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Reciprocal Sum of Palindromes

Phakhinkon Phunphayap and Prapanpong Pongsriiam[1]
Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom 73000
Thailand
phakhinkon@gmail.com
pongsriiam_p@silpakorn.edu, prapanpong@gmail.com

Abstract

A positive integer n is a b-adic palindrome if the representation of n in base b reads the same backward as forward. Let s_b be the reciprocal sum of all b-adic palindromes. In this article, we obtain upper and lower bounds, and an asymptotic formula for s_b. We also show that the sequence $(s_b)_{b\geq 2}$ is strictly increasing and log-concave.

1 Introduction

Let $n \geq 1$ and $b \geq 2$ be integers. We call n a *palindrome in base* b (or b-*adic palindrome*) if the b-adic expansion of $n = (a_k a_{k-1} \cdots a_0)_b$ with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for $0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$. As usual, if we write a number without specifying the base, then it is always in base 10. So, for example, $9 = (1001)_2 = (100)_3$ is a palindrome in bases 2 and 10 but not in base 3.

In recent years, there has been an increasing interest in the importance of palindromes in mathematics [1, 2, 3, 13, 17, 25], theoretical computer science [4, 9, 12], and theoretical physics [11, 14]. There are also some discussions on the reciprocal sum of palindromes on the internet but as far as we are aware, our observation has not appeared in the literature. Throughout this article, we let $b \geq 2$, s_b the reciprocal sum of all b-adic palindromes, and $s_{b,k}$ the reciprocal sum of all b-adic palindromes which have k digits in their b-adic expansions.

[1]Prapanpong Pongsriiam is the corresponding author.

The set of all b-adic palindromes is infinite but quite sparse, so it is not difficult to see that s_b converges. In fact, Shallit proposed the convergence of s_b as a problem in the *Fibonacci Quarterly* in 1980 [26, 27].

In this article, we obtain upper and lower bounds for s_b, which enable us to show that $s_{b+1} > s_b$ for all $b \geq 2$ and $s_b^2 - s_{b-1}s_{b+1} > 0$ for all $b \geq 3$. That is, the sequence $(s_b)_{b\geq 2}$ is strictly increasing and log-concave. Furthermore, we give an asymptotic formula for s_b of the form $s_b = g(b) + O(h(b))$ where the implied constant can be taken to be 1 and the order of magnitude of $h(b)$ is $\frac{\log b}{b^3}$ as $b \to \infty$. Our result $s_{b+1} > s_b$ for all $b \geq 2$ also implies that if $b_1 > b_2 \geq 2$ and if we use the logarithmic measure, then we can say that the palindromes in base b_1 occur more often than those in base b_2. On the other hand, if we use the usual counting measure, then we obtain from Pongsriiam and Subwattanachai's exact formula [22] that the number of palindromes in different bases which are less than or equal to N are not generally comparable. It seems that there are races between palindromes in different bases which may be similar to races between primes in different residue classes. We will get back to this problem in the near future.

The reciprocal sum of an integer sequence is also of general interest in mathematics and theoretical physics as proposed by Bayless and Klyve [8], and by Roggero, Nardelli, and Di Noto [24]. See also the work of Nguyen and Pomerance [19] on the reciprocal sum of the amicable numbers, the preprint of Kinlaw, Kobayashi, and Pomerance [15] on the reciprocal sum of the positive integers n satisfying $\varphi(n) = \varphi(n+1)$, and the article by Lichtman [16] on the reciprocal sum of primitive nondeficient numbers. In addition, Banks [5], Cilleruelo, Luca, and Baxter [10], and Rajasekaran, Shallit, and Smith [23] have recently investigated some additive properties of palindromes while Banks, Hart, and Sakata [6] and Banks and Shparlinski [7] show some of their multiplicative properties. For more information concerning palindromes, we refer the reader to the entry A002113 in the On-Line Encyclopedia of Integer Sequences (OEIS) [28].

2 Results

Throughout this section, a, c, m, n, k, ℓ denote positive integers, and x, y, z denote positive real numbers. Furthermore,

- $\lfloor x \rfloor$ is the greatest integer less than or equal to x;
- $\lceil x \rceil$ is the least integer greater than or equal to x;
- $\log x$ is the natural logarithm of x;
- $x_b = \sum_{m=1}^{b-1} \frac{1}{m}$;
- $y_b = \sum_{m=b}^{b^2-1} \frac{1}{m}$; and
- $z_b = \sum_{m=b+1}^{b^2} \frac{1}{m}$.

