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The Resultant, the Discriminant, and the Derivative of Generalized Fibonacci Polynomials

Rigoberto Flórez
Department of Mathematical Sciences
The Citadel
Charleston, SC 29409
USA
rigo.florez@citadel.edu

Robinson A. Higuita and Alexander Ramírez
Instituto de Matemáticas
Universidad de Antioquia
Medellín
Colombia
robinson.higuita@udea.edu.co
jalexander.ramirez@udea.edu.co

Abstract

A second-order polynomial sequence is of *Fibonacci-type* (*Lucas-type*) if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. Known examples of these types of sequences are Fibonacci polynomials, Pell polynomials, Fermat polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell-Lucas polynomials, Fermat-Lucas polynomials, and Chebyshev polynomials.

The *resultant* of two polynomials is the determinant of the Sylvester matrix and the *discriminant* of a polynomial p is the resultant of p and its derivative. We study the resultant, the discriminant, and the derivatives of Fibonacci-type polynomials and

Lucas-type polynomials as well the resultant of combinations of these two types of polynomials. As a corollary, we give explicit formulas for the resultant, the discriminant, and the derivative for the polynomials mentioned above.

1 Introduction

A second-order polynomial sequence is of *Fibonacci-type* $\mathcal{F}_n$ (*Lucas-type* $\mathcal{L}_n$), if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. These are known as *generalized Fibonacci polynomials* GFPs (see, for example [2, 9, 10, 13, 14]). Some known examples are Pell polynomials, Fermat polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell-Lucas polynomials, Fermat-Lucas polynomials, Chebyshev polynomials, Vieta polynomials and Vieta-Lucas polynomials.

The *resultant* of two polynomials p and q, denoted by $\operatorname{Res}(p,q)$, is the determinant of the Sylvester matrix (see (7), [1, 4, 12, 23] or [28, p. 426]). Very often in mathematics we ask the question whether or not two polynomials share a root. In particular, if p and q are two GFPs, we ask whether or not p and q have a common root. Since the resultant of p and q is also the product of p evaluated at each root of q, the resultant of two GFPs can be used to answer this question.

Several authors have been interested in the resultant. The first formula for the resultant of two cyclotomic polynomials was given by Apostol [3]. Recently Bzdega et al. [5] computed the resultant of two cyclotomic polynomials in a short way. Some other papers have been dedicated to the study of the resultant of Chebyshev polynomials [7, 20, 25, 29]. In this paper we deduce simple closed formulas for the resultants of a big family of GFPs. For example, the resultant of Fibonacci polynomials F_n and the resultant of Chebyshev polynomials of second kind U_n are given by

$$\operatorname{Res}(F_m, F_n) = 1 \text{ if } \gcd(m,n) = 1, \text{ and } \operatorname{Res}(F_m, F_n) = 0 \text{ otherwise,}$$

and

$$\operatorname{Res}(U_m, U_n) = (-4)^{(m-1)(n-1)/2} \text{ if } \gcd(m,n) = 1, \text{ and } \operatorname{Res}(U_m, U_n) = 0 \text{ otherwise.}$$

If $\nu_2(n)$ is the 2-adic valuation of n, then the resultant of Lucas polynomial $D_n(x)$ and the resultant of Chebyshev polynomials of the first kind $T_n(x)$ are given by

$$\operatorname{Res}(D_m, D_n) = 2^{\delta} \text{ if } \nu_2(m) \neq \nu_2(n), \text{ and } \operatorname{Res}(D_m, D_n) = 0 \text{ otherwise}$$

and

$$\operatorname{Res}(T_m, T_n) = (-1)^{\frac{mn}{2}} 2^{(m-1)(n-1)-1} 2^{\delta} \text{ if } \nu_2(m) \neq \nu_2(n), \text{ and } \operatorname{Res}(T_m, T_n) = 0 \text{ otherwise.}$$

The resultant for the family of Fibonacci-type polynomials is given in Theorem 1 and the resultant for the family of Lucas-type polynomials is given in Theorem 2. In Tables 1, 2, and 3 we give the resultants for some known polynomials.

Note that the resultant has been used to solve systems of polynomial equations (it encapsulates the solutions) [4, 21, 24, 27]. The resultant can also be used in combination with the elimination theory to answer other different types of questions about the multivariable polynomials.

The *discriminant* is the resultant of a polynomial and its derivative. If p is a GFP, we ask the question whether or not p has a repeated root. The discriminant helps to answer this question. In this paper we find simple closed formulas for the discriminant of both types of GFPs. For example, the discriminant of Fibonacci polynomials, the Chebyshev polynomials of the second kind, Lucas polynomials, and the Chebyshev polynomials first kind are given by

$$\operatorname{Disc}\big(F_n\big) = (-1)^{(n-2)(n-1)/2} 2^{n-1} n^{n-3}; \quad \operatorname{Disc}\big(U_n\big) = 2^{(n-1)^2} n^{n-3},$$

and

$$\operatorname{Disc}\big(D_n\big) = (-1)^{n(n-1)/2} 2^{n-1} n^{n}; \quad \operatorname{Disc}\big(T_n\big) = 2^{(n-1)^2} n^{n}.$$

The Theorems 4 and 5 give the discriminant of both families of Fibonacci-type polynomials and Lucas-type polynomials. Table 4 shows the discriminant for some known polynomials.

The following formulas generalize the formulas for the derivative of Fibonacci and Lucas polynomials given by several authors [2, 8, 15, 16, 17, 30] to Fibonacci-type polynomials and Lucas-type polynomials (for details see Theorem 6).

$$\mathcal{F}_n' = \frac{d'\left(n\alpha\mathcal{L}_n - d\mathcal{F}_n\right)}{(a-b)^2} \qquad \text{and} \qquad \mathcal{L}_n' = \frac{nd'\mathcal{F}_n}{\alpha}.$$

2 Main Results

In this section we present the main theorems and corollaries of this paper. Proofs appear elsewhere in the paper. We give some brief definitions (needed to make the theorems readable), all formal definitions are given in Section 3.

For brevity and if there is no ambiguity we present the polynomials without explicit use of "x". For example, instead of $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ we use $\mathcal{F}_n$ and $\mathcal{L}_n$.

We say that a polynomial recurrence relation is of *Fibonacci-type* if it satisfies that (see also (2))

$$\mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1,\ \text{and } \mathcal{F}_n = d\mathcal{F}_{n-1} + g\mathcal{F}_{n-2} \text{ for } n \geq 2,$$

where d, and g are fixed non-zero polynomials in $\mathbb{Q}[x]$. We say that a polynomial recurrence relation is of *Lucas-type* if it satisfies that (see also (3))

$$\mathcal{L}_0 = p_0,\ \mathcal{L}_1 = p_1,\ \text{and } \mathcal{L}_n = d\mathcal{L}_{n-1} + g\mathcal{L}_{n-2} \text{ for } n \geq 2,$$

where $|p_0| = 1$ or 2 and p_1, $d = \alpha p_1$, and g are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$. These are called generalized Fibonacci polynomials (GFPs).

If P is a polynomial, then we use $\deg(P)$ and $\operatorname{lc}(P)$ to mean the degree of P and the leading coefficient of P. Let

$$\beta = \operatorname{lc}(d), \quad \lambda = \operatorname{lc}(g), \quad \eta = \deg(d), \quad \omega = \deg(g), \quad \text{and} \quad \rho = \operatorname{Res}\big(g, d\big). \tag{1}$$

Let P and Q be polynomials with $a_n = \operatorname{lc}(P)$, $b_m = \operatorname{lc}(Q)$, $n = \deg(P)$, and $m = \deg(Q)$. If $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{m}$ are the roots of P and Q in $\mathbb{C}$, respectively, then the resultant of P and Q is given by $\operatorname{Res}\big(P, Q\big) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i - y_j) = a_n^m \prod_{i=1}^{n} Q(x_i) = b_m^n \prod_{j=1}^{m} P(y_j)$. The *discriminant* of P, $\operatorname{Disc}\big(P\big)$, is defined by $(-1)^{n(n-1)/2} a_n^{2n-2} \Pi_{i\neq j}(x_i - x_j)$.

We use $\nu_2(n)$ to represent the *integer exponent base two* of a positive integer n, which is defined to be the largest integer k such that $2^k \mid n$ (this concept is also known as the 2-*adic order* or 2-*adic valuation* of n).

2.1 Theorems of the resultant of GFPs

In this subsection we give simple expressions for the resultant of two GFPs of Fibonacci-type, $\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_m\big)$, the resultant of two GFPs of Lucas-type, $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_m\big)$, and the resultant of two equivalent polynomials (Lucas-type and Fibonacci-type), $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big)$.

The proof of Theorem 1 is in Section 4 on page 16, the proof of Theorem 2 is in Section 5 on page 17, and the proof of Theorem 3 is in Section 6 on page 18.

Theorem 1. *Let* $T_{\mathcal{F}} = \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(n-1)(m-1)}{2}}$, *where* $n, m \in \mathbb{Z}_{>0}$. *Then*

$$\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_m\big) = \begin{cases} 0, & \textit{if} \ \ \gcd(m,n) > 1; \\ T_{\mathcal{F}}, & \textit{otherwise.} \end{cases}$$

Theorem 2. *Let* $T_{\mathcal{L}} = \alpha^{-\eta(n+m)} 2^{\eta \gcd(m,n)} \big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{nm/2}$, *where* $m, n \in \mathbb{Z}_{>0}$. *Then*

$$\operatorname{Res}\big(\mathcal{L}_m, \mathcal{L}_n\big) = \begin{cases} 0, & \textit{if } \nu_2(n) = \nu_2(m); \\ T_{\mathcal{L}}, & \textit{if } \nu_2(n) \neq \nu_2(m). \end{cases}$$

Theorem 3. *Let* $T_{\mathcal{LF}} = 2^{\eta \gcd(m,n) - \eta} \alpha^{\eta(1-m)} \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{(n(m-1))/2}$, *where* $n, m \in \mathbb{Z}_{>0}$. *Then*

$$\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = \begin{cases} 0, & \textit{if } \nu_2(n) < \nu_2(m); \\ T_{\mathcal{LF}}, & \textit{if } \nu_2(n) \geq \nu_2(m). \end{cases}$$

2.2 Theorems of the discriminant of GFPs

As an application of the theorems in the previous subsection we give the discriminants of generalized Fibonacci polynomials. The proofs of Theorems 4 and 5 are in Section 8.2 on page 25.

Theorem 4. *If* $\deg(d) = 1$*,* g *is a constant, and* d' *is the derivative of* d*, then*

$$\mathrm{Disc}\big(\mathcal{F}_n\big) = (-g)^{(n-2)(n-1)/2} 2^{n-1} n^{n-3} \beta^{(n-1)(n-2)}.$$

Theorem 5. *If* $\deg(d) = 1$*,* g *is a constant, and* d' *is the derivative of* d*, then*

$$\mathrm{Disc}\big(\mathcal{L}_n\big) = (-g)^{n(n-1)/2} 2^{n-1} n^n \alpha^{2(1-n)} \beta^{n(n-1)}.$$

2.3 Theorem of the derivative of GFPs

The Theorem 6 gives formulas for the derivatives of Fibonacci-type polynomials and Lucas-type polynomials. Note that the formulas given here are restricted to the special case in which g is a constant. The derivatives of a Lucas-type polynomials is given in term of its equivalent polynomial and the derivative of a Fibonacci-type polynomial is given in terms of Fibonacci-type and its equivalent.

Here we use $\mathcal{F}_n'$, $\mathcal{L}_n'$, a', b' and d' to mean the derivatives of $\mathcal{F}_n$, $\mathcal{L}_n$, a, b and d with respect to x, where a and b are given in (4) and (5). The proof of the following theorem is in Section 7.

