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Counting Young Tableaux of Bounded Height*

François Bergeron and Francis Gascon
Departement de Mathematiques
Univerite du Quebec à Montreal
Email addresses: bergeron.francois@uqam.ca and gascon.francis@uqam.ca

Abstract

We show that formulae of Gessel for the generating functions for Young standard tableaux of height bounded by k (see [2]) satisfy linear differential equations, with polynomial coefficients, equivalent to P-recurrences conjectured by Favreau, Krob and the first author (see [1]) for the number of bounded height tableaux and pairs of bounded height tableaux.

1. Results

Let us first fix some notation. A partition λ of a positive integer n is a sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$$

such that $\sum_i \lambda_i = n$. We denote this by writing $\lambda \vdash n$, and say that k is the *height* $h(\lambda)$ of λ. The height of the empty partition (of 0) is 0. The (Ferrer's) diagram of a partition λ is the set of points $(i,j) \in \mathbf{Z}^2$ such that $1 \leq j \leq \lambda_i$. It is also denoted by λ. Clearly a partition is characterized by its diagram. The conjugate λ' of a partition λ is the partition with diagram $\{(j,i) \mid (i,j) \in \lambda\}$.

A standard Young tableau T is an injective labeling of a Ferrer's diagram by the elements of $\{1, 2, \ldots, n\}$ such that $T(i,j) < T(i+1,j)$ for $1 \leq i < k$ and $T(i,j) < T(i,j+1)$ for $1 \leq j < \lambda_i$. We further say that λ is the *shape* of the tableau T. For a given λ, the number f_λ of tableaux of shape λ is given by the *hook length* formula

$$f_\lambda = \frac{n!}{\prod_c h_c},$$

where $c = (i,j)$ runs over the set of points in the diagram of λ, and

$$h_c = \lambda_i - i + \lambda'_j - j + 1.$$

Other classical results in this context are

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!,$$

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and

$$\sum_{\lambda\vdash n} f_\lambda = \text{ coeff of } \frac{x^n}{n!} \text{ in } e^{x+x^2/2}.$$

We are interested in the enumeration of tableaux of height bounded by some integer k; that is to say we wish to compute the numbers

$$\tau_k(n) = \sum_{h(\lambda)\le k} f_\lambda$$

as well as

$$T_k(n) = \sum_{h(\lambda)\le k} f_\lambda^2.$$

For example, the first few sequences $\tau_k(n)$ for $n \ge 1$ are

$\tau_2(n) \to$ 1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, 924, 1716, 3432, 6435, 12870, 24310, 48620, 92378, ...

$\tau_3(n) \to$ 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, ...

$\tau_4(n) \to$ 1, 2, 4, 10, 25, 70, 196, 588, 1764, 5544, 17424, 56628, 184041, 613470, 2044900, ...

$\tau_5(n) \to$ 1, 2, 4, 10, 26, 75, 225, 715, 2347, 7990, 27908, 99991, 365587, 1362310, 5159208, ...

$\tau_6(n) \to$ 1, 2, 4, 10, 26, 76, 231, 756, 2556, 9096, 33231, 126060, 488488, 1948232, 7907185, ...

(These are sequences A001405, A001006, A005817, A049401, A007579 in [5].) For $T_k(n)$, we have

$T_2(n) \to$ 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, ...

$T_3(n) \to$ 1, 2, 6, 23, 103, 513, 2761, 15767, 94359, 586590, 3763290, 24792705, 167078577, ...

$T_4(n) \to$ 1, 2, 6, 24, 119, 694, 4582, 33324, 261808, 2190688, 19318688, 178108704, 1705985883, ...

$T_5(n) \to$ 1, 2, 6, 24, 120, 719, 5003, 39429, 344837, 3291590, 33835114, 370531683, 4285711539, ...

$T_6(n) \to$ 1, 2, 6, 24, 120, 720, 5039, 40270, 361302, 3587916, 38957991, 457647966, 5763075506, ...

(Sequences A000108, A005802, A052397, A052398, A052399 in [5].)

