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# Derangements and Applications

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**Abstract**

In this paper we introduce some formulas for the number of derangements. Then we define the derangement function and use the software package MAPLE to obtain some integrals related to the incomplete gamma function and also to some hypergeometric summations.

## 1 Introduction and motivation

A permutation of $S_n = \{1, 2, 3, \cdots, n\}$ that has no fixed points is a *derangement* of $S_n$. Let $D_n$ denote the number of derangements of $S_n$. It is well-known that

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}, \tag{1}$$

$$D_n = \| \frac{n!}{e} \| \quad (\| \ \| \text{ denotes the nearest integer}). \tag{2}$$

We can rewrite (2) as follows:

$$D_n = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor.$$

We can generalize the above formula replacing $\frac{1}{2}$ by every $m \in [\frac{1}{3}, \frac{1}{2}]$. In fact we have:

**Theorem 1.1** *Suppose $n \geq 1$ is an integer, we have*

$$D_n = \begin{cases} \lfloor \frac{n!}{e} + m_1 \rfloor, & n \text{ is odd}, m_1 \in [0, \frac{1}{2}]; \\ \lfloor \frac{n!}{e} + m_2 \rfloor, & n \text{ is even}, m_2 \in [\frac{1}{3}, 1]. \end{cases} \tag{3}$$

For a proof of this theorem, see Hassani [3]. At the end of the next section we give another proof of it.

On the other hand, the idea of proving (2) leads to a family of formulas for the number of derangements, as follows: we have

$$|\frac{n!}{e} - D_n| \leq \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots .$$

Let $M(n)$ denote the right side of above inequality. We have

$$M(n) < \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots = \frac{1}{n},$$

and therefore

$$D_n = \lfloor \frac{n!}{e} + \frac{1}{n} \rfloor \quad (n \geq 2). \tag{4}$$

Also we can get a better bound for $M(n)$ as follows

$$M(n) < \frac{1}{n+1}(1 + \frac{1}{(n+2)} + \frac{1}{(n+2)^2} + \cdots) = \frac{n+2}{(n+1)^2},$$

and similarly

$$D_n = \lfloor \frac{n!}{e} + \frac{n+2}{(n+1)^2} \rfloor \quad (n \geq 2). \tag{5}$$

The above idea is extensible, but before extending we recall a useful formula (see [2, 3]). For every positive integer $n \geq 1$, we have

$$\sum_{i=0}^{n} \frac{n!}{i!} = \lfloor en! \rfloor. \tag{6}$$

## 2 New families and some other formulas

**Theorem 2.1** *Suppose $m$ is an integer and $m \geq 3$. The number of derangements of $n$ distinct objects ($n \geq 2$) is*

$$D_n = \lfloor (\frac{\lfloor e(n+m-2)! \rfloor}{(n+m-2)!} + \frac{n+m}{(n+m-1)(n+m-1)!} + e^{-1})n! \rfloor - \lfloor en! \rfloor. \tag{7}$$

**Proof**: For $m \geq 3$ we have

$$|\frac{n!}{e} - D_n| < \frac{1}{(n+1)}(1 + \frac{1}{(n+2)}(\cdots 1 + \frac{1}{(n+m-1)}(\frac{n+m}{n+m-1})\cdots)).$$

Let $M_m(n)$ denote the right side of the above inequality; we have

$$(n+1)(n+2)(n+3)\cdots(n+m-1)M_m(n) =$$
$$(n+2)(n+3)\cdots(n+m-1) + (n+3)\cdots(n+m-1) + \cdots + (n+m-1) + \frac{n+m}{n+m-1},$$

and dividing by $(n+1)(n+2)(n+3)\cdots(n+m-1)$ we obtain

$$M_m(n) = n!(\frac{n+m}{(n+m-1)(n+m-1)!} + \sum_{i=n+1}^{n+m-2} \frac{1}{i!}).$$

Therefore

$$D_n = \lfloor \frac{n!}{e} + n!(\frac{n+m}{(n+m-1)(n+m-1)!} + \sum_{i=n+1}^{n+m-2} \frac{1}{i!})\rfloor. \tag{8}$$

Now consider (6) and rewrite (8) by using $\sum_{i=n+1}^{n+m-2} \frac{1}{i!} = \sum_{i=0}^{n+m-2} \frac{1}{i!} - \sum_{i=0}^{n} \frac{1}{i!}$. The proof is complete. □

**Corollary 2.2** *For $n \geq 2$, we have*

$$D_n = \lfloor (e + e^{-1})n! \rfloor - \lfloor en! \rfloor. \tag{9}$$

**Proof**: We give two proofs.
Method 1. Because (7) holds for all $m \geq 3$, we have

$$D_n = \lim_{m\to\infty} \lfloor (\frac{\lfloor e(n+m-2)! \rfloor}{(n+m-2)!} + \frac{n+m}{(n+m-1)(n+m-1)!} + e^{-1})n! \rfloor - \lfloor en! \rfloor$$
$$= \lfloor (e + e^{-1})n! \rfloor - \lfloor en! \rfloor.$$