Theorem 6. *If* g *is a constant, then*

(i)

$$\mathcal{F}_n' = \frac{d'\left(ng\mathcal{F}_{n-1} - d\mathcal{F}_n + n\mathcal{F}_{n+1}\right)}{(a-b)^2} = \frac{d'\left(n\alpha\mathcal{L}_n - d\mathcal{F}_n\right)}{(a-b)^2}.$$

(ii)

$$\mathcal{L}_n' = \frac{nd'\mathcal{F}_n}{\alpha}.$$

2.4 Resultants, discriminants, and derivatives of known GFPs

In this subsection we construct several tables with the resultant, the discriminant and the derivative of some known polynomials (see Table 6).

The Table 1 presents the resultants of some GFPs of Fibonacci-type. The Table 2 presents the resultants of some GFPs of Lucas-type. The Table 3 presents the resultants of two equivalent polynomials (Lucas-type and its equivalent polynomial of Fibonacci-type). The Table 4 gives the discriminants of GFPs of both types. The first half of Table 4 has GFPs of Fibonacci-type and the second half has GFPs of Lucas-type. The Table 5 gives the derivatives for GFPs.

Note that the following property can be used to find the discriminant of a product of GFPs (see [6]). If P and Q are polynomials in $\mathbb{Q}[x]$, then $\mathrm{Disc}\big(PQ\big) = \mathrm{Disc}\big(P\big)\mathrm{Disc}\big(Q\big)\mathrm{Res}\big(P, Q\big)$.

Polynomial	$\gcd(m,n)=1$	$\gcd(m,n)>1$
Fibonacci	$\mathrm{Res}\big(F_m,F_n\big)=1$	$\mathrm{Res}\big(F_m,F_n\big)=0$
Pell	$\mathrm{Res}\big(P_m,P_n\big)=2^{(m-1)(n-1)}$	$\mathrm{Res}\big(P_m,P_n\big)=0$
Fermat	$\mathrm{Res}\big(\Phi_m,\Phi_n\big)=(-18)^{(m-1)(n-1)/2}$	$\mathrm{Res}\big(\Phi_m,\Phi_n\big)=0$
Chebyshev 2nd kind	$\mathrm{Res}\big(U_m,U_n\big)=(-4)^{(m-1)(n-1)/2}$	$\mathrm{Res}\big(U_m,U_n\big)=0$
Morgan-Voyce	$\mathrm{Res}\big(B_m,B_n\big)=(-1)^{(m-1)(n-1)/2}$	$\mathrm{Res}\big(B_m,B_n\big)=0$

Table 1: Resultants of Fibonacci-type polynomials using Theorem 1.

Polynomial	$\nu_2(m)\neq\nu_2(n)$, $\delta=\gcd(m,n)$	$\nu_2(m)=\nu_2(n)$
Lucas	$\mathrm{Res}\big(D_m,D_n\big)=2^{\delta}$	$\mathrm{Res}\big(D_m,D_n\big)=0$
Pell-Lucas-prime	$\mathrm{Res}\big(Q'_m,Q'_n\big)=2^{(m-1)(n-1)-1}2^{\delta}$	$\mathrm{Res}\big(Q'_m,Q'_n\big)=0$
Fermat-Lucas	$\mathrm{Res}\big(\vartheta_m,\vartheta_n\big)=(-1)^{mn/2}18^{mn/2}\,2^{\delta}$	$\mathrm{Res}\big(\vartheta_m,\vartheta_n\big)=0$
Chebyshev 1st kind	$\mathrm{Res}\big(T_m,T_n\big)=(-1)^{\frac{mn}{2}}2^{(m-1)(n-1)-1}2^{\delta}$	$\mathrm{Res}\big(T_m,T_n\big)=0$
Morgan-Voyce	$\mathrm{Res}\big(C_m,C_n\big)=(-1)^{\frac{mn}{2}}\,2^{\delta}$	$\mathrm{Res}\big(C_m,C_n\big)=0$

Table 2: Resultants of Lucas-type polynomials using Theorem 2.

Polynomials	$\nu_2(n)\geq\nu_2(m)$, $\delta=\gcd(n,m)$	$\nu_2(n)<\nu_2(m)$
Lucas, Fibonacci	$\mathrm{Res}\big(D_n,F_m\big)=2^{\delta-1}$	$\mathrm{Res}\big(D_n,F_m\big)=0$
Pell-Lucas-prime, Pell	$\mathrm{Res}\big(Q'_n,P_m\big)=2^{(m-1)(n-1)}\,2^{\delta-1}$	$\mathrm{Res}\big(Q'_n,P_m\big)=0$
Fermat-Lucas, Fermat	$\mathrm{Res}\big(\vartheta_n,\Phi_m\big)=(-18)^{n(m-1)/2}\,2^{\delta-1}$	$\mathrm{Res}\big(\vartheta_n,\Phi_m\big)=0$
Chebyshev both kinds	$\mathrm{Res}\big(T_n,U_m\big)=(-1)^{n(m-1)/2}2^{(m-1)(n-1)}2^{\delta-1}$	$\mathrm{Res}\big(T_n,U_m\big)=0$
Morgan-Voyce both types	$\mathrm{Res}\big(C_n,B_m\big)=(-1)^{n(m-1)/2}\,2^{\delta-1}$	$\mathrm{Res}\big(C_n,B_m\big)=0$

Table 3: Resultants of two equivalent polynomials using Theorem 3.

Evaluating the derivative of Fibonacci polynomials and the derivative of Lucas polynomials at $x=1$ and $x=2$ we obtain numerical sequences that appear in [26]. Thus,

$$\left.\frac{d(F_n)}{dx}\right|_{x=1}=\text{A001629};\qquad \left.\frac{d(F_n)}{dx}\right|_{x=2}=\text{A006645};$$
$$\left.\frac{d(D_n)}{dx}\right|_{x=1}=\text{A045925};\qquad \left.\frac{d(D_n)}{dx}\right|_{x=2}=\text{A093967}.$$

For the sequences generated by the derivatives of the other familiar polynomials studied here see: A001871, A317404, A317405, A317408, A317451, 3(A045618), 2(A006645), and 2(A093967).

Polynomial	Discriminants of GFPs	The OEIS
Fibonacci	$\mathrm{Disc}\big(F_n\big) = (-1)^{(n-2)(n-1)/2}2^{n-1}n^{n-3}$	A317403
Pell	$\mathrm{Disc}\big(P_n\big) = (-1)^{(n-2)(n-1)/2}2^{(n-1)^2}n^{n-3}$	A317450
Fermat	$\mathrm{Disc}\big(\Phi_n\big) = 2^{n(n-1)/2}3^{(n-1)(n-2)}n^{n-3}$	A318184
Chebyshev 2nd kind	$\mathrm{Disc}\big(U_n\big) = 2^{(n-1)^2}n^{n-3}$	A086804
Morgan-Voyce	$\mathrm{Disc}\big(B_n\big) = 2^{n-1}n^{n-3}$	A127670
Lucas	$\mathrm{Disc}\big(D_n\big) = (-1)^{n(n-1)/2}2^{n-1}n^n$	A193678
Pell-Lucas-prime	$\mathrm{Disc}\big(Q'_n\big) = (-1)^{n(n-1)/2}2^{(n-1)^2}n^n$	A007701
Fermat-Lucas	$\mathrm{Disc}\big(\vartheta_n\big) = 2^{(n-1)(n+2)/2}3^{n(n-1)}n^n$	A318197
Chebyshev 1st kind	$\mathrm{Disc}\big(T_n\big) = 2^{(n-1)^2}n^n$	A007701
Morgan-Voyce	$\mathrm{Disc}\big(C_n\big) = 2^{n-1}n^n$	A193678

Table 4: Discriminants of GFPs using Theorems 4 and 5.

Fibonacci-type	Derivative	Lucas-Type	Derivative
Fibonacci	$\frac{d(F_n)}{dx} = \frac{nD_n - xF_n}{4+x^2}$	Lucas	$\frac{d(D_n)}{dx} = nF_n$
Pell	$\frac{d(P_n)}{dx} = \frac{nQ_n - 2xP_n}{2(1+x^2)}$	Pell-Lucas-prime	$\frac{d(Q_n)}{dx} = 2nP_n$
Fermat	$\frac{d(\Phi_n)}{dx} = \frac{3(n\vartheta_n - 3x\Phi_n)}{-8+9x^2}$	Fermat-Lucas	$\frac{d(\vartheta_n)}{dx} = 3n\Phi_m$
Chebyshev 2nd kind	$\frac{d(U_n)}{dx} = \frac{2nT_n - 2xU_n}{2(x^2-1)}$	Chebyshev 1st kind	$\frac{d(T_n)}{dx} = nU_m$
Morgan-Voyce	$\frac{d(B_n)}{dx} = \frac{nC_n - (x+2)B_n}{x(x+4)}$	Morgan-Voyce	$\frac{d(C_n)}{dx} = nB_m$

Table 5: Derivatives of GFPs using Theorem 6.

3 Definitions, background, and basic results

In this section we give formal definitions of the concepts that we are going to use in this paper. Throughout the paper we consider polynomials in $\mathbb{Q}[x]$.

3.1 Second order polynomial sequences

In this section we introduce the generalized Fibonacci polynomial sequences. This definition gives rise to some known polynomial sequences (see, for example, Table 6 or [9, 10, 14, 22]). The polynomials in this subsection are presented in a formal way (with explicit use of "x").

For the remaining part of this section we reproduce the definitions given by Flórez et al. [9, 10] for generalized Fibonacci polynomials. We now give the two second-order polynomial recurrence relations in which we divide the generalized Fibonacci polynomials.

$$\mathcal{F}_0(x) = 0,\ \mathcal{F}_1(x) = 1,\ \text{and } \mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x) \text{ for } n \geq 2, \tag{2}$$

where $d(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$.

We say a polynomial recurrence relation is of *Fibonacci-type* if it satisfies the relation given in (2), and of *Lucas-type* if:

$$\mathcal{L}_0(x) = p_0,\ \mathcal{L}_1(x) = p_1(x),\ \text{and } \mathcal{L}_n(x) = d(x)\mathcal{L}_{n-1}(x) + g(x)\mathcal{L}_{n-2}(x) \text{ for } n \geq 2, \tag{3}$$

where $|p_0| = 1$ or 2 and $p_1(x)$, $d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$.

To use similar notation for (2) and (3) on certain occasions we write $p_0 = 0$, $p_1(x) = 1$ to indicate the initial conditions of Fibonacci-type polynomials. Some known examples of Fibonacci-type polynomials and Lucas-type polynomials are in Table 6 or in [9, 14, 18, 19, 22].

If G_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$ for all $n \geq 0$ and $d^2(x) + 4g(x) > 0$, then the explicit formula for the recurrence relations in (2) and (3) is given by

$$G_n(x) = t_1 a^n(x) + t_2 b^n(x),$$

where $a(x)$ and $b(x)$ are the solutions of the quadratic equation associated with the second-order recurrence relation $G_n(x)$. That is, $a(x)$ and $b(x)$ are the solutions of $z^2 - d(x)z - g(x) = 0$. If $\alpha = 2/p_0$, then the Binet formula for Fibonacci-type polynomials is stated in (4) and the Binet formula for Lucas-type polynomials is stated in (5) (for details on the construction of the two Binet formulas see [9])

$$\mathcal{F}_n(x) = \frac{a^n(x) - b^n(x)}{a(x) - b(x)} \tag{4}$$

and

$$\mathcal{L}_n(x) = \frac{a^n(x) + b^n(x)}{\alpha}. \tag{5}$$

Note that for both types of sequence:

$$a(x) + b(x) = d(x), \quad a(x)b(x) = -g(x), \quad \text{and} \quad a(x) - b(x) = \sqrt{d^2(x) + 4g(x)},$$

where $d(x)$ and $g(x)$ are the polynomials defined in (2) and (3).

A sequence of Lucas-type (Fibonacci-type) is *equivalent* or *conjugate* to a sequence of Fibonacci-type (Lucas-type), if their recursive sequences are determined by the same polynomials $d(x)$ and $g(x)$. Notice that two equivalent polynomials have the same $a(x)$ and $b(x)$ in their Binet representations. In [9, 10] there are examples of some known equivalent polynomials with their Binet formulas. The polynomials in Table 6 are organized by pairs of equivalent polynomials. For instance, Fibonacci and Lucas, Pell and Pell-Lucas, and so on.