In [2] Gessel deduces the following formulae from a result of Gordon:

$$y_k(x) := \sum_{n=0}^{\infty} \frac{\tau_k(n)x^n}{n!} = \begin{cases} \det\left[J_{i-j}(x) - J_{i+j-1}(x)\right]_{1\le i,j\le k/2} & \text{if } k \text{ is even,} \\ e^x \det\left[J_{i-j}(x) - J_{i+j}(x)\right]_{1\le i,j\le (k-1)/2} & \text{if } k \text{ is odd,} \end{cases}$$

and

$$Y_k(x) := \sum_{n=0}^{\infty} \frac{T_k(n)x^n}{(n!)^2} = \det\left[I_{i-j}(x)\right]_{1\le i,j\le k},$$

where

$$J_k(x) = \sum_{n=0}^{\infty} \frac{x^{2n+k}}{n!\,(n+k)!}$$

and

$$I_k(x) = \sum_{n=0}^{\infty} \frac{x^{n+k/2}}{n!\,(n+k)!}$$

If k is positive integer, we set $J_{-k} := J_k$ and $I_{-k} := I_k$. The resulting expressions rapidly become unwieldy. For example,

$$y_2(x) = J_0(x) + J_1(x)$$

$$y_3(x) = e^x\,(J_0(x) - J_2(x))$$

$$y_4(x) = J_0(x)^2 + J_0(x)\,J_1(x) + J_0(x)\,J_3(x) - J_1(x)^2 - 2\,J_1(x)\,J_2(x) + J_1(x)\,J_3(x) - J_2(x)^2$$

$$y_5(x) = e^x\left(J_0(x)^2 - J_0(x)\,J_2(x) - J_0(x)\,J_4(x) - J_1(x)^2 + 2\,J_1(x)\,J_3(x) + J_2(x)\,J_4(x) - J_3(x)^2\right)$$

$$\begin{aligned} y_6(x) = {} & J_0(x)^3 + J_0(x)^2\,J_1(x) + J_0(x)^2\,J_3(x) + J_0(x)^2\,J_5(x) - 2\,J_0(x)\,J_1(x)^2 - 2\,J_0(x)\,J_1(x)\,J_2(x) \\ & + J_0(x)\,J_1(x)\,J_3(x) - 2\,J_0(x)\,J_1(x)\,J_4(x) + J_0(x)\,J_1(x)\,J_5(x) - 2\,J_0(x)\,J_2(x)^2 \\ & - 2\,J_0(x)\,J_2(x)\,J_3(x) - J_0(x)\,J_3(x)^2 + J_0(x)\,J_3(x)\,J_5(x) - J_0(x)\,J_4(x)^2 - J_1(x)^3 \\ & + 2\,J_1(x)^2\,J_2(x) + 2\,J_1(x)^2\,J_3(x) - 2\,J_1(x)^2\,J_4(x) - J_1(x)^2\,J_5(x) + 2\,J_1(x)\,J_2(x)^2 \\ & + 2\,J_1(x)\,J_2(x)\,J_3(x) + 2\,J_1(x)\,J_2(x)\,J_4(x) - 2\,J_1(x)\,J_2(x)\,J_5(x) + 2\,J_1(x)\,J_3(x)\,J_4(x) \\ & + J_1(x)\,J_3(x)\,J_5(x) - J_1(x)\,J_4(x)^2 - J_2(x)^2\,J_3(x) + 2\,J_2(x)^2\,J_4(x) - J_2(x)^2\,J_5(x) \\ & - 2\,J_2(x)\,J_3(x)^2 + 2\,J_2(x)\,J_3(x)\,J_4(x) - J_3(x)^3 \end{aligned}$$