Method 2. By using (6), we have

$$M(n) = n!(e - \sum_{i=0}^{n} \frac{1}{i!}) = en! - \lfloor en! \rfloor = \{en!\} \quad (n \geq 1, \{\ \} \text{ denotes the fractional part}),$$

and the proof follows. □

Now

$$\lim_{m\to\infty} M_m(n) = M(n),$$

and if we put $M_1(n) = \frac{1}{n}$ and $M_2(n) = \frac{n+2}{(n+1)^2}$ (see formulas (4) and (5)), then

$$M_{m+1}(n) < M_m(n) \quad (n \geq 1).$$

Now we find bounds sharper than $\{en!\}$ for $e^{-1}n! - D_n$ and consequently another family of formulas for $D_n$. This family is an extension of (9).

**Theorem 2.3** *Suppose $m$ is an integer and $m \geq 1$. The number of derangements of $n$ distinct objects ($n \geq 2$) is*

$$D_n = \lfloor (\frac{\{e(n+2m)!\}}{(n+2m)!} + \sum_{i=1}^{m} \frac{n+2i-1}{(n+2i)!} + e^{-1}) n! \rfloor. \tag{10}$$

**Proof**: Since $m \geq 1$ we have

$$\frac{e^{-1}n! - D_n}{(-1)^{n+1}} = n! \sum_{i=1}^{\infty} (\frac{1}{(n+2i-1)!} - \frac{1}{(n+2i)!}) < n! (\sum_{i=1}^{m} \frac{n+2i-1}{(n+2i)!} + \sum_{i=2m+1}^{\infty} \frac{1}{(n+i)!}).$$

Let $N_m(n)$ denote the right member of above inequality. Considering (6), we have

$$N_m(n) = n! (\sum_{i=1}^{m} \frac{n+2i-1}{(n+2i)!} + \frac{\{e(n+2m)!\}}{(n+2m)!}),$$

and for ($n \geq 2$), $D_n = \lfloor e^{-1} n! + N_m(n) \rfloor$. This completes the proof. $\square$

**Corollary 2.4** *For all integers $m, n \geq 1$, we have*

$$N_{m+1}(n) < N_m(n), \quad N_1(n) < \{en!\}.$$

Therefore we have the following chain of bounds for $|\frac{n!}{e} - D_n|$

$$|\frac{n!}{e} - D_n| < \cdots < N_2(n) < N_1(n) < \{en!\} < \cdots < M_2(n) < M_1(n) < 1 \quad (n \geq 2).$$

**Question 1.** Can we find the following limit?

$$\lim_{m \to \infty} N_m(n).$$

Before going to the next section we give our proof of Theorem 1. The idea of present proof is hidden in Apostol's analysis [1], where he proved the irrationality of $e$ by using (11). And now,

**Proof**: (Proof of Theorem 1) Suppose $k \geq 1$ be an integer, we have

$$0 < \frac{1}{e} - \sum_{i=0}^{2k-1} \frac{(-1)^i}{i!} < \frac{1}{(2k)!} \tag{11}$$

so, for every $m_1$, we have

$$m_1 < \frac{(2k-1)!}{e} + m_1 - \sum_{i=0}^{2k-1} \frac{(-1)^i (2k-1)!}{i!} < m_1 + \frac{1}{2}$$

if $0 \le m_1 \le \frac{1}{2}$ , then

$$\sum_{i=0}^{2k-1} \frac{(-1)^i(2k-1)!}{i!} = \lfloor \frac{(2k-1)!}{e} + m_1 \rfloor.$$

Similarly since (11), for every $m_2$ we have

$$m_2 - 1 < \frac{(2k)!}{e} + m_2 - \sum_{i=0}^{2k} \frac{(-1)^i(2k)!}{i!} < m_2.$$

Now, if $m_2 \ge \frac{1}{3}$, then

$$0 < \frac{(2k)!}{e} + m_2 - \sum_{i=0}^{2k} \frac{(-1)^i(2k)!}{i!}$$

therefore, if $\frac{1}{3} \le m_2 \le 1$, we obtain

$$\sum_{i=0}^{2k} \frac{(-1)^i(2k)!}{i!} = \lfloor \frac{(2k)!}{e} + m_2 \rfloor.$$

This completes the proof. $\square$

In the next section there are some applications of the proven results.