Most of the following conditions were required in the papers that we are citing. Therefore, we require here that $\gcd(d(x), g(x)) = 1$ and $\deg(g(x)) < \deg(d(x))$ for both types of sequences. (For instance these conditions hold for polynomial in Table 6.) The conditions in (6) also hold for Lucas-type polynomials;

$$\gcd(p_0, p_1(x)) = 1, \gcd(p_0, d(x)) = 1, \gcd(p_0, g(x)) = 1, \text{ and that degree of } \mathcal{L}_1 \geq 1. \tag{6}$$

Notice that in the definition of Pell-Lucas we have $Q_0(x) = 2$ and $Q_1(x) = 2x$. Thus, the $\gcd(2, 2x) = 2 \neq 1$. Therefore, Pell-Lucas does not satisfy the extra conditions that we imposed in (6). So, to resolve this inconsistency we use $Q'_n(x) = Q_n(x)/2$ instead of $Q_n(x)$.

Polynomial	Initial value $G_0(x) = p_0(x)$	Initial value $G_1(x) = p_1(x)$	Recursive Formula $G_n(x) = d(x)G_{n-1}(x) + g(x)G_{n-2}(x)$
Fibonacci	0	1	$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$
Lucas	2	x	$D_n(x) = xD_{n-1}(x) + D_{n-2}(x)$
Pell	0	1	$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$
Pell-Lucas	2	$2x$	$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$
Pell-Lucas-prime	1	x	$Q'_n(x) = 2xQ'_{n-1}(x) + Q'_{n-2}(x)$
Fermat	0	1	$\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x) = 3x\vartheta_{n-1}(x) - 2\vartheta_{n-2}(x)$
Chebyshev second kind	0	1	$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
Morgan-Voyce	0	1	$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$
Morgan-Voyce	2	$x+2$	$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$
Vieta	0	1	$V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$
Vieta-Lucas	2	x	$v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$

Table 6: Recurrence relation of some GFPs.

3.2 The resultant and the discriminant

In this section we use the Sylvester determinant to define the discriminant of two polynomials. For a complete development of the theory of the resultant of polynomials see [12].

Let P and Q be non-zero polynomials of degree n and m in $\mathbb{Q}[x]$, with

$$P = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \text{and} \quad Q = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0.$$

The *resultant* of P and Q, denoted by $\operatorname{Res}(P, Q)$, is the determinant of $\operatorname{Syl}(P, Q)$ (see, for example [1, 4, 20, 28]).

$$\operatorname{Syl}(P,Q) = \begin{bmatrix} a_n & a_{n-1} & \cdots & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & \cdots & a_0 & \cdots & 0 \\ \vdots & \ddots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & a_n & \cdots & & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & \cdots & b_0 & \cdots & 0 \\ \vdots & \ddots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & b_m & b_{m-1} & \cdots & \cdots & b_0 \end{bmatrix}. \tag{7}$$

If P' is the derivative of P, then the *discriminant* of P is given by

$$\operatorname{Disc}(P) = (-1)^{\frac{n(n-1)}{2}} a_n^{-1} \operatorname{Res}(P, P').$$

Note that the discriminant can also be written as a Vandermonde determinant (see [23]).

3.3 Classic results

In this subsection we give some classic properties of the resultant needed to prove the main theorems. Most of the parts of the following lemma can be found in [4, 23].

Let f and h be polynomials, where $a_n = \text{lc}(f)$, $b_m = \text{lc}(h)$, $n = \deg(f)$ and $m = \deg(h)$. Note that if k is a constant, then $\text{Res}\big(k, f\big) = \text{Res}\big(f, k\big) = k^{\deg(f)}$. The following lemma summarize some classic result about the resultant (see, for example, Jocobs et al. [20]).

Lemma 7. *Let f, h, p, and q be polynomials in $\mathbb{Q}[x]$. If $n = \deg(f)$, $m = \deg(h)$, and $a_n = \text{lc}(f)$, then*

(i) $\text{Res}\big(f, h\big) = (-1)^{nm}\text{Res}\big(h, f\big)$,

(ii) $\text{Res}\big(f, ph\big) = \text{Res}\big(f, p\big)\text{Res}\big(f, h\big)$,

(iii) $\text{Res}\big(f, p^k\big) = \text{Res}\big(f, p\big)^k$,

(iv) if $G = fq + h$ and $r = \deg(G)$, then $\text{Res}\big(f, G\big) = a_n^{r-m}\text{Res}\big(f, h\big)$,

(v) $\text{Res}\big(f, h\big) = 0$ *if and only if f and h have a common divisor of positive degree.*

3.4 The GCD of two GFPs and other properties

Most of the results in this subsection are in [9]. Proposition 8 is a result that is in the proof of [9, Proposition 6] therefore its proof is omitted.

In this paper we use $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ to mean the set of non-negative integers and positive integers, respectively. Recall that β, λ, η, ω, and ρ are defined in (1) on page 4 and that d, g and α are defined on page 8.

Proposition 8. *Let m, n, r, and q be positive integers. If $n = mq + r$, then there is a polynomial T such that $\mathcal{F}_n = \mathcal{F}_m T + g\mathcal{F}_{mq-1}\mathcal{F}_r$.*

Proposition 9. *If m, r, and q are positive integers, then if $r < m$, then there is a polynomial T such that for $t = \left\lceil \frac{q}{2} \right\rceil$ we have*

$$\mathcal{L}_{mq+r} = \begin{cases} \mathcal{L}_m T + (-1)^{m(t-1)+t+r}(g)^{(t-1)m+r}\mathcal{L}_{m-r}, & \text{if } q \text{ is odd;} \\ \mathcal{L}_m T + (-1)^{(m+1)t}(g)^{mt}\mathcal{L}_r, & \text{if } q \text{ is even.} \end{cases}$$

Proposition 10. *If m, q, and r are nonnegative integers with $q > 0$, then*

(i)

$$\mathcal{F}_{mq+r} = \begin{cases} \alpha\mathcal{L}_m\mathcal{F}_{m(q-1)+r} - (-g)^m\mathcal{F}_{m(q-2)+r}, & \text{if } q > 1; \\ \alpha\mathcal{L}_m\mathcal{F}_r + (-g)^r\mathcal{F}_{m-r}, & \text{if } q = 1. \end{cases}$$

(ii)

$$\alpha\mathcal{L}_{mq+r} = \begin{cases}(a-b)^2\mathcal{F}_m\mathcal{F}_{m(q-1)+r} + \alpha(-g)^m\mathcal{L}_{m(q-2)+r}, & \text{if } q>1;\\ (a-b)^2\mathcal{F}_m\mathcal{F}_r + \alpha(-g)^r\mathcal{L}_{m-r}, & \text{if } q=1.\end{cases}$$

Proof. We prove Part (i), the proof of Part (ii) is similar and it is omitted. If $q = 1$, then the proof follows from [9, Proposition 3]. We now prove the case in which $q > 1$. Using Binet formulas (4) and (5) we obtain

$$\alpha\mathcal{L}_m\mathcal{F}_{m(q-1)+r} = \alpha\frac{(a^m+b^m)}{\alpha}\frac{(a^{m(q-1)+r} - b^{m(q-1)+r})}{a-b}.$$

Expanding and simplifying we have

$$\alpha\mathcal{L}_m\mathcal{F}_{m(q-1)+r} = \mathcal{F}_{mq+r} + (ab)^m\frac{a^{m(q-2)+r} - b^{m(q-2)+r}}{a-b} = \mathcal{F}_{mq+r} + (-g)^m\mathcal{F}_{m(q-2)+r}.$$

Solving this equation for $\mathcal{F}_{mq+r}$ we have $\mathcal{F}_{mq+r} = \alpha\mathcal{L}_m\mathcal{F}_{m(q-1)+r} - (-g)^m\mathcal{F}_{m(q-2)+r}$. This completes the proof. □

Lemma 11. *Let $k, n \in \mathbb{Z}_{>0}$. Then*

(i) $\deg(\mathcal{F}_k) = \eta(k-1)$ *and* $\operatorname{lc}(\mathcal{F}_k) = \beta^{k-1}$.

(ii) $\deg(\mathcal{L}_n) = \eta n$ *and* $\operatorname{lc}(\mathcal{L}_n) = \beta^n/\alpha$.

Proof. We use mathematical induction to prove all parts. We prove Part (i). Let $P(k)$ be the statement:

$$\deg(\mathcal{F}_k) = \eta(k-1) \text{ for every } k \geq 1.$$

The basis step, $P(1)$, is clear, so we suppose that $P(k)$ is true for $k = t$, where $t > 1$. Thus, we suppose that $\deg(\mathcal{F}_t) = \eta(t-1)$ and we prove $P(t+1)$. We know that $\deg(\mathcal{F}_n) \geq \deg(\mathcal{F}_{n-1})$ for $n \geq 1$. This, $\deg(d) > \deg(g)$, and (2) imply

$$\deg(\mathcal{F}_{t+1}) = \deg(d\mathcal{F}_t) = \deg(d) + \deg(\mathcal{F}_t) = \eta + \eta(t-1) = \eta t.$$

We now prove the second half of Part (i). Let $Q(k)$ be the statement:

$$\operatorname{lc}(\mathcal{F}_k) = \beta^{k-1} \text{ for every } k \geq 1.$$

The basis step, $Q(1)$, is clear, so we suppose that $Q(k)$ is true for $k = t$, where $t > 1$. Thus, we suppose that $\operatorname{lc}(\mathcal{F}_t) = \beta^{t-1}$ and we prove $Q(t+1)$. We know that $\deg(\mathcal{F}_n) \geq \deg(\mathcal{F}_{n-1})$ for $n \geq 1$. This, $\deg(d) > \deg(g)$, and (2) imply $\operatorname{lc}(\mathcal{F}_{t+1}) = \operatorname{lc}(d)\operatorname{lc}(\mathcal{F}_t) = \operatorname{lc}(d)\beta^{t-1} = \beta\beta^{t-1} = \beta^t$.

We prove Part (ii). Let $H(n)$ be the statement: $\deg(\mathcal{L}_n) = \eta n$ for every $n > 0$. It is easy to see that $H(1)$ is true. Suppose that $H(n)$ is true for some $n = k > 1$. Thus, suppose that $\deg(\mathcal{L}_k) = \eta k$ and we prove $H(k+1)$. Since $\mathcal{L}_{k+1} = d\mathcal{L}_k + g\mathcal{L}_{k-1}$ and $\deg(d) > \deg(g)$, we have $\deg(\mathcal{L}_{k+1}) = \deg(d) + \deg(\mathcal{L}_k) = \eta + \eta k = \eta(k+1)$. This proves the first half of Part (ii).

We now prove the second half of Part (ii). Let $N(n)$ be the statement: $\operatorname{lc}(\mathcal{L}_n) = \beta^n$ for every $n > 0$. (for simplicity we suppose that $\alpha = 1$). It is easy to verify that $\operatorname{lc}(\mathcal{L}_1) = \beta^1$. Suppose that $N(n)$ is true for some $n = k > 1$. Thus, suppose that $\operatorname{lc}(\mathcal{L}_k) = \beta^k$. Since $\mathcal{L}_{k+1} = d\mathcal{L}_k + g\mathcal{L}_{k-1}$ and $\deg(d) > \deg(g)$, we have $\operatorname{lc}(\mathcal{L}_{k+1}) = \operatorname{lc}(d)\operatorname{lc}(\mathcal{L}_k)$. This and the inductive hypothesis imply that $\operatorname{lc}(\mathcal{L}_{k+1}) = \beta\beta^k = \beta^{k+1}$. □

Proposition 12 plays an important role in this paper. This in connection with Lemma 7 Part (v) gives criterions to determine whether or not the resultant of two GFPs is equal to zero (see Corollary 14, Corollary 16, and 18).

Recall that definition of $\nu_2(n)$ was given in Section 2 on page 4.