Note that $x_b = s_{b,1}$ and $x_b/(b+1) = s_{b,2}$ and that

$$z_b - y_b = \frac{1}{b^2} - \frac{1}{b} = \frac{1-b}{b^2}.$$

Theorem 1. *We have*

$$\frac{y_b}{b} - \frac{x_b}{b^3} \le s_{b,3} \le \frac{y_b}{b} \quad \textit{and} \quad \frac{z_b}{b^{\lfloor \frac{k}{2} \rfloor}} \le s_{b,k} \le \frac{y_b}{b^{\lfloor \frac{k}{2} \rfloor}} \quad \textit{for every } k \ge 4.$$

Proof. We first consider the case $k = 3$. The b-adic palindromes which have 3 digits are of the form $(aca)_b$ where $1 \le a \le b-1$ and $0 \le c \le b-1$. Since

$$\left| \frac{1}{(aca)_b} - \frac{1}{(ac0)_b} \right| = \left| \frac{-a}{(ab^2+cb+a)(ab^2+cb)} \right| \le \frac{a}{(ab^2)^2} = \frac{1}{ab^4},$$

we obtain

$$\frac{1}{(ac0)_b} - \frac{1}{ab^4} \le \frac{1}{(aca)_b} \le \frac{1}{(ac0)_b}. \tag{1}$$

Observe that

$$\sum_{\substack{1 \le a \le b-1 \\ 0 \le c \le b-1}} \frac{1}{(ac0)_b} = \frac{1}{b} \sum_{\substack{1 \le a \le b-1 \\ 0 \le c \le b-1}} \frac{1}{ab+c} = \frac{y_b}{b} \quad \text{and} \quad \sum_{\substack{1 \le a \le b-1 \\ 0 \le c \le b-1}} \frac{1}{ab^4} = \frac{x_b}{b^3}.$$

So by summing (1) over all $a = 1, 2, \ldots, b-1$ and $c = 0, 1, \ldots, b-1$, we obtain the inequality $\frac{y_b}{b} - \frac{x_b}{b^3} \le s_{b,3} \le \frac{y_b}{b}$. For $k = 4$, $1 \le a \le b-1$, and $0 \le c \le b-1$, we have

$$\frac{1}{b^2(ab+c+1)} = \frac{1}{(ac00)_b + b^2} \le \frac{1}{(acca)_b} \le \frac{1}{(ac00)_b} = \frac{1}{b^2(ab+c)}.$$

Summing over all $a = 1, 2, \ldots, b-1$ and $c = 0, 1, \ldots, b-1$ leads to

$$\frac{z_b}{b^2} = \frac{1}{b^2} \sum_{\substack{1 \le a \le b-1 \\ 0 \le c \le b-1}} \frac{1}{ab+c+1} \le s_{b,4} \le \frac{1}{b^2} \sum_{\substack{1 \le a \le b-1 \\ 0 \le c \le b-1}} \frac{1}{ab+c} = \frac{y_b}{b^2}.$$

Let $k \ge 5$. The b-adic palindromes which have k digits are of the form $(aa_1a_2 \cdots a_{k-2}a)_b$ where $1 \le a \le b-1$, $0 \le a_i \le b-1$ for all $i \in \{1, 2, \ldots, k-2\}$ with the usual symmetric property on a_i. We fix a and a_1 and count the number of palindromes in this form. There are b choices for $a_2 \in \{0, 1, 2, \ldots, b-1\}$ and so there is only 1 choice for $a_{k-3} = a_2$. Similarly, there are b choices for a_3 and 1 choice for a_{k-4}. By continuing this counting, we see that the number of palindromes in this form (when a and a_1 are already chosen) is equal to $b^{\lceil \frac{k-4}{2} \rceil}$. Therefore the reciprocal sum of such palindromes satisfies

$$\sum_{a_2, \ldots, a_{k-3}} \frac{1}{(aa_1a_2 \cdots a_{k-2}a)_b} \le \frac{b^{\lceil \frac{k-4}{2} \rceil}}{(aa_10 \cdots 0a_1a)_b} \le \frac{b^{\lceil \frac{k-4}{2} \rceil}}{b^{k-2}(ab+a_1)} = \frac{1}{b^{\lfloor \frac{k}{2} \rfloor}(ab+a_1)},$$

where $a_2, \ldots, a_{k-3}$ run over all integers $0, 1, 2, \ldots, b-1$ with the symmetric condition of palindromes. Hence

$$s_{b,k} = \sum_{\substack{1 \le a \le b-1 \\ 0 \le a_1 \le b-1}} \sum_{a_2, \ldots, a_{k-3}} \frac{1}{(aa_1a_2 \cdots a_{k-2}a)_b} \le \sum_{\substack{1 \le a \le b-1 \\ 0 \le a_1 \le b-1}} \frac{1}{b^{\lfloor \frac{k}{2} \rfloor}(ab+a_1)} = \frac{y_b}{b^{\lfloor \frac{k}{2} \rfloor}}.$$