# 3 The derangement function, incomplete gamma and hypergeometric functions

Let's find other formulas for $D_n$. The computer algebra program MAPLE yields that

$$D_n = (-1)^n \texttt{hypergeom}([1, -n], [\ \ ], 1),$$

and

$$D_n = e^{-1}\Gamma(n+1, -1),$$

where $\texttt{hypergeom}([1, -n], [\ \ ], 1)$ is MAPLE's notation for a hypergeometric function. More generally, $\texttt{hypergeom}([a_1 \ \ a_2 \ \ \cdots \ \ a_p], [b_1 \ \ b_2 \ \ \cdots \ \ b_q], x)$ is defined as follows (see [4]),

$${}_pF_q \left[ \begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix} ; x \right] = \sum_{k \ge 0} t_k x^k$$

where

$$\frac{t_{k+1}}{t_k} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)} x.$$

Also $\Gamma(n+1, -1)$ is an incomplete gamma function and generally defined as follows:

$$\Gamma(a, z) = \int_z^\infty e^{-t} t^{a-1} dt \quad (Re(a) > 0),$$

Now, because we know the value of $D_n$, we can estimate some summations and integrals. To do this, we define the *derangement function*, a natural generalization of derangements, denoted by $D_n(x)$, for every integer $n \geq 0$ and every real $x$ as follows:

$$D_n(x) = \begin{cases} n! \sum_{i=0}^{n} \frac{x^i}{i!}, & x \neq 0; \\ n!, & x = 0. \end{cases}$$

It is easy to obtain the following generalized recursive relations:

$$D_n(x) = (x+n)D_{n-1}(x) - x(n-1)D_{n-2}(x) = x^n + nD_{n-1}(x), \quad (D_0(x) = 1, D_1(x) = x+1).$$

Note that $D_n(x)$ is a nice polynomial. Its value for $x = -1$ is $D_n$, for $x = 0$ is the number of permutations of $n$ distinct objects and for $x = 1$ is $w_{n+2}$ = the number of distinct paths between every pair of vertices in a complete graph on $n+2$ vertices, and

$$D_n(1) = \lfloor en! \rfloor \quad (n \geq 1), \quad \text{(see [3])}.$$

A natural question is

**Question 2.** Is there any combinatorial meaning for the value of $D_n(x)$ for other values of $x$?

The above definitions yield

$$D_n(x) = x^n \, {}_2F_0 \left[ \begin{matrix} 1 & -n \\ - & \end{matrix} ; -\frac{1}{x} \right] \quad (x \neq 0),$$

and

$$D_n(x) = e^x \Gamma(n+1, x). \tag{12}$$

We obtain

$${}_2F_0 \left[ \begin{matrix} 1 & -n \\ - & \end{matrix} ; -1 \right] = \lfloor en! \rfloor,$$

and

$${}_2F_0 \left[ \begin{matrix} 1 & -n \\ - & \end{matrix} ; 1 \right] = (-1)^n \left\lfloor \frac{n!+1}{e} \right\rfloor .$$

Also we have some corollaries.

**Corollary 3.1** *For every real $x \neq 0$ we have*

$${}_1F_1 \left[ \begin{matrix} n+1 \\ n+2 \end{matrix} ; -x \right] = \frac{(n+1)(n! - e^{-x} D_n(x))}{x^{n+1}}.$$

**Proof**: Obvious. $\square$

**Corollary 3.2** *For every integer* $n \geq 1$ *we have*

$$\int_{-1}^{\infty} e^{-t} t^n dt = e \left\lfloor \frac{n!+1}{e} \right\rfloor,$$

$$\int_{0}^{\infty} e^{-t} t^n dt = n!,$$

$$\int_{1}^{\infty} e^{-t} t^n dt = \frac{\lfloor en! \rfloor}{e},$$

*and*

$$\int_{0}^{1} e^{-t} t^n dt = \frac{\{en!\}}{e},$$

$$\int_{-1}^{0} e^{-t} t^n dt = \begin{cases} -e\{\frac{n!}{e}\} & n \text{ is odd,} \\ e - e\{\frac{n!}{e}\} & n \text{ is even.} \end{cases}$$

$$\int_{-1}^{1} e^{-t} t^n dt = e\lfloor (e+e^{-1})n! \rfloor - (e+e^{-1})\lfloor en! \rfloor,$$

**Proof**: Use relations (3), (6), (9), (12) and the definition of derangement function in the case $x = 0$. $\square$

**Question 3.** Are there any similar formulas for ${}_2F_0 \left[ \begin{matrix} 1 & -n \\ - & \end{matrix} ; -\frac{1}{x} \right]$? In other words, given any real number $x$, is there an interval $I$ (dependent on $x$) such that

$$n! \sum_{i=0}^{n} \frac{x^i}{i!} = \lfloor e^x n! + m \rfloor \quad (m \in I_x)?$$

# 4 Acknowledgements

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# References

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(Concerned with sequence A000166.)

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