K)$ to be the maximum/minimum over all choices of a_i and K with $0 \le a_1, a_2, \ldots, a_n, K \le m-1$ and m fixed. Furthermore, we let the function S_m be as defined in the first section. Suppose now

$$M(n) := \begin{cases} \max\ S_m(A,K), & n \text{ odd}; \\ \min\ S_m(A,K), & n \text{ even}. \end{cases}$$

Then for $n \ge 4$, we conjecture the following result in two parts.

1. $n = 4k-1, 4k$ or $4k+1$, where $k \in \mathbb{Z}^+$: Under the condition that m is a multiple of $2k+1$, the values of $M(n)$ occur exactly at

$$A = \left\{\frac{km}{2k+1}, \frac{km}{2k+1}, \ldots, \frac{km}{2k+1}\right\}, K = \frac{km}{2k+1} - 1,$$
$$\text{or } A = \left\{\frac{(k+1)m}{2k+1}, \frac{(k+1)m}{2k+1}, \ldots, \frac{(k+1)m}{2k+1}\right\}, K = \frac{(k+1)m}{2k+1} - 1.$$

2. $n = 4k+2$, where $k \in \mathbb{Z}^+$: Under the condition that m is a multiple of $2k+1$ and $2k+3$, the values of $M(n)$ occur at (among other places)

$$A = \left\{\frac{km}{2k+1}, \frac{km}{2k+1}, \ldots, \frac{km}{2k+1}\right\}, K = \frac{km}{2k+1} - 1,$$
$$\text{or } A = \left\{\frac{(k+1)m}{2k+1}, \frac{(k+1)m}{2k+1}, \ldots, \frac{(k+1)m}{2k+1}\right\}, K = \frac{(k+1)m}{2k+1} - 1,$$
$$\text{or } A = \left\{\frac{(k+1)m}{2k+3}, \frac{(k+1)m}{2k+3}, \ldots, \frac{(k+1)m}{2k+3}\right\}, K = \frac{(k+1)m}{2k+3} - 1,$$
$$\text{or } A = \left\{\frac{(k+2)m}{2k+3}, \frac{(k+2)m}{2k+3}, \ldots, \frac{(k+2)m}{2k+3}\right\}, K = \frac{(k+2)m}{2k+3} - 1.$$

Moreover, $M(n)$ can be calculated directly from a formula similar to the equations from Propositions 5 and 11, or by

$$M(n) = m \cdot f(n),$$

where $f(n)$ satisfies the recurrence relation

$$\begin{aligned}-5(n+3)(n-2)f(n) &= 10(n^2+n-8)f(n-1) - 4(2n^2-10n+3)f(n-2)\\ &\quad - 24(2n-11)f(n-3) - 32(2n^2-10n-1)f(n-4)\\ &\quad - 192(n-1)(n-5)f(n-5) + 64(2n^2-22n+51)f(n-6)\\ &\quad + 384(2n-13)f(n-7) - 256(n-3)(n-8)f(n-8)\\ &\quad + 512(n-9)(n-8)f(n-9)\end{aligned}$$

for $n \geq 11$, with the initial conditions

$$f(2) = 0, f(3) = 1/3, f(4) = -1, f(5) = 2, f(6) = -3, f(7) = 8, f(8) = -18,$$
$$f(9) = 36, \text{ and } f(10) = -65.$$

For convenience, we give examples of some of the bounds (and the set A for which the values of those bounds occur) produced from the conjectures above.

- Suppose $n = 4, 5$ and m is a multiple of 3.

  Case $n = 4$:
  $$-3 \cdot \left\lfloor \frac{m}{3} \right\rfloor \leq S_m.$$
  Case $n = 5$:
  $$S_m \leq 6 \cdot \left\lfloor \frac{m}{3} \right\rfloor.$$

  For these cases, $M(n)$ occurs at

  $$A = \{m/3, m/3, \dots, m/3\}, K = m/3 - 1$$
  $$\text{or } A = \{2m/3, 2m/3, \dots, 2m/3\}, K = 2m/3 - 1.$$

- Suppose $n = 6$.
  - If m is a multiple of 3, then
    $$-9 \cdot \left\lfloor \frac{m}{3} \right\rfloor \leq S_m,$$
    with the minimum at (among other places)
    $$A = \{m/3, m/3, \dots, m/3\}, K = m/3 - 1$$
    $$\text{or } A = \{2m/3, 2m/3, \dots, 2m/3\}, K = 2m/3 - 1.$$

- If m is a multiple of 5, then

$$-15 \cdot \left\lfloor \frac{m}{5} \right\rfloor \leq S_m,$$

with the minimum at (among other places)

$$\begin{aligned} A &= \{2m/5, 2m/5, \ldots, 2m/5\}, K = 2m/5 - 1 \\ \text{or } A &= \{3m/5, 3m/5, \ldots, 3m/5\}, K = 3m/5 - 1. \end{aligned}$$

- Suppose $n = 7, 8, 9$ and m is a multiple of 5.

  Case $n = 7$:
  $$S_m \leq 40 \cdot \left\lfloor \frac{m}{5} \right\rfloor.$$

  Case $n = 8$:
  $$-90 \cdot \left\lfloor \frac{m}{5} \right\rfloor \leq S_m.$$

  Case $n = 9$:
  $$S_m \leq 180 \cdot \left\lfloor \frac{m}{5} \right\rfloor.$$

  For these cases, $M(n)$ occurs at

$$\begin{aligned} A &= \{2m/5, 2m/5, \ldots, 2m/5\}, K = 2m/5 - 1 \\ \text{or } A &= \{3m/5, 3m/5, \ldots, 3m/5\}, K = 3m/5 - 1. \end{aligned}$$

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