Similarly, if a and a_1 are fixed, then

$$\sum_{a_2, \ldots, a_{k-3}} \frac{1}{(aa_1a_2 \cdots a_{k-2}a)_b} \ge \frac{b^{\lceil \frac{k-4}{2} \rceil}}{ab^{k-1} + a_1 b^{k-2} + b^{k-2}} = \frac{1}{b^{\lfloor \frac{k}{2} \rfloor}(ab + a_1 + 1)}.$$

Summing the above over all $a = 1, 2, \ldots, b-1$ and $a_1 = 0, 1, \ldots, b-1$, we obtain the desired lower bound for $s_{b,k}$. This completes the proof. □

Theorem 2. *For every $b, \ell \ge 2$, we have*

$$\left(\frac{b+2}{b+1}\right) x_b + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2z_b}{(b-1)b^{\ell-1}} \le s_b \le \left(\frac{b+2}{b+1}\right) x_b + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2y_b}{(b-1)b^{\ell-1}}.$$

In particular,

$$\left(\frac{b+2}{b+1}\right) x_b + \frac{y_b}{b} - \frac{x_b}{b^3} + \frac{2z_b}{b(b-1)} \le s_b \le \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b(b-1)}\right) y_b. \tag{2}$$

Proof. For simplicity, we write x, y, z instead of x_b, y_b, z_b, respectively. We consider $s_{b,k}$ for each k as follows. Obviously $s_{b,1} = 1 + \frac{1}{2} + \cdots + \frac{1}{b-1} = x$. For $k = 2$, $s_{b,k}$ is

$$\sum_{a=1}^{b-1} \frac{1}{(aa)_b} = \sum_{a=1}^{b-1} \frac{1}{a(b+1)} = \frac{x}{b+1}.$$

By writing $s_b = x + \frac{x}{b+1} + \sum_{k=3}^{2\ell-1} s_{b,k} + \sum_{k=2\ell}^{\infty} s_{b,k}$ and applying Theorem 1, we obtain

$$s_b \le \frac{b+2}{b+1} x + \sum_{k=3}^{2\ell-1} s_{b,k} + \sum_{k=2\ell}^{\infty} \frac{y}{b^{\lfloor \frac{k}{2} \rfloor}} = \frac{b+2}{b+1} x + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2y}{(b-1)b^{\ell-1}}.$$

Similarly,

$$s_b \ge \frac{b+2}{b+1} x + \sum_{k=3}^{2\ell-1} s_{b,k} + \sum_{k=2\ell}^{\infty} \frac{z}{b^{\lfloor \frac{k}{2} \rfloor}} = \left(\frac{b+2}{b+1}\right) x + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2z}{(b-1)b^{\ell-1}}.$$

This proves the first part of this theorem. The second part follows from Theorem 1 and the substitution $\ell = 2$ in the first part. □

Theorem 3. *The sequence $(s_b)_{b\geq 2}$ is strictly increasing.*

Proof. We first verify that $s_{b+1} > s_b$ for $2 \leq b \leq 16$. Myers gives the decimal expansion of s_2 in the entry A244162 in the OEIS [28]. Myers also describe the algorithm in his calculation, which can be found in the web page [18]. So we know that $s_2 < 2.3787957$. Alternatively, substituting $\ell = 3$ and $b = 2, 3$ in Theorem 2 and running the computation in a computer, we obtain $2.32137259 \leq s_2 \leq 2.44637260$ and $2.60503980 \leq s_3 \leq 2.62973117$, which implies that $s_2 < s_3$. Similarly, we apply (2) to obtain upper and lower bounds for s_b and we see that $s_b < s_{b+1}$ for $3 \leq b \leq 16$. So we assume throughout that $b \geq 16$. We first observe that

$$y_b = z_b + \frac{1}{b} - \frac{1}{b^2} \quad \text{and} \quad \frac{y_b}{b} - \frac{z_b}{b} = \frac{b-1}{b^3} \geq \frac{x_b}{b^3}.$$