We can simplify these using properties of Bessel functions. Recalling the easily deduced relations

$$J_k(x) = J_{k-2}(x) - \frac{1}{x}(n-k-1)\,J_{k-1}(x), \qquad k \ge 2,$$

we get, after some computation, the much simpler expressions

$$y_3(x) = x^{-1} e^x\,J_1(x)$$

$$y_4(x) = x^{-2}\,\left(-2\,x\,J_0(x)^2 + 2\,J_0(x)\,J_1(x) + (2\,x+1)\,J_1(x)^2\right)$$

$$y_5(x) = x^{-4} e^x\,\left(-4\;x^2\,J_0(x)^2 + 2\,x\,J_0(x)\,J_1(x) + 2\,\left(2\,x^2+1\right)\,J_1(x)^2\right)$$

$$\begin{aligned} y_6(x) = {} & x^{-6}\left(-4\,x^2\,(4\,x-3)\,J_0(x)^3 - 4\,x\,\left(4\,x^2 - 3\,x + 6\right)\,J_0(x)^2\,J_1(x)\right. \\ & \left. + 4\,\left(4\,x^3 - x^2 + 3\right)\,J_0(x)\,J_1(x)^2 + 4\,\left(4\,x^3 - x^2 + 5\,x + 1\right)\,J_1(x)^3\right) \end{aligned}$$

Similarly

$$Y_2(x) = I_0(x)^2 - I_1(x)^2$$

$$Y_3(x) = x^{-1}\left(2\,\sqrt{x}\,I_0(x)^2\,I_1(x) - I_0(x)\,I_1(x)^2 - 2\,\sqrt{x}\,I_1(x)^3\right)$$

$$\begin{aligned} Y_4(x) = {} & x^{-3}\big(-4\,x^2\,I_0(x)^4 + 8\,x\,\sqrt{x}\,I_0(x)^3\,I_1(x) + 4\,x\,(2\,x-1)\,I_0(x)^2\,I_1(x)^2 \\ & - 8\,x\,\sqrt{x}\,I_0(x)\,I_1(x)^3 - x\,(4\,x-1)\,I_1(x)^4\big) \end{aligned}$$

A theoretical argument (see [2]) shows that the generating functions $y_k(x)$ and $Y_k(x)$ are *D-finite*. That is to say, they satisfy linear differential equations with polynomial coefficients. In fact, it is well known and classical that one can translate such linear differential equations into recurrences with polynomial coefficients. More precisely, a *P-recurrence* for a sequence a_n is one of the form

$$p_0(n)\,a_n + p_1(x)\,a_{n-1} + \ldots + p_k(n)\,a_{n-k} = q(n),$$

where all $p_i(n)$, $1 \le i \le k$, and $q(n)$ are polynomials in n. We say that a sequence is P-recursive if it satisfies a P-recurrence. The class of P-recursive sequences is closed under point-wise products. Since $1/n!$ is easily seen to be P-recursive, it follows that, if a_n is P-recursive, then so are $a_n/n!$ and $a_n/n!^2$. The algorithmic translation from D-finite to P-recursive (and back) has been implemented in the package GFUN in Maple (see [4]), which also contains many other nice tools for handling recurrences and generating functions.

Computer experiments made by Krob, Favreau and the first author led to conjectures (see [1]) for an explicit form for P-recurrences for $\tau_h(n)$ and $T_h(n)$. These conjectures can be easily (and automatically) reformulated as linear differential equations for $y_k(x)$ and $Y_k(x)$. We first observe that it is not hard to show the existence of a linear differential equation of order bounded by

$$\ell(k) := \left\lfloor \frac{k}{2} \right\rfloor + 1$$

with polynomial coefficients, admitting $y_k(x)$ as a solution. In fact, this follows readily from the following proposition.

Proposition 1. *Let $\mathcal{V}_k$ denote the vector space over the field $\mathbf{C}(x)$ of rational functions in x spanned by $y_k(x)$ and all its derivatives. Then*

$$\dim \mathcal{V}_k \leq \ell(k).$$

Proof. Setting $n := \ell(k) - 1$, it is clear from our previous discussion that y_k lies in the span $\mathcal{W}_k$ of the set of $\ell(k)$ elements given by

$$\{J_0(x)^m \, J_1(x)^{n-m} \mid 0 \leq m \leq n \}$$

if k is even, and by

$$\{e^x \, J_0(x)^m \, J_1(x)^{n-m} \mid 0 \leq m \leq n \}$$

if k is odd. $\mathcal{W}_k$ is clearly closed under differentiation, since we easily see that

$$\begin{aligned} \frac{d}{dx} J_0(x) &= 2\, J_1(x), \\ \frac{d}{dx} J_1(x) &= 2\, J_0(x) - \frac{1}{x}\, J_1(x), \end{aligned} \tag{1}$$