Proposition 12 ([9])**.** *If $m, n \in \mathbb{Z}_{>0}$ and $\delta = \gcd(m,n)$, then*

(i) $\gcd(\mathcal{F}_m, \mathcal{F}_n) = 1$ *if and only if* $\delta = 1$.

(ii)

$$\gcd(\mathcal{L}_m, \mathcal{L}_n) = \begin{cases} \mathcal{L}_\delta, & \text{if } \nu_2(m) = \nu_2(n); \\ \gcd(\mathcal{L}_\delta, \mathcal{L}_0), & \text{otherwise.} \end{cases}$$

(iii)

$$\gcd(\mathcal{L}_n, \mathcal{F}_m) = \begin{cases} \mathcal{L}_\delta, & \text{if } \nu_2(m) > \nu_2(n); \\ 1, & \text{otherwise.} \end{cases}$$

3.5 Some resultants of GFPs of Fibonacci-type polynomials

In this subsection we give some properties of the resultant of two GFPs and some results needed to prove Theorem 1.

Proposition 13. *For m and n in $\mathbb{Z}_{\geq 0}$ these hold*

(i) *If $n > 0$, then* $\operatorname{Res}(g, \mathcal{F}_n) = \rho^{n-1}$,

(ii) $\operatorname{Res}(\mathcal{F}_m, g\mathcal{F}_n) = (-1)^{\omega\eta(m-1)}\rho^{m-1}\operatorname{Res}(\mathcal{F}_m, \mathcal{F}_n)$,

(iii) $\operatorname{Res}(\mathcal{L}_m, g\mathcal{L}_n) = (-1)^{\omega\eta m}\rho^{m}\operatorname{Res}(\mathcal{L}_m, \mathcal{L}_n)$.

Proof. We prove Part (i) using mathematical induction. Let $P(n)$ be the statement:

$$\operatorname{Res}(g, \mathcal{F}_n) = \rho^{n-1} \text{ for every } n \geq 1.$$

Since $\mathcal{F}_1 = 1$ the basis step, $P(1)$, is clear. Suppose that $P(n)$ is true for $n = k$, where $k > 1$. Thus, suppose that $\operatorname{Res}(g, \mathcal{F}_k) = \rho^{k-1}$, and we prove $P(k+1)$. From (2) and Lemma 7 Parts (ii) and (iv) we have

$$\operatorname{Res}(g, \mathcal{F}_{k+1}) = \operatorname{Res}(g, d\mathcal{F}_k + g\mathcal{F}_{k-1}) = \lambda^{\eta k - (\eta + \eta(k-1))}\operatorname{Res}(g, d)\operatorname{Res}(g, \mathcal{F}_k).$$

This and $P(k)$ imply $\text{Res}(g, \mathcal{F}_n) = \text{Res}(g, d)\text{Res}(g, \mathcal{F}_{n-1}) = \rho^{n-1}$.

We prove Part (ii), the proof of Part (iii) is similar and it is omitted. From Lemma 7 Part (i) and Part (ii), and Lemma 11 Part (i), we have

$$\begin{aligned}
\text{Res}(\mathcal{F}_m, g\mathcal{F}_n) &= \text{Res}(\mathcal{F}_m, g)\text{Res}(\mathcal{F}_m, \mathcal{F}_n) \\
&= (-1)^{\omega\eta(m-1)}\text{Res}(g, \mathcal{F}_m)\text{Res}(\mathcal{F}_m, \mathcal{F}_n) \\
&= (-1)^{\omega\eta(m-1)}\rho^{m-1}\text{Res}(\mathcal{F}_m, \mathcal{F}_n).
\end{aligned}$$

This completes the proof. □

The proof of the following corollary is straightforward from Proposition 12 Part (i) and Lemma 7 Part (v).

Corollary 14. *Let $m, n \in \mathbb{Z}_{>0}$. Then $\gcd(m, n) = 1$ if and only if $\text{Res}(\mathcal{F}_m, \mathcal{F}_n) \neq 0$.*

Proposition 15. *If m, n and q are positive integers, with $n > 1$ and $mq > 1$, then*

(i) $\text{Res}(\mathcal{F}_n, \mathcal{F}_{n-1}) = ((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho)^{(n-2)(n-1)/2}$,

(ii) $\text{Res}(\mathcal{F}_m, \mathcal{F}_{mq-1}) = ((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{(m-1)(mq-2)/2}$.

Proof. We prove all parts by mathematical induction. Proof of Part (i). Let $Q(n)$ be the statement:

$$\text{Res}(\mathcal{F}_n, \mathcal{F}_{n-1}) = \left((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\right)^{(n-2)(n-1)/2} \text{ for every } n \geq 2.$$

Since $\mathcal{F}_1 = 1$, the basis step, $Q(2)$, is clear. Suppose that $Q(n)$ is true for $n = k-1$, where $k > 2$. Thus, suppose that $\text{Res}(\mathcal{F}_{k-1}, \mathcal{F}_{k-2}) = ((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho)^{(k-3)(k-2)/2}$. We prove $Q(k)$. Using Lemma 11 Part (i) and Lemma 7 Part (i) we get

$$\begin{aligned}
\text{Res}(\mathcal{F}_n, \mathcal{F}_{n-1}) &= (-1)^{\eta^2(n-1)(n-2)}\text{Res}(\mathcal{F}_{n-1}, \mathcal{F}_n) \\
&= \text{Res}(\mathcal{F}_{n-1}, d\mathcal{F}_{n-1} + g\mathcal{F}_{n-2}).
\end{aligned}$$

This, Lemma 7 Part (iv), Lemma 11 Part (i) and Proposition 13 Part (ii) imply

$$\begin{aligned}
\text{Res}(\mathcal{F}_n, \mathcal{F}_{n-1}) &= (\beta^{n-2})^{\eta(n-1)-(\omega+\eta(n-3))}\text{Res}(\mathcal{F}_{n-1}, g\mathcal{F}_{n-2}) \\
&= (-1)^{\omega\eta(n-2)}\beta^{(n-2)(2\eta-\omega)}\rho^{n-2}\text{Res}(\mathcal{F}_{n-1}, \mathcal{F}_{n-2}).
\end{aligned}$$

Simplifying we have

$$\text{Res}(\mathcal{F}_n, \mathcal{F}_{n-1}) = \left((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\right)^{n-2}\text{Res}(\mathcal{F}_{n-1}, \mathcal{F}_{n-2}).$$

This and $Q(k-1)$ give

$$\begin{aligned}
\text{Res}(\mathcal{F}_n, \mathcal{F}_{n-1}) &= \left((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\right)^{n-2}\left(\beta^{2\eta-\omega}(-1)^{\omega\eta}\rho\right)^{\frac{(n-3)(n-2)}{2}} \\
&= \left((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\right)^{\frac{(n-2)(n-1)}{2}}.
\end{aligned}$$

Proof of Part (ii). The proof is straightforward when $m=1$ and $q>1$. Therefore, we suppose that $m>1$. Let $W(q)$ be the statement: for a fixed integer m we have

$$\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{mq-1}\big)=\big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{(m-1)(mq-2)/2}\ \text{ for every } q\geq 1.$$

From Part (i) it follows that $W(1)$ is true. Suppose that $W(q)$ is true for $q=k$, where $k>1$. Thus, suppose that $\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{mk-1}\big)=((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{(m-1)(mk-2)/2}$. We prove $W(k+1)$. From Proposition 8 we know that there is a polynomial T such that $\mathcal{F}_{m(k+1)-1}=\mathcal{F}_mT+g\mathcal{F}_{km-1}\mathcal{F}_{m-1}$. This, Lemma 7 Part (ii) and Part (iv) and Proposition 13 Part (ii) imply

$$\begin{aligned}
\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{m(k+1)-1}\big) &= \operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_mT+g\mathcal{F}_{km-1}\mathcal{F}_{m-1}\big)\\
&= \big(\beta^{m-1}\big)^{\eta(mk+m-2)-(\omega+\eta(mk+m-4))}\operatorname{Res}\big(\mathcal{F}_m,g\mathcal{F}_{km-1}\mathcal{F}_{m-1}\big)\\
&= \beta^{(m-1)(2\eta-\omega)}\operatorname{Res}\big(\mathcal{F}_m,g\mathcal{F}_{km-1}\mathcal{F}_{m-1}\big)\\
&= (-1)^{\omega\eta(m-1)}\beta^{(m-1)(2\eta-\omega)}\rho^{m-1}\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{m-1}\big)\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{km-1}\big).
\end{aligned}$$

From this, Part (i), and $W(k)$ we conclude

$$\begin{aligned}
\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{mk+m-1}\big) &= \big((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\big)^{m-1}\big((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-1)(m-2)}{2}}\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{km-1}\big)\\
&= \big((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-1)m}{2}}\operatorname{Res}\big(\mathcal{F}_m,\mathcal{F}_{km-1}\big)\\
&= \big((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-1)m}{2}}\big((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-1)(mk-2)}{2}}.
\end{aligned}$$

Simplifying the last expression, we have

$$ress\mathcal{F}_m\mathcal{F}_{mk+m-1}=\big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-1)(m(k+1)-2)}{2}}.$$

This completes the proof. □

3.6 Some resultants of GFPs of Lucas-type polynomials

In this subsection we give some properties of the resultant of two GFPs of Lucas-type and some results needed to prove Theorem 2.

Recall that a GFP of Lucas-type is a polynomial sequence such that $\mathcal{L}_0\in\{1,2\}$, $\mathcal{L}_1=2^{-1}\mathcal{L}_0d$, and $\mathcal{L}_n=d\mathcal{L}_{n-1}+g\mathcal{L}_{n-2}$ for $n>1$.

Note that if we take the particular case of the Lucas-type sequence $\mathcal{L}_n$ in which $\mathcal{L}_0=1$ and $\mathcal{L}_1=2^{-1}d$, then using the initial conditions we define a new Lucas-type sequence as follows: Let $\overline{\mathcal{L}_0}=2\mathcal{L}_0$, $\overline{\mathcal{L}_1}=2\mathcal{L}_1=d$ and $\overline{\mathcal{L}_n}=d\overline{\mathcal{L}_{n-1}}+g\overline{\mathcal{L}_{n-2}}$ for $n>1$. It is easy to verify that $\overline{\mathcal{L}_n}=2\mathcal{L}_n$ for $n\geq 0$. Therefore, to find the resultant of a polynomial of Lucas-type $\mathcal{L}_n$, it is enough to find the resultant for $\mathcal{L}_n$ in which $\mathcal{L}_0=2$.

The following corollary is a direct consequence of Proposition 12 Part (ii), Lemma 7 Part (v), and Lemma 11 Part (ii). So, we omit its proof.

Corollary 16. *Let $m, n \in \mathbb{Z}_{>0}$. $\nu_2(n) = \nu_2(m)$ if and only if* $\text{Res}\big(\mathcal{L}_m, \mathcal{L}_n\big) = 0$.

Proposition 17. *If $n \in \mathbb{Z}_{>0}$ and $\mathcal{L}_0 = 2$, then*

(i) $\text{Res}\big(g, \mathcal{L}_n\big) = \rho^n$,

(ii)

$$\text{Res}\big(\mathcal{L}_1, \mathcal{L}_n\big) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 2^\eta \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We prove Part (i) by mathematical induction on n. Since $\text{Res}\big(g, \mathcal{L}_1\big) = \text{Res}\big(g, d\big) = \rho$, it holds that the result is true for $n = 1$. Suppose that for some integer $n = k > 1$, $\text{Res}\big(g, \mathcal{L}_k\big) = \rho^k$ holds. From (3) and Lemma 7 Parts (ii) and (iv) we have

$$\text{Res}\big(g, \mathcal{L}_{k+1}\big) = \text{Res}\big(g, d\mathcal{L}_k + g\mathcal{L}_{k-1}\big) = \alpha^{\eta(k+1)-\eta(k+1)}\text{Res}\big(g, d\big)\text{Res}\big(g, \mathcal{L}_k\big).$$

This and the inductive hypothesis imply that

$$\text{Res}\big(g, \mathcal{L}_{k+1}\big) = \text{Res}\big(g, d\big)\text{Res}\big(g, \mathcal{L}_k\big) = \text{Res}\big(g, d\big)^{k+1},$$

which is our claim.