Therefore $\frac{y_b}{b} - \frac{x_b}{b^3} \geq \frac{z_b}{b}$ for all $b \geq 2$. So the term $\frac{y_b}{b} - \frac{x_b}{b^3}$ in (2) can be replaced by $\frac{z_b}{b}$. Therefore we obtain by (2) that $s_{b+1} - s_b$ is larger than

$$\left(\frac{b+3}{b+2}\right) x_{b+1} - \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b+1} + \frac{2}{(b+1)b}\right) z_{b+1} - \left(\frac{1}{b} + \frac{2}{b(b-1)}\right) y_b. \tag{3}$$

In addition, $z_{b+1} - z_b$ is equal to

$$-\frac{1}{b+1} + \sum_{m=b^2+1}^{b^2+2b+1} \frac{1}{m} \geq -\frac{1}{b+1} + \frac{2b+1}{b^2+2b+1} = \frac{b}{(b+1)^2} > 0.$$

Since $x_{b+1} = x_b + \frac{1}{b}$ and $z_{b+1} > z_b = y_b - \frac{1}{b} + \frac{1}{b^2}$, we obtain from (3) that $s_{b+1} - s_b$ is larger than or equal to

$$\begin{aligned}
&\left(\frac{b+3}{b+2}\right)\left(\frac{1}{b}\right) + x_b\left(\frac{b+3}{b+2} - \frac{b+2}{b+1}\right) + y_b\left(\frac{1}{b+1} + \frac{2}{b(b+1)} - \frac{1}{b} - \frac{2}{b(b-1)}\right)\\
&\quad + \left(\frac{1}{b^2} - \frac{1}{b}\right)\left(\frac{1}{b+1} + \frac{2}{b(b+1)}\right)\\
&= \frac{1}{b} + \frac{1}{b(b+2)} - \frac{x_b}{(b+1)(b+2)} - \frac{(b+3)y_b}{b(b-1)(b+1)} - \frac{(b-1)(b+2)}{b^3(b+1)}.
\end{aligned}$$

Recall that if a and b are integers, $a < b$, and f is monotone on $[a, b]$, then

$$\min\{f(a), f(b)\} \leq \sum_{n=a}^{b} f(n) - \int_a^b f(t)dt \leq \max\{f(a), f(b)\}. \tag{4}$$

From (4), we obtain

$$x_b = \sum_{m=1}^{b-1} \frac{1}{m} \leq 1 + \log(b-1) \leq \frac{3}{2}\log b,$$

$$y_b = -\frac{1}{b-1} + \sum_{m=b-1}^{b^2-1} \frac{1}{m} \leq \log(b+1) \leq \frac{5}{4}\log b.$$

In addition, it is straightforward to verify that

$$\frac{1}{b(b+2)} - \frac{(b-1)(b+2)}{b^3(b+1)} > -\frac{1}{b^2}.$$

Therefore $s_{b+1} - s_b$ is larger than

$$\frac{1}{b} - \frac{3\log b}{2(b+1)(b+2)} - \frac{5(b+3)\log b}{4b(b-1)(b+1)} - \frac{1}{b^2} > \frac{1}{b} - \frac{3\log b}{2b^2} - \frac{3\log b}{2b^2} - \frac{1}{b^2} = \frac{1}{b} - \frac{1}{b^2} - \frac{3\log b}{b^2}. \tag{5}$$

Observe that the function $x \mapsto \frac{\log x}{x}$ is decreasing on $[3, \infty)$. Since $b \geq 16$, we obtain

$$\frac{3\log b}{b} \leq \frac{3\log 16}{16} < \frac{7}{10} \quad \text{and} \quad \frac{1}{b} < \frac{1}{10}.$$

Hence we obtain from (5) that

$$s_{b+1} - s_b > \frac{1}{b} - \frac{1}{b^2} - \frac{3\log b}{b^2} > \frac{1}{b} - \frac{1}{10b} - \frac{7}{10b} = \frac{1}{5b} > 0.$$

This completes the proof. □

Recall that if we write $f(b) = g(b) + O^*(h(b))$, then it means that $f(b) = g(b) + O(h(b))$ and the implied constant can be taken to be 1. In addition, $f(b) = g(b) + \Omega_+(h(b))$ means $\limsup_{b\to\infty} \frac{f(b)-g(b)}{h(b)} > 0$. From this point on, we use (4) without further reference.