from which we deduce that

$$\frac{d}{dx} J_0(x)^a \, J_1(x)^b = \frac{2\,a}{x}\, J_0(x)^{a-1}\, J_1(x)^{b+1}\, x + 2\, b\, J_0(x)^{a+1}\, J_1(x)^{b-1} - \frac{b}{x}\, J_0(x)^a \, J_1(x)^b, \tag{2}$$

as well as a similar expression for the derivative of $e^x \, J_0(x)^a \, J_1(x)^b$. Thus $\mathcal{V}_k$ is contained in $\mathcal{W}_k$, and hence its dimension is bounded by $\ell(k)$. ∎

Setting for the moment $n := \ell(k)$ and $y := y_k(x)$, it clearly follows from the above proposition that

$$y,\ y',\ y'',\ \ldots\ , y^{(n)}$$

are linearly dependent, hence $y_k(x)$ satisfies a homogeneous linear differential equation of order (at most) $\ell(k)$ with polynomial coefficients (in x). However, it appears that a stronger result holds.

Conjecture (Bergeron–Favreau–Krob, [1]). For each k, there are polynomials $p_m(x)$ of degree at most $\ell - 1$ such that $y_k(x)$ is a solution of

$$\sum_{m=0}^{\ell} p_m(x) y^{(m)} \ = \ 0,$$

where $\ell = \ell(k)$. Moreover, for $m \geq 1$, $p_m(x) = q_m(x)\, x^{m-1}$, and $p_\ell(x) = x^{\ell-1}$.

The first few cases for $y_k(x)$ are*

* Here $\rightarrow$ means “is a solution of”.

$$y_2(x) \;\rightarrow\; x\,y'' + 2\,y' - 2\,(2\,x+1)\,y = 0$$

$$y_3(x) \;\rightarrow\; x\,y'' - (2\,x-3)\,y' - 3\,(x+1)\,y = 0$$

$$y_4(x) \;\rightarrow\; x^2\,y''' + 10\,x\,y'' - 4\,\left(4\,x^2+2\,x-5\right)y' - 4\,(8\,x+5)\,y = 0$$

$$y_5(x) \;\rightarrow\; x^2\,y''' - (3\,x-13)\,x\,y'' - \left(13\,x^2+26\,x-35\right)y' + 5\,\left(3\,x^2-7\,x-7\right)y = 0$$

Equating coefficients of $x^n/n!$ on both hand sides of these differential equations, one finds that they are equivalent to the recurrences

$$(n+1)\,\tau_2(n) - 2\,\tau_2(n-1) - 4\,(n-1)\,\tau_2(n-2) = 0$$

$$(n+2)\,\tau_3(n) - (2\,n+1)\,\tau_3(n-1) - 3\,(n-1)\,\tau_3(n-2) = 0$$

$$(n+3)\,(n+4)\,\tau_4(n) - 16\,(n-1)\,\tau_4(n-2)\,n - (8\,n+12)\,\tau_4(n-1) = 0$$

$$(n+4)\,(n+6)\,\tau_5(n) - \left(3\,n^2+17\,n+15\right)\tau_5(n-1) - (n-1)(13\,n+9)\,\tau_5(n-2) + 15\,(n-1)\,(n-2)\,\tau_5(n-3) = 0$$

Up to now, only these recurrences (that is, for $k \le 5$), had been implicitly known (see [3]). However, using the simplified expressions for $y_k(x)$ given here, and a reformulation in term of linear differential equations (with the help of GFUN [4]) we have been able to check (in the form of a computer algebra proof) that the conjecture above is true for $k \le 11$, from which it follows that the corresponding recurrences hold. This computer verification simply uses the derivation rules (1) for $J_0(x)$ and $J_1(x)$ to simplify the expressions obtained by substitution of Gessel's formulae in the following differential equations.