We prove Part (ii) by induction on n. Let $Q(n)$ be the statement:

$$\text{Res}\big(\mathcal{L}_1, \mathcal{L}_n\big) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 2^\eta \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Since $\text{Res}\big(\mathcal{L}_1, \mathcal{L}_1\big) = 0$, $Q(1)$ holds. Note that $\mathcal{L}_1 = (p_0/2)d = d$. This and Lemma 7 Parts (ii) and (iv) imply that

$$\text{Res}\big(\mathcal{L}_1, \mathcal{L}_2\big) = \text{Res}\big(\mathcal{L}_1, d\mathcal{L}_1 + g\mathcal{L}_0\big) = \beta^{2\eta-\omega}\text{Res}\big(d, 2g\big) = 2^\eta(-1)^{\eta\omega}\beta^{2\eta-\omega}\rho.$$

This proves $Q(2)$.

Suppose that $Q(k-2)$ and $Q(k-1)$ is true and we prove $Q(k)$. Note that if k is odd by Corollary 16 we have that $\text{Res}\big(\mathcal{L}_1, \mathcal{L}_n\big) = 0$. We suppose that k is even. Lemma 7 Parts (ii) and (iv), $\mathcal{L}_1 = d$, Lemma 11 Part (ii), and (5) imply

$$\begin{aligned} \text{Res}\big(\mathcal{L}_1, \mathcal{L}_k\big) &= \text{Res}\big(d, d\mathcal{L}_{k-1} + g\mathcal{L}_{k-2}\big) \\ &= \beta^{2\eta-\omega}\text{Res}\big(d, g\mathcal{L}_{k-2}\big) \\ &= \beta^{2\eta-\omega}\text{Res}\big(d, g\big)\text{Res}\big(d, \mathcal{L}_{k-2}\big) \\ &= \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)\text{Res}\big(\mathcal{L}_1, \mathcal{L}_{k-2}\big). \end{aligned}$$

Note that $k-2$ and k have the same parity. This and $Q(k-2)$ imply that

$$\text{Res}\big(\mathcal{L}_1, \mathcal{L}_k\big) = \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right) 2^\eta \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{k-2}{2}} = 2^\eta \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{k}{2}}.$$

This proves $Q(k)$. ∎

4 Proof of Theorem 1

In this section we prove the theorem about the resultant of two GFPs of Fibonacci-type (see Section 2).

Proof of Theorem 1. Let A be the set of all $i \in \mathbb{Z}_{>0}$ such that for every $j \in \mathbb{Z}_{>0}$ we have

$$\operatorname{Res}\big(\mathcal{F}_i, \mathcal{F}_j\big) = \begin{cases} 0, & \text{if } \gcd(m,n) > 1; \\ \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(i-1)(j-1)}{2}}, & \text{otherwise.} \end{cases} \tag{8}$$

Since $1 \in A$, we have that $A \neq \emptyset$. The following claim completes the proof of the Theorem.

Claim. $A = \mathbb{Z}_{>0}$.

Proof of Claim. Suppose $B := \mathbb{Z}_{>0} \setminus A$ is a non-empty set. Let $n \neq 1$ be the least element of B. So, there is $h \in \mathbb{Z}_{>0}$ such that $\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_h\big)$ does not satisfy Property (8) (if $m = n$, then $\operatorname{Res}\big(\mathcal{F}_i, \mathcal{F}_j\big) = 0$). Let m be the least element of the non-empty set $H = \{h \in \mathbb{Z}_{>0} \mid \operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_h\big)$ does not satisfy (8)$\}$. Note that Corollary 14 and (8) imply that $\gcd(m,n) = 1$. We now consider two cases.

Case $m < n$**.** Since n is the minimum element of B, $m \in A$. Either m or n is odd, because $\gcd(m,n) = 1$. We know, from Lemma 11 Part (i), that $\deg(\mathcal{F}_m) = \eta(m-1)$. This implies that $\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_m\big) = \operatorname{Res}\big(\mathcal{F}_m, \mathcal{F}_n\big)$. Since $m \in A$, we have that (8) holds for $j \in \mathbb{Z}_{>0}$, in particular (8) holds when $j = n$. That is a contradiction.

Case $n < m$**.** The Euclidean algorithm and $\gcd(m,n) = 1$ guarantee that there are $q, r \in \mathbb{Z}$ such that $m = nq + r$ with $0 < r < n$. We now can proceed analogously to the proof of Proposition 15 Part (ii). From the Euclidean algorithm, Proposition 8 and Lemma 7 Part (iv) we have

$$\begin{aligned} \operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_m\big) &= \operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_{nq+r}\big) \\ &= \operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_n T + g\mathcal{F}_{nq-1}\mathcal{F}_r\big) \\ &= \left(\beta^{n-1}\right)^{\eta(m-1)-(\omega+\eta(nq-2+r-1))}\operatorname{Res}\big(\mathcal{F}_n, g\mathcal{F}_{nq-1}\mathcal{F}_r\big). \end{aligned}$$

This, Lemma 7 Part (ii) and Proposition 13 Part (ii) imply

$$\begin{aligned} \operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_m\big) &= \beta^{(n-1)(2\eta-\omega)}\operatorname{Res}\big(\mathcal{F}_n, g\mathcal{F}_r\big)\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_{nq-1}\big) \\ &= (-1)^{\omega\eta(n-1)}\beta^{(n-1)(2\eta-\omega)}\rho^{n-1}\left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(n-1)(nq-2)}{2}}\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_r\big) \\ &= \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(n-1)nq}{2}}\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_r\big). \end{aligned} \tag{9}$$

Since $\gcd(n,m) = \gcd(n,r) = 1$, either n or r is odd. So, $\operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_r\big) = \operatorname{Res}\big(\mathcal{F}_r, \mathcal{F}_n\big)$. It is easy to verify that $r \in A$, because $r < n$ and n is the minimum element of B. Set $j = n$, so $\operatorname{Res}\big(\mathcal{F}_r, \mathcal{F}_n\big) = ((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{(n-1)(r-1)/2}$. This and (9) imply that

$$\begin{aligned} \operatorname{Res}\big(\mathcal{F}_n, \mathcal{F}_m\big) &= \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(m-1)(n-r)}{2}}\left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(r-1)(m-1)}{2}} \\ &= \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{\frac{(n-1)(m-1)}{2}}. \end{aligned}$$

That is a contradiction. This implies that $A = \mathbb{Z}_{>0}$. $\square$

5 Proof of Theorem 2

In this section we prove the theorem about the resultant of two GFPs of Lucas-type (see Section 2).

Proof of Theorem 2. We consider two cases: $\mathcal{L}_0 = 2$ and $\mathcal{L}_0 = 1$. We first prove the case $\mathcal{L}_0 = 2$. Let $A = \{i \in \mathbb{Z}_{>0} \mid \forall j \in \mathbb{Z}_{>0}, \text{Property (10) holds for } \operatorname{Res}\big(\mathcal{L}_i, \mathcal{L}_j\big)\}$.

$$\operatorname{Res}\big(\mathcal{L}_i, \mathcal{L}_j\big) = \begin{cases} 0, & \text{if } \nu_2(i) = \nu_2(j); \\ 2^{\eta \gcd(i,j)} \left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{ij/2}, & \text{if } \nu_2(i) \neq \nu_2(j). \end{cases} \tag{10}$$

From Proposition 17 Part (ii) we have that $i = 1 \in A$. So, $A \neq \emptyset$. The following claim completes the proof part $\mathcal{L}_0 = 2$.

Claim. $A = \mathbb{Z}_{>0}$.

Proof of Claim. Suppose $B := \mathbb{Z}_{>0} \backslash A$ is a non-empty set. Let $n \neq 1$ be the least element of B. So, there is $h \in \mathbb{Z}_{>0}$ such that $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_h\big)$ does not satisfy Property (10). Let m be the least element of the non-empty set $H = \{h \in \mathbb{Z}_{>0} \mid \operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_h\big) \text{ does not satisfy Property (10)}\}$. Note that if $\nu_2(n) = \nu_2(m)$, then $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_m\big) = 0$ (by Corollary 16). That is a contradiction by the definition of H. Therefore, we have that $\nu_2(n) \neq \nu_2(m)$. So, $n \neq m$, where at least one of them is even. This implies that $\operatorname{Res}\big(\mathcal{L}_m, \mathcal{L}_n\big) = (-1)^{\eta^2 mn}\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_m\big) = \operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_m\big)$. Therefore, $\operatorname{Res}\big(\mathcal{L}_m, \mathcal{L}_n\big)$ does not satisfy (10). So, $m \notin A$. Since $n \neq m$ is the least element of B, we have $m > n$. From the Euclidean algorithms we know that there are $q, r \in \mathbb{Z}_{\geq 0}$ such that $m = nq + r$ with $0 \leq r < n$.

We now proceed by cases over q.

Case q odd Suppose that $q = 2t - 1$. Note that $t = \lceil q/2 \rceil$ and that $(m - n + r)/2 = (t-1)n + r$. Since $\nu_2(n) \neq \nu_2(m)$, $r \neq 0$ and $n(n-r)$ is even. This, Proposition 9 for odd case, and Lemma 7 Part (ii) and Part (iv) imply that $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_m\big)$ equals

$$\begin{aligned} \operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_{nq+r}\big) &= \operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_n T + (-1)^{t(n+1)+r-n} g^{(t-1)n+r}\mathcal{L}_{n-r}\big) \\ &= \operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_n T + (-1)^{t(n+1)+r-n} g^{\frac{m-n+r}{2}}\mathcal{L}_{n-r}\big) \\ &= (\beta^n)^{\eta m - \frac{\omega(m-n+r)}{2} - \eta(n-r)}\operatorname{Res}\big(\mathcal{L}_n, (-1)^{t(n+1)+r-n} g^{\frac{m-n+r}{2}}\mathcal{L}_{n-r}\big) \\ &= \beta^{\frac{n(m-n+r)(2\eta-\omega)}{2}}\operatorname{Res}\big(\mathcal{L}_n, (-1)^{t(n+1)+r-n} g^{\frac{m-n+r}{2}}\mathcal{L}_{n-r}\big). \end{aligned}$$

Note that $\operatorname{Res}\big(\mathcal{L}_n, (-1)^{t(n+1)+r-n}\big) = (-1)^{\eta n(t(n+1)+r-n)} = (-1)^{\eta n(r-n)} = 1$. This, Lemma 7 Parts (iii) and (iv) and Proposition 17 Part (i) imply

$$\begin{aligned} \operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_m\big) &= \beta^{\frac{n(m-n+r)(2\eta-\omega)}{2}}\operatorname{Res}\big(\mathcal{L}_n, g^{\frac{m-n+r}{2}}\mathcal{L}_{n-r}\big) \\ &= \beta^{\frac{n(m-n+r)(2\eta-\omega)}{2}}\operatorname{Res}\big(\mathcal{L}_n, g\big)^{\frac{n-m+r}{2}}\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_{n-r}\big) \\ &= \beta^{\frac{n(m-n+r)(2\eta-\omega)}{2}}\left((-1)^{\eta n\omega}\rho^n\right)^{\frac{m-n+r}{2}}\operatorname{Res}\big(\mathcal{L}_n, \mathcal{L}_{n-r}\big). \end{aligned}$$

Thus,

$$\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big) = \big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-n+r)}{2}n}\operatorname{Res}\big(\mathcal{L}_{n-r},\mathcal{L}_n\big). \tag{11}$$

Since $n(n-r)$ is even, we have that $\operatorname{Res}\big(\mathcal{L}_{n-r},\mathcal{L}_n\big) = \operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_{n-r}\big)$. This and $m > n-r$ imply that $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_{n-r}\big)$ satisfies (10). From this and (11) we have

$$\begin{aligned}\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big) &= \big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{(m-n+r)}{2}n} 2^{\eta\gcd(n-r,n)}\big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{n(n-r)}{2}}\\ &= 2^{\eta\gcd(n,m)}\big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{nm}{2}}.\end{aligned}$$

Thus, $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big)$ satisfies (10). That is a contradiction.