Theorem 4. *Uniformly for $b \geq 2$,*

$$s_b = \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b^2}\right) y_b + O^*\left(\frac{5\log b}{b^3}\right).$$

This estimate is sharp in the sense that $O^*\left(\frac{5\log b}{b^3}\right)$ can be replaced by $\Omega_+\left(\frac{\log b}{b^3}\right)$.

Proof. Let $g(b) = \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b^2}\right) y_b$ be the main term above. Since $y_b \leq \log(b+1)$, we obtain by (2) that

$$s_b - g(b) \leq \frac{2y_b}{b^2(b-1)} \leq \frac{2\log(b+1)}{b^2(b-1)}.$$

If $b = 2$, then it is easy to check that $\frac{2y_b}{b^2(b-1)} = \frac{5}{12} < \frac{5\log b}{b^3}$. If $b \geq 3$, then we assert that $\frac{2\log(b+1)}{b^2(b-1)} \leq \frac{5\log b}{b^3}$. To verify this assertion, we observe that it is equivalent to $\left(\frac{5}{2}\log b\right)\left(\frac{b-1}{b}\right) \geq \log(b+1)$. Since $b \geq 3$, we obtain

$$\left(\frac{5}{2}\log b\right)\left(\frac{b-1}{b}\right) \geq \frac{5}{3}\log b \geq \log 2 + \log b = \log(2b) \geq \log(b+1), \text{ as desired.}$$

So in any case,

$$s_b - g(b) \le \frac{5\log b}{b^3}. \tag{6}$$

We also obtain by (2) that $s_b - g(b)$ is larger than or equal to

$$\frac{2z_b}{b(b-1)} - \frac{2y_b}{b^2} - \frac{x_b}{b^3} = \frac{2z_b}{b(b-1)} - \frac{2\left(z_b + \frac{1}{b} - \frac{1}{b^2}\right)}{b^2} - \frac{x_b}{b^3} = \frac{2z_b}{b^2(b-1)} - \frac{x_b}{b^3} - \frac{2}{b^3} + \frac{2}{b^4}. \tag{7}$$

We have

$$x_b = \sum_{m=1}^{b-1} \frac{1}{m} \le 1 + \log(b-1) \le 2\log b, \tag{8}$$

$$z_b = \sum_{m=b}^{b^2} \frac{1}{m} - \frac{1}{b} \ge \int_b^{b^2} \frac{1}{t} dt + \frac{1}{b^2} - \frac{1}{b} \ge \log b - \frac{1}{b} \ge \frac{\log b}{4}. \tag{9}$$

Therefore (7) implies that

$$s_b - g(b) \ge \frac{\log b}{2b^2(b-1)} - \frac{2\log b}{b^3} - \frac{2}{b^3} \ge \frac{\log b}{2b^3} - \frac{2\log b}{b^3} - \frac{3\log b}{b^3} > -\frac{5\log b}{b^3}. \tag{10}$$

By (6) and (10), we obtain $|s_b - g(b)| \le \frac{5\log b}{b^3}$. This proves the first part of this theorem. For the Ω_+ result, we only need to observe that as $b \to \infty$, (7) and the inequalities $x_b \le 1 + \log(b-1)$ and $z_b \ge \log b - \frac{1}{b}$ given in (8) and (9) imply that

$$s_b - g(b) \ge \frac{2\left(\log b - \frac{1}{b}\right)}{b^2(b-1)} - \frac{1 + \log(b-1)}{b^3} - \frac{2}{b^3} + \frac{2}{b^4} > \frac{3\log b}{2b^2(b-1)} - \frac{11\log b}{10b^3} - \frac{2}{b^3} + \frac{2}{b^4},$$

so

$$\limsup_{b\to\infty} \frac{s_b - g(b)}{\left(\frac{\log b}{b^3}\right)} \ge \frac{3}{2} - \frac{11}{10} > 0.$$

This completes the proof. □

Recall that by applying Euler-Maclaurin summation formula, we get

$$\sum_{m\le n} \frac{1}{m} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\theta_n}{60n^4}, \tag{11}$$

where γ is Euler's constant and $\theta_n \in [0,1]$. The calculation of (11) can be found in Tenenbaum [30, p. 6]. From this, we obtain another form of Theorem 4 as follows.