$$\begin{aligned} y_6(x) \;\rightarrow\;& x^3\,y^{(4)} + 28\,x^2\,y''' - 10\,\left(4\,x^2+2\,x-23\right)x\,y'' \\ & - 4\,\left(108\,x^2+61\,x-135\right)y' + 36\,(2\,x+5)\left(2\,x^2-3\,x-3\right)y = 0 \end{aligned}$$

$$\begin{aligned} y_7(x) \;\rightarrow\;& x^3\,y^{(4)} - 2\,(2\,x-17)\,x^2\,y''' - \left(34\,x^2+102\,x-343\right)x\,y'' \\ & + \left(76\,x^3-450\,x^2-686\,x+1001\right)y' + 7\,\left(15\,x^3+74\,x^2-143\,x-143\right)y = 0 \end{aligned}$$

$$\begin{aligned} y_8(x) \;\rightarrow\;& x^4\,y^{(5)} + 60\,x^3\,y^{(4)} - 2\,\left(40\,x^2+20\,x-619\right)x^2\,y''' - 4\,\left(608\,x^2+331\,x-2567\right)x\,y'' \\ & + 8\,\left(128\,x^4+128\,x^3-2480\,x^2-1527\,x+3536\right)y' + 128\,\left(64\,x^3+72\,x^2-286\,x-221\right)y = 0 \end{aligned}$$

$$\begin{aligned} y_9(x) \;\rightarrow\;& x^4\,y^{(5)} - 5\,(x-14)\,x^3\,y^{(4)} - \left(70\,x^2+280\,x-1693\right)x^2\,y''' \\ & + \left(230\,x^3-2492\,x^2-5079\,x+16535\right)x\,y'' \\ & + \left(789\,x^4+5544\,x^3-24073\,x^2-33070\,x+53865\right)y' \\ & - 27\,\left(35\,x^4-274\,x^3-1017\,x^2+1995\,x+1995\right)y = 0 \end{aligned}$$

$$\begin{aligned} y_{10}(x) \;\rightarrow\;& x^5\,y^{(6)} + 110\,x^4\,y^{(5)} - 2\,\left(70\,x^2+35\,x-2269\right)x^3\,y^{(4)} \\ & - 4\,\left(2268\,x^2+1211\,x-21752\right)x^2\,y''' \\ & + 4\,\left(1036\,x^4+1036\,x^3-48033\,x^2-27900\,x+191477\right)x\,y'' \\ & + 8\,\left(14300\,x^4+15542\,x^3-185404\,x^2-121352\,x+303875\right)y' \\ & - 200\,\left(72\,x^5+108\,x^4-3262\,x^3-3987\,x^2+14960\,x+12155\right)y = 0 \end{aligned}$$

$$
\begin{aligned}
y_{11}(x) \;\rightarrow\;\; & x^5\, y^{(6)} - (6\,x - 125)\, x^4\, y^{(5)} - \left(125\,x^2 + 625\,x - 5873\right) x^3\, y^{(4)} \\
& + 2\,\left(270\,x^3 - 4611\,x^2 - 11746\,x + 64252\right) x^2\, y''' \\
& + \left(3319\,x^4 + 30166\,x^3 - 223422\,x^2 - 385512\,x + 1293125\right) x\, y'' \\
& - \left(7734\,x^5 - 104329\,x^4 - 493828\,x^3 + 1987124\,x^2 + 2586250\,x - 4697275\right) y' \\
& - 11\,\left(945\,x^5 + 11343\,x^4 - 62023\,x^3 - 204012\,x^2 + 427025\,x + 427025\right) y = 0
\end{aligned}
$$

However, these verifications rapidly become (computer) time consuming. For example, with $k = 11$, we have to substitute in this last differential equation the following expression

$$
\begin{aligned}
y_{11}(x) = \frac{138240\, e^x}{x^{25}} \Big(& -14\left(32\,x^6 + 177\,x^4 + 198\,x^2 - 72\right) x^5\, J_0(x)^5 \\
& + 8\,\left(16\,x^8 + 256\,x^6 + 825\,x^4 + 585\,x^2 - 495\right) x^4\, J_1(x)\, J_0(x)^4 \\
& + 4\,\left(192\,x^8 + 833\,x^6 + 495\,x^4 + 135\,x^2 + 1440\right) x^3\, J_1(x)^2\, J_0(x)^3 \\
& - \left(256\,x^{10} + 3648\,x^8 + 10799\,x^6 + 9690\,x^4 + 1980\,x^2 + 3600\right) x^2\, J_1(x)^3\, J_0(x)^2 \\
& - 5\,\left(64\,x^{10} + 190\,x^8 - 77\,x^6 + 114\,x^4 + 504\,x^2 - 144\right) x\, J_1(x)^4\, J_0(x) \\
& + \left(128\,x^{12} + 1632\,x^{10} + 4557\,x^8 + 5482\,x^6 + 4158\,x^4 + 2052\,x^2 + 72\right) J_1(x)^5 \Big)
\end{aligned}
$$