Case q is even Let $q = 2t$. Note that $t = \lceil q/2 \rceil$. Using Proposition 9 for the even case, Lemma 7 Parts (ii) and (iv) and following a similarly procedure as in the proof of the case $q = 2t+1$ we obtain $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big) = \beta^{(2\eta-\omega)(m-r)n/2}\operatorname{Res}\big(\mathcal{L}_n,(-1)^{(n+1)t}\big)\operatorname{Res}\big(\mathcal{L}_n,g^{nt}\mathcal{L}_r\big)$. This, the fact that $\operatorname{Res}\big(\mathcal{L}_n,(-1)^{(n+1)t}\big) = 1$ and following a similar procedure as in the proof of the case $q = 2t+1$ we obtain that $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big) = ((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{(m-r)n/2}\operatorname{Res}\big(\mathcal{L}_r,\mathcal{L}_n\big)$. Since $r < n$, we have $r \notin B$. Therefore, $r \in A$. This implies that

$$\begin{aligned}\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big) &= ((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{\frac{(m-r)n}{2}}\operatorname{Res}\big(\mathcal{L}_r,\mathcal{L}_n\big)\\ &= ((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{\frac{(m-r)n}{2}}\big((-1)^{\eta\omega}2^{\eta\gcd(r,n)}\beta^{2\eta-\omega}\rho\big)^{\frac{nr}{2}}\\ &= 2^{\eta\gcd(n,m)}\big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{mn}{2}}.\end{aligned}$$

Thus, $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{L}_m\big)$ satisfies (10). That is a contradiction. This completes the proof that $A = \mathbb{Z}_{>0}$.

We now prove the case $\mathcal{L}_0 = 1$. It is easy to see that $\overline{\mathcal{L}_n} = 2\mathcal{L}_n$ is a GFP sequence of Lucas-type, where $\overline{\mathcal{L}_0} = 2$. This and the previous case imply

$$\operatorname{Res}\big(\overline{\mathcal{L}_m},\overline{\mathcal{L}_n}\big) = \begin{cases}0, & \text{if } \nu_2(n) = \nu_2(m);\\ 2^{\eta\gcd(m,n)}\big((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\big)^{\frac{nm}{2}}, & \text{if } \nu_2(n) \neq \nu_2(m).\end{cases}$$

Since $\operatorname{Res}\big(\overline{\mathcal{L}_m},\overline{\mathcal{L}_n}\big) = \operatorname{Res}\big(2\mathcal{L}_m,2\mathcal{L}_n\big)$, we have

$$\operatorname{Res}\big(\overline{\mathcal{L}_m},\overline{\mathcal{L}_n}\big) = \operatorname{Res}\big(2,\mathcal{L}_n\big)\operatorname{Res}\big(\mathcal{L}_m,2\big)\operatorname{Res}\big(\mathcal{L}_m,\mathcal{L}_n\big) = 2^{(n+m)\eta}\operatorname{Res}\big(\mathcal{L}_m,\mathcal{L}_n\big).$$

Therefore, $\operatorname{Res}\big(\mathcal{L}_m,\mathcal{L}_n\big) = 2^{-(n+m)\eta}\operatorname{Res}\big(\overline{\mathcal{L}_m},\overline{\mathcal{L}_n}\big)$, completing the proof. □

6 Proof of Theorem 3

In this section we prove the theorem about the resultant of two equivalent polynomials (see Section 2).

The following corollary is a consequence of Proposition 12 Part (iii). So, we omit its proof.

Corollary 18. *Let $m, n \in \mathbb{Z}_{>0}$. $\nu_2(n) < \nu_2(m)$ if and only if* $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = 0$.

Proof of Theorem 3. We consider two cases: $\alpha = 1$ and $\alpha = 2$. We prove the case $\alpha = 1$, the case $\alpha = 2$ is similar and it is omitted. Let

$$A = \{i \in \mathbb{Z}_{>0} \mid \forall j \in \mathbb{Z}_{>0},\ \text{Property (12) holds for } \operatorname{Res}\big(\mathcal{L}_j, \mathcal{F}_i\big)\}.$$

$$\operatorname{Res}\big(\mathcal{L}_j, \mathcal{F}_i\big) = \begin{cases} 0, & \text{if } \nu_2(j) < \nu_2(i); \\ 2^{\eta \gcd(i,j) - \eta}\left((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right)^{(j(i-1))/2}, & \text{if } \nu_2(j) \geq \nu_2(i). \end{cases} \tag{12}$$

Since $\operatorname{Res}\big(\mathcal{L}_j, 1\big) = 1$, we have $i = 1 \in A$. So, $A \neq \emptyset$. The following claim completes the proof.

Claim. $A = \mathbb{Z}_{>0}$.

Proof of Claim. Suppose $B := \mathbb{Z}_{>0} \setminus A$ is a non-empty set. Let $m \neq 1$ be the least element of B. So, there is $h \in \mathbb{Z}_{>0}$ such that $\operatorname{Res}\big(\mathcal{L}_h, \mathcal{F}_m\big)$ does not satisfy (12). Let n be the least element of the non-empty set $H = \{h \in \mathbb{Z}_{>0} \mid \operatorname{Res}\big(\mathcal{L}_h, \mathcal{F}_m\big) \text{ does not satisfy (12)}\}$.

If $\nu_2(n) < \nu_2(m)$, then by Corollary 18 it holds $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = 0$. This and (12) imply that $m \in A$. That is a contradiction. Let us suppose that $\nu_2(n) \geq \nu_2(m)$.

We now analyze cases on m.

Case $m = n$**.** Note that $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_n\big) = \operatorname{Res}\big(\mathcal{F}_n, \mathcal{L}_n\big)$. From Proposition 10 Part (i) with $r = q = \alpha = 1$ (if $\alpha \neq 1$ is similar) we have $\mathcal{L}_n = \mathcal{F}_{n+1} + g\mathcal{F}_{n-1} = d\mathcal{F}_n + 2g\mathcal{F}_{n-1}$. Therefore, $\operatorname{Res}\big(\mathcal{F}_n, \mathcal{L}_n\big)$ equals

$$\begin{aligned} \operatorname{Res}\big(\mathcal{F}_n, d\mathcal{F}_n + 2g\mathcal{F}_{n-1}\big) &= (\beta^{n-1})^{\eta n - (\omega + \eta(n-2))}\operatorname{Res}\big(\mathcal{F}_n, 2g\mathcal{F}_{n-1}\big) \\ &= \beta^{(n-1)(2-\omega)} 2^{\eta(n-1)}\operatorname{Res}\big(\mathcal{F}_n, g\mathcal{F}_{n-1}\big). \end{aligned}$$

By Lemma 7 Part (ii) and Proposition 15 Part (i), we have

$$\begin{aligned} \operatorname{Res}\big(\mathcal{F}_n, d\mathcal{F}_n + 2g\mathcal{F}_{n-1}\big) &= 2^{\eta(n-1)}(-1)^{\omega\eta(n-1)}(\beta^{(n-1)(2\eta-\omega)})\rho^{n-1}\left((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\right)^{\frac{(n-2)(n-1)}{2}}. \\ &= 2^{\eta(n-1)}\left((-1)^{\omega\eta}\beta^{2\eta-\omega}\rho\right)^{\frac{n(n-1)}{2}}. \end{aligned}$$

So, $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_n\big)$ satisfies Property (12). That is contradiction. Therefore $m \neq n$.

Case $m > n$**.** From the Euclidean algorithm we know that $m = nq + r$ for $0 \leq r < n$. We consider two sub-cases on q.

Sub-case $q = 1$**.** Note that $0 < r < n$. So, $m = n + r$ and $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = \operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_{n+r}\big)$. This and Proposition 10 Part (i) imply that $\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = \operatorname{Res}\big(\mathcal{L}_n, \alpha\mathcal{L}_n\mathcal{F}_r + (-g)^r\mathcal{F}_{n-r}\big)$. Therefore,

$$\begin{aligned} \operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) &= \beta^{n\eta(n+r-1) - (\omega r + \eta(n-r-1))}\operatorname{Res}\big(\mathcal{L}_n, (-g)^r\mathcal{F}_{n-r}\big) \\ &= \beta^{n(2\eta-\omega)r}\operatorname{Res}\big(\mathcal{L}_n, (-g)^r\mathcal{F}_{n-r}\big) \\ &= \beta^{n(2\eta-\omega)r}\operatorname{Res}\big(\mathcal{L}_n, (-g)^r\big)\operatorname{Res}\big(\mathcal{L}_n, \mathcal{F}_{n-r}\big). \end{aligned}$$

This and Lemma 7 Part (iii) imply

$$\begin{aligned}\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) &= \beta^{n(2\eta-\omega)r}\left(\operatorname{Res}\big(\mathcal{L}_n,-g\big)\right)^r\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n-r}\big)\\ &= \beta^{n(2\eta-\omega)r}\left(\operatorname{Res}\big(\mathcal{L}_n,-1\big)\right)^r\left(\operatorname{Res}\big(\mathcal{L}_n,g\big)\right)^r\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n-r}\big)\\ &= \beta^{n(2\eta-\omega)r}(-1)^{r\eta n}(-1)^{\eta n\omega r}\operatorname{Res}\big(g,\mathcal{L}_n\big)^r\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n-r}\big).\end{aligned}$$

Therefore,

$$\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) = (-1)^{\eta nr(\omega+1)}\beta^{n(2\eta-\omega)r}\rho^{nr}\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n-r}\big). \tag{13}$$

Since $m > n-r$ is the least element of B, we have $n-r\in A$. Therefore, it holds $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n-r}\big) = 2^{\eta(\gcd(n,n-r)-1)}\left((-1)^{\eta\omega}\beta^{2\eta-w}\rho\right)^{n(n-r-1)/2}$. This and (13) (after simplifications) imply that

$$\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) = 2^{\eta(\gcd(n,n-r)-1)}\left[\beta^{2\eta-\omega}(-1)^{\eta\omega}\rho\right]^{\frac{n(n+r-1)}{2}}.$$

Therefore, $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big)$ satisfies Property (12). That is a contradiction. Thus, $m\neq n+r$.