Theorem 5. *Uniformly for $b \ge 2$,*

$$s_b = \log b + \gamma + \left(\frac{1}{b} + \frac{1}{b+1}\right)\log b + \frac{\gamma}{b+1} - \frac{1}{2b} + \frac{2\log b}{b^2} - \frac{1}{12b(b+1)} + O^*\left(\frac{6\log b}{b^3}\right). \tag{12}$$

This estimate is sharp in the sense that $O^*\left(\frac{6\log b}{b^3}\right)$ is also $\Omega_+\left(\frac{\log b}{b^3}\right)$.

Proof. By (11), we have

$$x_b = \sum_{m \le b} \frac{1}{m} - \frac{1}{b} = \log b + \gamma - \frac{1}{2b} - \frac{1}{12b^2} + \frac{\theta_b}{60b^4},$$

$$z_b = \sum_{m \le b^2} \frac{1}{m} - \sum_{m \le b} \frac{1}{m} = \left(\log b^2 + \gamma + \frac{1}{2b^2} - \frac{1}{12b^4} + \frac{\theta_{b^2}}{60b^8}\right) - \left(\log b + \gamma + \frac{1}{2b} - \frac{1}{12b^2} + \frac{\theta_b}{60b^4}\right)$$

$$= \log b - \frac{1}{2b} + \frac{7}{12b^2} - \frac{5+\theta_b}{60b^4} + \frac{\theta_{b^2}}{60b^8},$$

$$y_b = z_b + \frac{1}{b} - \frac{1}{b^2} = \log b + \frac{1}{2b} - \frac{5}{12b^2} - \frac{5+\theta_b}{60b^4} + \frac{\theta_{b^2}}{60b^8}.$$

Writing $\frac{b+2}{b+1} = 1 + \frac{1}{b+1}$ and substituting x_b and y_b in Theorem 4, we obtain

$$s_b = h(b) + h_1(b) + O^*\left(\frac{5\log b}{b^3}\right) \tag{13}$$

where $h(b)$ is the main term given in (12) and

$$h_1(b) = \frac{11b^2 - 3b - 10}{12b^4(b+1)} + \frac{\theta_b}{60b^4} + \frac{\theta_b}{60b^4(b+1)} + \frac{\theta_{b^2}}{60b^9} + \frac{\theta_{b^2}}{30b^{10}} - \frac{5+\theta_b}{60b^5} - \frac{5+\theta_b}{30b^6}.$$

It is not difficult to see that $h_1(b) \ge 0$ and

$$h_1(b) \le \frac{11b^2}{12b^4(b+1)} + \frac{1}{60b^4} + \frac{1}{60b^4(b+1)} \le \frac{11\log b}{12b^3} + \frac{\log b}{60b^3} + \frac{\log b}{60b^3} \le \frac{\log b}{b^3}. \tag{14}$$

Therefore (13) implies that $s_b = h(b) + O^*\left(\frac{6\log b}{b^3}\right)$, which is the same as (12). In addition, by the first inequality given in (14), we see that $h_1(b) \ll \frac{1}{b^3}$. Since $O^*\left(\frac{5\log b}{b^3}\right)$ in (13) is $\Omega_+\left(\frac{\log b}{b^3}\right)$ and $h_1(b) \ll \frac{1}{b^3}$, $h_1(b) + O^*\left(\frac{5\log b}{b^3}\right)$ in (13) is $\Omega_+\left(\frac{\log b}{b^3}\right)$. This completes the proof. □

Corollary 6. *The sequence $(s_b)_{b\ge 2}$ diverges to $+\infty$ and the sequence $(s_b - s_{b-1})_{b\ge 3}$ converges to zero as $b \to \infty$.*

Proof. The first assertion follows immediately from Theorem 5. Recall that $\log(b-1) = \log b + O\left(\frac{1}{b}\right)$. So we obtain by Theorem 5 that as $b \to \infty$,

$$0 < s_b - s_{b-1} = \log b - \log(b-1) + O\left(\frac{\log b}{b}\right) \ll \frac{\log b}{b},$$

which implies our assertion. □

Recall that a sequence $(a_n)_{n\ge 0}$ is said to be log-concave if $a_n^2 - a_{n-1}a_{n+1} > 0$ for every $n \ge 1$ and is said to be log-convex if $a_n^2 - a_{n-1}a_{n+1} < 0$ for every $n \ge 1$. For a survey article concerning the log-concavity and log-convexity of sequences, we refer the reader to Stanley [29]. See also Pongsriiam [21] for some combinatorial sequences which are log-concave or log-convex, and some open problems concerning the log-properties of a certain sequence.