and simplify. Clearly we could go on to larger cases, but the point seems to be made that the conjectures are reasonable.

Similar considerations for the enumeration of pairs of tableaux, with the following differential equations, settle the corresponding conjectures for the cases $k \le 7$:

$$
Y_2(x) \;\rightarrow\;\; x^2\, y''' + 4\,x\, y'' - 2\,(2\,x - 1)\, y' - 2\,y = 0
$$

$$
Y_3(x) \;\rightarrow\;\; x^3\, y^{(4)} + 10\,x^2\, y''' - (10\,x - 23)\, x\, y'' - (32\,x - 9)\, y' + 9\,(x - 1)\, y = 0
$$

$$
\begin{aligned}
Y_4(x) \;\rightarrow\;\; & x^4\, y^{(5)} + 20\,x^3\, y^{(4)} - 2\,(10\,x - 59)\, x^2\, y''' - 2\,(91\,x - 110)\; x\, y'' \\
& + 4\,\left(16\,x^2 - 87\,x + 20\right) y' + 16\,(8\,x - 5)\, y = 0
\end{aligned}
$$

$$
\begin{aligned}
Y_5(x) \;\rightarrow\;\; & x^5\, y^{(6)} + 35\,x^4\, y^{(5)} - 7\,(5\,x - 59)\, x^3\, y^{(4)} - 2\,(336\,x - 979)\, x^2\, y''' + \left(259\,x^2 - 3650\,x + 3383\right) x\, y'' \\
& + \left(1917\,x^2 - 5708\,x + 1225\right) y' - 25\,\left(9\,x^2 - 93\,x + 49\right) y = 0
\end{aligned}
$$

$$
\begin{aligned}
Y_6(x) \;\rightarrow\;\; & x^6\, y^{(7)} + 56\,x^5\, y^{(6)} - 28\,(2\,x - 41)\, x^4\, y^{(5)} - 4\,(483\,x - 2684)\, x^3\, y^{(4)} \\
& + 4\,\left(196\,x^2 - 5480\,x + 11543\right) x^2\, y''' + 8\,\left(1686\,x^2 - 11941\,x + 9830\right) x\, y'' \\
& - 4\,\left(576\,x^3 - 14931\,x^2 + 34438\,x - 7290\right) y' - 72\,\left(144\,x^2 - 821\,x + 405\right) y = 0
\end{aligned}
$$

$$
\begin{aligned}
Y_7(x) \;\rightarrow\;\; & x^7\, y^{(8)} + 84\,x^6\, y^{(7)} - 42\,(2\,x - 65)\, x^5\, y^{(6)} - 2\,(2352\,x - 21881)\, x^4\, y^{(5)} \\
& + 3\,\left(658\,x^2 - 31606\,x + 121455\right) x^3\, y^{(4)} + 2\,\left(31986\,x^2 - 424260\,x + 754183\right) x^2\, y''' \\
& - \left(12916\,x^3 - 648834\,x^2 + 3329230\,x - 2610671\right) x\, y'' \\
& - \left(175704\,x^3 - 2292734\,x^2 + 4684008\,x - 1002001\right) y' \\
& + 49\,\left(225\,x^3 - 9630\,x^2 + 42313\,x - 20449\right) y = 0
\end{aligned}
$$

2. Acknowledgments

The Maple package *gfun* (available as a shared library) was used extensively in the elaboration of the conjectures and results in this note.

3. References

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(Concerned with sequences A000108, A001006, A001405, A005802, A005817, A007579, A049401, A052397, A052398, A052399.)

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