Sub-case $q>1$**.** Since $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) = \operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{nq+r}\big)$, by Proposition 10 Part (i) we have $\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) = \operatorname{Res}\big(\mathcal{L}_n,\alpha\mathcal{L}_n\mathcal{F}_{n(q-1)+r}-(-g)^n\mathcal{F}_{n(q-2)+r}\big)$. This, Lemma 7 Part (iv) and the fact that $n\eta(nq+r-1)-(\omega n+\eta(n(q-2)+r-1)) = (2\eta-\omega)n^2$, imply that

$$\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) = \beta^{n(2\eta-\omega)n}\operatorname{Res}\big(\mathcal{L}_n,-(-g)^n\mathcal{F}_{n(q-2)+r}\big).$$

So, by Lemma 7 we have

$$\begin{aligned}\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) &= \beta^{(2\eta-\omega)n^2}\operatorname{Res}\big(\mathcal{L}_n,-1\big)\operatorname{Res}\big(\mathcal{L}_n,(-g)^n\big)\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n(q-2)+r}\big)\\ &= \beta^{(2\eta-\omega)n^2}(-1)^{\eta n}\left(\operatorname{Res}\big(\mathcal{L}_n,-g\big)\right)^n\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n(q-2)+r}\big)\\ &= \beta^{(2\eta-\omega)n^2}(-1)^{\eta n}(-1)^{\eta n^2}\left(\operatorname{Res}\big(\mathcal{L}_n,g\big)\right)^n\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n(q-2)+r}\big)\\ &= \beta^{(2\eta-\omega)n^2}(-1)^{\eta n(n+1)}\left(\operatorname{Res}\big(\mathcal{L}_n,g\big)\right)^n\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n(q-2)+r}\big)\\ &= \beta^{(2\eta-\omega)n^2}\left(\operatorname{Res}\big(\mathcal{L}_n,g\big)\right)^n\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n(q-2)+r}\big).\end{aligned}$$

Since m is the least element of B and $n(q-2)+r<m$, we have that $n(q-2)+r\in A$. This and $\nu_2(n)\geq\nu_2(m)=\nu_2(n(q-2)+r)$ imply that

$$\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_{n(q-2)+r}\big) = 2^{\eta(\gcd(n,n(q-2)+r)-1)}\left[(-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right]^{\frac{n(n(q-2)+r-1)}{2}}.$$

Note that $\gcd(n,n(q-2)+r)=\gcd(n,nq+r)$. Therefore,

$$\begin{aligned}\operatorname{Res}\big(\mathcal{L}_n,\mathcal{F}_m\big) &= \beta^{(2\eta-\omega)n^2}(-1)^{\eta n\omega}\rho^{n^2}2^{\eta(\gcd(n,n(q-2)+r)-1)}\left[(-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right]^{\frac{n(n(q-2)+r-1)}{2}}\\ &= \beta^{\frac{(2\eta-\omega)n(nq+r-1)}{2}}(-1)^{\frac{\eta\omega n(nq+r-1)}{2}}\rho^{\frac{n(nq+r-1)}{2}}2^{\eta(\gcd(n,nq+r)-1)}\\ &= 2^{\eta(\gcd(n,nq+r)-1)}\left[(-1)^{\eta\omega}\beta^{2\eta-\omega}\rho\right]^{\frac{n(nq+r-1)}{2}}.\end{aligned}$$

From this we conclude that $\text{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = 2^{\eta(\gcd(n,m)-1)}\left[\beta^{2\eta-\omega}(-1)^{\eta\omega}\rho\right]^{\frac{n(m-1)}{2}}$. Therefore, $m \in A$. That is is a contradiction.

Case $m < n$**.** From the Euclidean algorithm we know that $n = mq + r$ for $0 \le r < m$. There are two sub-cases to consider, $q = 1$ and $q > 1$. However, we prove only the case in which $q = 1$, the other case is similar and it is omitted.

Sub-case $q = 1$**.** In this case $r \neq 0$, for $r = 0$ see the case $m = n$. Therefore, $\text{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = \text{Res}\big(\mathcal{L}_{m+r}, \mathcal{F}_m\big)$. This, Proposition 10 Part (ii) and Lemma 7 Parts (i) imply that

$$\begin{aligned}
\text{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) &= \text{Res}\big(\left((a-b)^2/\alpha\right)\mathcal{F}_m\mathcal{F}_r + (-g)^r\mathcal{L}_{m-r}, \mathcal{F}_m\big) \\
&= (-1)^{\eta^2(m+r)(m-1)}\text{Res}\Big(\mathcal{F}_m, \frac{(a-b)^2}{\alpha}\mathcal{F}_m\mathcal{F}_r + (-g)^r\mathcal{L}_{m-r}\Big).
\end{aligned}$$

Note that $(m \pm r)(m-1)$ and $r(m-1)$ are even (it is clear if m is odd), if m is even, then $1 \le \nu_2(m) \le \nu_2(n = m + r)$. So, both n and r are even. Therefore, we have $(-1)^{\eta^2(m+r)(m-1)} = (-1)^{\eta(m-1)r} = (-1)^{\eta(m-1)\omega r} = 1$.

From Lemma 7 Parts (ii), (iii), and (iv) we have

$$\begin{aligned}
\text{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) &= (\beta^{m-1})^{\eta(m+r)-(\eta(m-r)+\omega r)}\text{Res}\big(\mathcal{F}_m, (-g)^r\mathcal{L}_{m-r}\big) \\
&= \beta^{(m-1)(2\eta-\omega)r}\text{Res}\big(\mathcal{F}_m, (-g)^r\big)\text{Res}\big(\mathcal{F}_m, \mathcal{L}_{m-r}\big) \\
&= \beta^{(m-1)(2\eta-\omega)r}\text{Res}\big(\mathcal{F}_m, (-g)^r\big)\text{Res}\big(\mathcal{F}_m, \mathcal{L}_{m-r}\big) \\
&= \beta^{(m-1)(2\eta-\omega)r}\text{Res}\big(\mathcal{F}_m, (-1)^r\big)\text{Res}\big(\mathcal{F}_m, g^r\big)\text{Res}\big(\mathcal{F}_m, \mathcal{L}_{m-r}\big) \\
&= \beta^{(m-1)(2\eta-\omega)r}(-1)^{\eta(m-1)r}(-1)^{\eta(m-1)\omega r}\text{Res}\big(g, \mathcal{F}_m\big)^r\text{Res}\big(\mathcal{F}_m, \mathcal{L}_{m-r}\big) \\
&= \beta^{(m-1)(2\eta-\omega)r}\rho^{r(m-1)}(-1)^{\eta^2(m-1)(m-r)}\text{Res}\big(\mathcal{L}_{m-r}, \mathcal{F}_m\big) \\
&= (-1)^{\eta(m-1)(1+\omega)r}\beta^{(m-1)(2\eta-\omega)r}\rho^{r(m-1)}\text{Res}\big(\mathcal{L}_{m-r}, \mathcal{F}_m\big).
\end{aligned}$$

Since $n = m + r$ is the least element of H, we have that $(m - r) \notin H$. Therefore, $\text{Res}\big(\mathcal{L}_{m-r}, \mathcal{F}_m\big)$ satisfies (12). So,

$$\text{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = \beta^{(m-1)(2\eta-\omega)r}\rho^{r(m-1)}2^{\eta(\gcd(m-r,m)-1)}\left(\beta^{2\eta-\omega}\rho\right)^{(m-r)(m-1)/2}.$$

Since $0 < r < m$, the $\gcd(m - r, m) = 1$. This (after some simplifications) implies that $\text{Res}\big(\mathcal{L}_n, \mathcal{F}_m\big) = ((-1)^{\eta\omega}\beta^{2\eta-\omega}\rho)^{(m-1)(m+r)/2}$. Therefore, $m \in A$. That is a contradiction. This completes the proof of the claim. □

7 Proof of Theorem 6

In this section we prove a theorem that gives closed formulas for the derivatives of GFPs.

Recall that from (4) and (5) we have $d = a + b$, $b = -g/a$, where d and g are the polynomials defined in (2) and (3). This implies that $a - b = a + ga^{-1}$. Here $\mathcal{F}'_n$, $\mathcal{L}'_n$, a', b' and d' are the derivatives of $\mathcal{F}_n$, $\mathcal{L}_n$, a, b and d with respect to x.

Proof of Theorem 6. We prove Part (i). From Binet formula (4) and $b = -g/a$ we have

$$\mathcal{F}_n = \left(a^n - (-g)^n a^{-n}\right) / \left(a - (-g)a^{-1}\right).$$

Differentiating $\mathcal{F}_n$ with respect to x, using $a - b = a + ga^{-1}$ and simplifying we have

$$\mathcal{F}'_n = \frac{na'\left(a^{n-1} + (-g)^n a^{-n-1}\right)}{(a-b)^2} - \frac{a'(1 - ga^{-2})\left(a^n - (-g)^n a^{-n}\right)}{(a-b)^2}. \tag{14}$$

Since $d = a + b$, and $b = -g/a$, we have $a' + b' = d'$, and $b' = ga^{-2}a'$. These imply that

$$a' = \frac{ad'}{a + ga^{-1}} \quad \text{and} \quad 1 - ga^{-2} = \frac{d}{a}.$$

Substituting these results in (14) and simplifying we have

$$\mathcal{F}'_n = \frac{nad'\left(a^{n-1} + (-g)^n a^{-n-1}\right)}{(a-b)^2} - \frac{d \cdot d'}{(1 + ga^{-1})^2} \frac{\left(a^n - (-g)^n a^{-n}\right)}{(a-b)}.$$

Thus,

$$\mathcal{F}'_n = \frac{nd'\left(a^n + b^n\right)}{(a-b)^2} - \frac{d \cdot d'}{(a-b)^2} \frac{\left(a^n - (-g)^n a^{-n}\right)}{(a-b)}.$$

It is known that (see, for example [11]) $a^n + b^n = g\mathcal{F}_{n-1} + \mathcal{F}_{n+1}$. So,

$$\mathcal{F}'_n = \frac{nd'\left(g\mathcal{F}_{n-1} + \mathcal{F}_{n+1}\right) - d \cdot d'\mathcal{F}_n}{(a-b)^2}.$$

This completes the proof of Part (i).

We now prove Part (ii). From [11] we know that $\mathcal{L}_n = \left(g\mathcal{F}_{n-1} + \mathcal{F}_{n+1}\right)/\alpha$. Differentiating $\mathcal{L}_n$ with respect to x, we have (recall that g is constant) $\mathcal{L}'_n = \left(g\mathcal{F}'_{n-1} + \mathcal{F}'_{n+1}\right)/\alpha$. This and Part (i) imply that

$$\mathcal{L}'_n = \frac{gd'\left((n-1)\alpha\mathcal{L}_{n-1} - d\mathcal{F}_{n-1}\right)}{\alpha(a-b)^2} + \frac{d'\left((n+1)\alpha\mathcal{L}_{n+1} - d\mathcal{F}_{n+1}\right)}{\alpha(a-b)^2}.$$

Simplifying we have

$$\mathcal{L}'_n = \frac{d'}{\alpha(a-b)^2}\left((n-1)\alpha g\mathcal{L}_{n-1} + (n+1)\alpha\mathcal{L}_{n+1} - d\alpha\frac{g\mathcal{F}_{n-1} + \mathcal{F}_{n+1}}{\alpha}\right).$$

This and $\mathcal{L}_n = \left(g\mathcal{F}_{n-1} + \mathcal{F}_{n+1}\right)/\alpha$ imply that

$$\mathcal{L}'_n = \frac{d'\left((n-1)g\mathcal{L}_{n-1} + (n+1)\mathcal{L}_{n+1} - d\mathcal{L}_n\right)}{(a-b)^2}.$$

Therefore,

$$\mathcal{L}'_n = \frac{d'\left(n(g\mathcal{L}_{n-1} + \mathcal{L}_{n+1}) + (\mathcal{L}_{n+1} - g\mathcal{L}_{n-1}) - d\mathcal{L}_n\right)}{(a-b)^2}. \tag{15}$$

From [11] we know that

$$g\mathcal{L}_{n-1} + \mathcal{L}_{n+1} = (a-b)^2\mathcal{F}_n/\alpha, \quad \mathcal{L}_{n+1} - g\mathcal{L}_{n-1} = \alpha\mathcal{L}_n\mathcal{L}_1, \quad \text{and} \quad \alpha\mathcal{L}_1 - d = 0.$$

Substituting these identities in (15) completes the proof. □

8 Proof of Theorems 4 and 5

In this section we prove Theorems 4 and 5. We first prove some basic lemmas needed for the main proof.

8.1 Basic lemmas for the discriminant

Recall that one of the expressions for the discriminant of a polynomial f is given by $\operatorname{Disc}(f) = (-1)^{n(n-1)/2}a^{-1}\operatorname{Res}(f, f')$, where $a = \operatorname{lc}(f)$, $n = \deg(f)$ and f' the derivative of f.

Lemma 19. *For $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\mathcal{F}_n \bmod d^2 + 4g \equiv \begin{cases} n(-g)^{(n-1)/2}, & \text{if } n \text{ is odd;} \\ (-1)^{(n+2)/2}\left(ndg^{(n-2)/2}\right)/2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We use mathematical induction. Let $S(k)$ be statement:

$$\mathcal{F}_k \bmod d^2 + 4g \equiv \begin{cases} (-1)^{(k-1)/2}kg^{(k-1)/2}, & \text{if } k \text{ is odd;} \\ (-1)^{(k+2)/2}\left(kdg^{(k-2)/2}\right)/2, & \text{if } k \text{ is even.} \end{cases}$$

Since $\mathcal{F}_1 = 1$ and $\mathcal{F}_2 = d$, we have $S(1)$ and $S(2)$ are true. Suppose that the statement is true for some $k = n-1$ and $k = n$. Thus, suppose that $S(n-1)$ and $S(n)$ are true and we prove $S(n+1)$. We consider two cases on the parity of n.