Theorem 7. *The sequence $(s_b)_{b\geq 2}$ is log-concave.*

Proof. For each $b = 2, 3, \ldots, 15$, we use Theorem 2 with $\ell = 4$ to get an upper bound C_b and a lower bound D_b for s_b. In addition, for each $b \geq 13$, let U_b and L_b be the upper and lower bounds of s_b given in (2), respectively. Then

$s_b^2 - s_{b-1}s_{b+1} > D_b^2 - C_{b-1}C_{b+1}$ for $3 \leq b \leq 14$ and $s_b^2 - s_{b-1}s_{b+1} > L_b^2 - U_{b-1}U_{b+1}$ for $b \geq 14$.

We use MATLAB to check that $D_b^2 - C_{b-1}C_{b+1} > 0$ for $3 \leq b \leq 14$ and $L_b^2 - U_{b-1}U_{b+1} > 0$ for $14 \leq b \leq 1500$. So $s_b^2 - s_{b-1}s_{b+1} > 0$ for $3 \leq b \leq 1500$. So we assume throughout that $b > 1500$. Then

$$\begin{aligned} U_{b-1}U_{b+1} &= \frac{(b+1)(b+3)}{b(b+2)}x_{b-1}x_{b+1} + \frac{b+2}{(b-2)(b-1)(b+1)}y_{b-1}y_{b+1} \\ &\quad + \frac{b(b+3)}{(b-2)(b-1)(b+2)}x_{b+1}y_{b-1} + \frac{b+2}{b^2}x_{b-1}y_{b+1} \\ &= A_1 + A_2 + A_3 + A_4, \text{ say.} \end{aligned} \tag{15}$$

Since

$$z_b = y_b + \frac{1}{b^2} - \frac{1}{b}, \quad b^4 + 2b^3 - b - 1 \geq b^4 + 2b^3 - 2b - 1 = (b-1)(b+1)^3, \text{ and}$$
$$L_b = \frac{b^4 + 2b^3 - b - 1}{b^3(b+1)}x_b + \frac{y_b}{b} + \frac{2z_b}{b(b-1)},$$

we obtain

$$L_b = \frac{b^4 + 2b^3 - b - 1}{b^3(b+1)}x_b + \frac{b+1}{b(b-1)}y_b - \frac{2}{b^3} \geq \frac{(b-1)(b+1)^2}{b^3}x_b + \frac{b+1}{b(b-1)}y_b - \frac{2}{b^3}.$$

Therefore L_b^2 is larger than or equal to

$$\begin{aligned} &\frac{(b-1)^2(b+1)^4}{b^6}x_b^2 + \frac{(b+1)^2}{b^2(b-1)^2}y_b^2 + \frac{4}{b^6} + \frac{2(b+1)^3}{b^4}x_b y_b - \frac{4(b-1)(b+1)^2}{b^6}x_b - \frac{4(b+1)}{b^4(b-1)}y_b \\ &\geq \frac{(b-1)^2(b+1)^4}{b^6}x_b^2 + \frac{(b+1)^2}{b^2(b-1)^2}y_b^2 + \frac{2(b+1)^3}{b^4}x_b y_b - \frac{4(b-1)(b+1)^2}{b^6}x_b - \frac{4(b+1)}{b^4(b-1)}y_b \\ &= B_1 + B_2 + B_3 - B_4 - B_5, \text{ say.} \end{aligned} \tag{16}$$

In addition, we see that

$$y_{b-1} = y_b + \frac{1}{b-1} - \sum_{m=(b-1)^2}^{b^2-1}\frac{1}{m} \leq y_b + \frac{1}{b-1} - \frac{2b-1}{b^2-1} \leq y_b - \frac{b-2}{b^2},$$

$$y_{b+1} = y_b - \frac{1}{b} + \sum_{m=b^2}^{b^2+2b}\frac{1}{m} \leq y_b - \frac{1}{b} + \frac{2b+1}{b^2} = y_b + \frac{b+1}{b^2},$$

$$x_{b-1} = x_b - \frac{1}{b-1}, \quad \text{and} \quad x_{b+1} = x_b + \frac{1}{b}.$$

From these, we obtain the following inequalities:

$$
\begin{aligned}
A_1 &= \frac{(b+1)(b+3)}{b(b+2)}x_b^2 - \frac{(b+1)(b+3)}{b^2(b-1)(b+2)}x_b - \frac{(b+1)(b+3)}{b^2(b-1)(b+2)} \\
&\le \frac{(b+1)(b+3)}{b(b+2)}x_b^2 - \frac{(b+1)(b+3)}{b^2(b-1)(b+2)}x_b - \frac{1}{b^2}, \\
A_2 &\le \frac{b+2}{(b-2)(b-1)(b+1)}y_b^2 + \frac{3(b+2)}{b^2(b-2)(b-1)(b+1)}y_b - \frac{b+2}{b^4(b-1)}, \\
A_3 &\le \frac{b(b+3)}{(b-2)(b-1)(b+2)}x_by_b + \frac{b+3}{(b-2)(b-1)(b+2)}y_b - \frac{b+3}{b(b-1)(b+2)}x_b - \frac{b+3}{b^2(b-1)(b+2)} \\
&\le \frac{b(b+3)}{(b-2)(b-1)(b+2)}x_by_b + \frac{b+3}{(b-2)(b-1)(b+2)}y_b - \frac{b+3}{b(b-1)(b+2)}x_b, \\
A_4 &\le \frac{b+2}{b^2}x_by_b - \frac{b+2}{b^2(b-1)}y_b + \frac{(b+1)(b+2)}{b^4}x_b - \frac{(b+1)(b+2)}{b^4(b-1)} \\
&\le \frac{b+2}{b^2}x_by_b - \frac{b+2}{b^2(b-1)}y_b + \frac{(b+1)(b+2)}{b^4}x_b.
\end{aligned}
$$

Since $b > 1500$, it is not difficult to verify that

$$
\begin{aligned}
B_1 - B_4 - A_1 &\ge -\frac{(b+1)(6b^3+3b^2-3b-2)}{b^6(b+2)}x_b^2 + \frac{b^6-5b^4+8b^3+16b^2-4b-8}{b^6(b-1)(b+2)}x_b + \frac{1}{b^2} \\
&\ge -\frac{7}{b^3}x_b^2 + \frac{1}{b^2}, \\
B_2 - B_5 - A_2 &\ge -\frac{b^2+5b+2}{b^2(b-2)(b-1)^2(b+1)}y_b^2 - \frac{7b^3+6b^2-12b-8}{b^4(b-2)(b-1)(b+1)}y_b + \frac{b+2}{b^4(b-1)} \\
&\ge -\frac{1}{b^3}y_b^2 - \frac{1}{b^3}y_b, \\
B_3 - A_3 - A_4 &\ge -\frac{2(b^4+8b^3+5b^2-8b-4)}{b^4(b-2)(b-1)(b+2)}x_by_b - \frac{b^2+4b+8}{b^2(b-2)(b-1)(b+2)}y_b \\
&\quad - \frac{b^3+3b^2-4b-4}{b^4(b-1)(b+2)}x_b \\
&\ge -\frac{4}{b^3}x_by_b - \frac{3}{b^3}y_b - \frac{3}{b^3}x_b.
\end{aligned}
$$

From (15), (16), and the above inequalities, we obtain

$$
L_b^2 - U_{b-1}U_{b+1} \ge \frac{1}{b^2} - \frac{7}{b^3}x_b^2 - \frac{1}{b^3}y_b^2 - \frac{4}{b^3}x_by_b - \frac{4}{b^3}y_b - \frac{3}{b^3}x_b.
$$

Since $x_b \le \frac{4}{3}\log b$ and $y_b \le \frac{5}{4}\log b$,

$$
L_b^2 - U_{b-1}U_{b+1} \ge \frac{1}{b^2} - \frac{2977}{144}\frac{(\log b)^2}{b^3} - \frac{9\log b}{b^3} \ge \frac{b - 23(\log b)^2}{b^3}.
$$

Observe that the function $x \mapsto x - 23(\log x)^2$ is strictly increasing on $[300, \infty)$. Since $b \geq 1500$,

$$b - 23(\log b)^2 \geq 1500 - 23(\log 1500)^2 > 0.$$

Therefore $s_b^2 - s_{b-1}s_{b+1} \geq L_b^2 - U_{b-1}U_{b+1} > 0$. Hence $(s_b)_{b\geq 2}$ is log-concave, as desired. □

Remark 8. We have uploaded the numerical data on the computation of s_b in the second author's ResearchGate account [20] which are freely downloadable by everyone.

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