Case n even. Recall that $\mathcal{F}_{n+1} = d\mathcal{F}_n + g\mathcal{F}_{n-1}$. This and $S(n-1)$ and $S(n)$ (with n even and $n-1$ odd) imply that $\mathcal{F}_{n+1} \equiv (-1)^{(n+2)/2}\left(nd^2g^{(n-2)/2}/2\right) + (n-1)(-g)^{(n-2)/2}g \bmod d^2 + 4g$. Simplifying

$$\mathcal{F}_{n+1} \equiv (-1)^{(n+2)/2}\frac{nd^2g^{(n-2)/2}}{2} + (2n - (n+1))(-1)^{(n-2)/2}g^{n/2} \bmod d^2 + 4g.$$

It is easy to see that

$$(-1)^{(n+2)/2}\frac{nd^2g^{(n-2)/2}}{2} + 2n(-1)^{(n-2)/2}g^{n/2} = (-1)^{(n+2)/2}\frac{ng^{(n-2)/2}}{2}\left(d^2 + 4g\right).$$

Thus,

$$\mathcal{F}_{n+1} \equiv (-1)^{(n+2)/2}\frac{ng^{(n-2)/2}}{2}\left(d^2 + 4g\right) + (n+1)(-g)^{n/2} \bmod d^2 + 4g.$$

This implies that $\mathcal{F}_{n+1} \equiv (n+1)(-g)^{n/2} \bmod d^2 + 4g$.

Case n odd. $S(n-1)$ and $S(n)$ (with n odd and $n-1$ even) and $\mathcal{F}_{n+1} = d\mathcal{F}_n + g\mathcal{F}_{n-1}$, imply that

$$\begin{aligned} \mathcal{F}_{n+1} &\equiv n(-g)^{(n-1)/2}d + (-1)^{(n+1)/2}\left(\frac{(n-1)dg^{(n-3)/2}}{2}\right)g \bmod d^2 + 4g \\ &\equiv dg^{(n-1)/2}\left(\frac{(-1)^{(n-1)/2}2n - (-1)^{(n-1)/2}(n-1)}{2}\right) \bmod d^2 + 4g \\ &\equiv \frac{(-1)^{(n+3)/2}(n+1)dg^{(n-1)/2}}{2} \bmod d^2 + 4g. \end{aligned}$$

This completes the proof. ☐

Lemma 20. *If $n \in \mathbb{Z}_{\geq 0}$, then* $\mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big) = (\beta^{2\eta-\omega}\rho)^{(n-1)}n^{2\eta}$.

Proof. From [11] we know that $(a-b)^2 = d^2 + 4g$. This and Lemma 19 imply that there is a polynomial T such that

$$\mathcal{F}_n = \begin{cases} (a-b)^2T + n(-g)^{(n-1)/2}, & \text{if } n \text{ is odd;} \\ (a-b)^2T + (-1)^{(n+2)/2}2^{-1}dg^{(n-2)/2}n, & \text{if } n \text{ is even.} \end{cases} \quad (16)$$

Using Lemma 7 Parts (i), (iii) and (iv) and simplifying we have

$$\mathrm{Res}\big(d^2+4g, g^m\big) = \mathrm{Res}\big(d^2+4g, g\big)^m = \mathrm{Res}\big(d^2+4g, g\big)^m = (\lambda^{2\eta-2\eta}\mathrm{Res}\big(g, d\big)^2)^m = \rho^{2m}. \quad (17)$$

To find $\mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big)$ we consider two cases, depending on the parity of n.

Case n is even. From (16) we have

$$\mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big) = \mathrm{Res}\big((a-b)^2, (a-b)^2T + (-1)^{(n+2)/2}2^{-1}dg^{(n-2)/2}n\big).$$

This and Lemma 7 Parts (i), (ii) and (iv) imply that

$$\begin{aligned} \mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big) &= \beta^{(2\eta-\omega)(n-2)}\mathrm{Res}\big((a-b)^2, (-1)^{(n+2)/2}2^{-1}n\big)\mathrm{Res}\big((a-b)^2, dg^{(n-2)/2}\big) \\ &= \beta^{(2\eta-\omega)(n-2)}(2^{-1}n)^{2\eta}\mathrm{Res}\big((a-b)^2, d\big)\mathrm{Res}\big((a-b)^2, g^{(n-2)/2}\big) \\ &= \beta^{(2\eta-\omega)(n-2)}(2^{-1}n)^{2\eta}\mathrm{Res}\big(d, d^2+4g\big)\mathrm{Res}\big((a-b)^2, g^{(n-2)/2}\big). \end{aligned}$$

Using similar analysis as in (17) we have $\mathrm{Res}\big((a-b)^2, g^{(n-2)/2}\big) = \rho^{n-2}$. It is easy to see that $\mathrm{Res}\big((a-b)^2, d\big) = \mathrm{Res}\big(d, d^2+4g\big) = \beta^{2n-\omega}2^{\eta}$. Therefore,

$$\mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big) = \beta^{(2\eta-\omega)(n-1)}n^{2\eta}\rho\rho^{n-2} = (\beta^{2\eta-\omega}\rho)^{(n-1)}n^{2\eta}.$$

Case n is odd. From (16) we have

$$\mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big) = \mathrm{Res}\big((a-b)^2, (a-b)^2T + (-1)^{(n+2)/2}2^{-1}dg^{(n-2)/2}n\big).$$

This and Lemma 7 Parts (i), (ii) and (iv) imply that

$$\begin{aligned} \mathrm{Res}\big((a-b)^2, \mathcal{F}_n\big) &= (\beta^2)^{\eta(n-1)-\omega(n-1)/2}\mathrm{Res}\big(d^2+4g, n(-g)^{(n-1)/2}\big) \\ &= \beta^{(2\eta-\omega)(n-1)}n^{2\eta}\mathrm{Res}\big(d^2+4g, g^{(n-1)/2}\big) \\ &= (\beta^{2\eta-\omega}\rho)^{(n-1)}n^{2\eta}. \end{aligned}$$

This completes the proof. ☐

8.2 Proof of main theorems about discriminant

Proof of Theorem 4. From Theorem 6 we have

$$\text{Res}\big(\mathcal{F}_n, (a-b)^2\mathcal{F}_n'\big) = \text{Res}\big(\mathcal{F}_n, d'(n\alpha\mathcal{L}_n - d\mathcal{F}_n)\big).$$

Since $\deg(d) = 1$, we have that d' is a constant. (Recall that when $\mathcal{F}_n$ and $\mathcal{L}_n$ are together in a resultant, they are equivalent.) Therefore,

$$\text{Res}\big(\mathcal{F}_n, (a-b)^2\mathcal{F}_n'\big) = (d')^{n-1}\text{Res}\big(\mathcal{F}_n, n\alpha\mathcal{L}_n - d\mathcal{F}_n\big).$$

Since $\deg(\mathcal{F}_n) = \eta(n-1)$ and $\deg(\mathcal{L}_n) = \eta n$, we have $\text{Res}\big(\mathcal{F}_n, n\alpha\mathcal{L}_n - d\mathcal{F}_n\big) = \text{Res}\big(\mathcal{F}_n, n\alpha\mathcal{L}_n\big)$. So, $\text{Res}\big(\mathcal{F}_n, (a-b)^2\mathcal{F}_n'\big) = (\alpha d' n)^{n-1}\text{Res}\big(\mathcal{F}_n, \mathcal{L}_n\big) = (\alpha d' n)^{n-1}\text{Res}\big(\mathcal{L}_n, \mathcal{F}_n\big)$. This and Theorem 3 imply that

$$\text{Res}\big(\mathcal{F}_n, (a-b)^2\mathcal{F}_n'\big) = (\alpha d' n)^{n-1} 2^{n-1}\alpha^{1-n}\big(\beta^2\rho\big)^{n(n-1)/2} = (2d'n)^{n-1}\big(\beta^2\rho\big)^{n(n-1)/2}. \tag{18}$$

On the other hand, from Lemma 20 and the fact that $\deg(a-b)^2$ is even we have

$$\text{Res}\big(\mathcal{F}_n, (a-b)^2\mathcal{F}_n'\big) = \text{Res}\big(\mathcal{F}_n, (a-b)^2\big)\text{Res}\big(\mathcal{F}_n, \mathcal{F}_n'\big) = n^2(\beta^2\rho)^{(n-1)}\text{Res}\big(\mathcal{F}_n, \mathcal{F}_n'\big).$$

This and (18) imply that

$$\text{Res}\big(\mathcal{F}_n, \mathcal{F}_n'\big) = \frac{(2d'n)^{n-1}\big(\beta^2\rho\big)^{n(n-1)/2}}{n^2(\beta^2\rho)^{(n-1)}} = n^{n-3}(2d')^{n-1}(\beta^2\rho)^{(n-1)(n-2)/2}.$$

Therefore,

$$\text{Disc}\big(\mathcal{F}_n\big) = (-1)^{\frac{(n-1)(n-2)}{2}}\beta^{1-n}\text{Res}\big(\mathcal{F}_n, \mathcal{F}_n'\big) = \beta^{1-n}n^{n-3}(2d')^{n-1}(-\beta^2\rho)^{(n-1)(n-2)/2}.$$

This completes the proof. □

Proof of Theorem 5. From the definition of the discriminant we have

$$\text{Disc}\big(\mathcal{L}_n\big) = (-1)^{n(n-1)/2}\alpha\beta^{-n}\text{Res}\big(\mathcal{L}_n, \mathcal{L}_n'\big).$$

This and Theorem 6 imply that $\text{Disc}\big(\mathcal{L}_n\big) = (-1)^{n(n-1)/2}\alpha\beta^{-n}\text{Res}\big(\mathcal{L}_n, (nd'\mathcal{F}_n)/\alpha\big)$. Since $(nd')/\alpha$ is a constant, $\text{Disc}\big(\mathcal{L}_n\big) = (-1)^{n(n-1)/2}\alpha\beta^{-n}(nd'/\alpha)^n\text{Res}\big(\mathcal{L}_n, \mathcal{F}_n\big)$. This and Theorem 3 imply that

$$\begin{aligned}\text{Disc}\big(\mathcal{L}_n\big) &= (-1)^{\frac{n(n-1)}{2}}\alpha\beta^{-n}\Big(\frac{nd'}{\alpha}\Big)^n 2^{n-1}\alpha^{1-n}\big(\beta^2\rho\big)^{(n(n-1))/2}\\ &= \beta^{n(n-2)}\Big(nd'\Big)^n 2^{n-1}\alpha^{2-2n}\big(-\rho\big)^{(n(n-1))/2}.\end{aligned}$$

Completing the proof. □

Theorems 4 and 5 generalize as follows.
If $\deg(d) = m$, g is a constant and d' is the derivative of d, then

$$\operatorname{Disc}\big(\mathcal{F}_n\big) = (-\rho)^{(n-2)(n-1)/2} 2^{n-1} n^{n-3} \beta^{(n-1)(n-3)} \operatorname{Res}\big(\mathcal{F}_n, d'\big)^{n-1}.$$

If $\deg(d) = m$, g is a constant and d' is the derivative of d, then

$$\operatorname{Disc}\big(\mathcal{L}_n\big) = (-\rho)^{n(n-1)/2}\, 2^{n-1} n^n \alpha^{2-2n} \beta^{n(n-2)} \operatorname{Res}\big(\mathcal{L}_n, d'\big)^n.$$

Open question. In this paper we did not investigate the case $\deg(g) \geq \deg(d)$. This property is satisfied by Jacobsthal polynomials.

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