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Meanders and Motzkin Words

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Abstract

We study the construction of closed meanders and systems of closed meanders, using Motzkin words with four letters. These words are generated by applying binary operation on the set of Dyck words. The procedure is based on the various kinds of intersection of the meandric curve with the horizontal line.

1 Introduction

Among various efforts to study and to generate meanders, Jensen [4] has used sequences related to the intervals between the crossing points along the horizontal line, Franz and Earnshaw [3] have used noncrossing partitions, whereas the authors [8] as well as Barraud et al. [1] have used planar permutations which follow the meandric curve.

This paper refers to the study and construction of closed meanders and systems of closed meanders, using Motzkin words.

The following definitions and notation refer to notions that are necessary for the development of the paper.

A word $u \in \{a, \bar{a}\}^*$ is called a *Dyck word* if $|u|_a = |u|_{\bar{a}}$ and for every factorization $u = pq$ we have $|p|_a \geq |p|_{\bar{a}}$ where $|u|_a$, $|p|_a$ (resp. $|u|_{\bar{a}}$, $|p|_{\bar{a}}$) denote the number of occurrences of a (resp. $\bar{a}$) in the words u,p.

A word $w \in \{a, \bar{a}, x, y\}^*$ is called a *Motzkin word* if $|w|_a = |w|_{\bar{a}}$ and for every factorization $w = pq$ we have $|p|_a \geq |p|_{\bar{a}}$, or equivalently if the word obtained by deleting every occurence of x, y from w is a Dyck word of $\{a, \bar{a}\}^*$.

Let $\mathcal{D}_{2n}$ denote the set of all Dyck words of length $2n$. It is well known that the cardinality of $\mathcal{D}_{2n}$ equals to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (A000108); Panayotopoulos and Sapounakis [6] have presented a construction of $\mathcal{D}_{2n}$.

Let $u = u_1 u_2 \cdots u_{2n}$ with $u \in D_{2n}$. Two indices i, j such that $i < j$, $u_i = a, u_j = \bar{a}$ are called *conjugates* with respect to u if j is the smallest element of $\{i{+}1, i{+}2, \ldots, 2n\}$ for which the subword $u_i u_{i+1} \cdots u_j$ is a Dyck word.

There exists a bijection between D_{2n} and N_{2n}, since for each $u \in D_{2n}$ we can determine its corresponding nested set $S_u \in N_{2n}$ as follows : $\{i, j\} \in S_u$ if and only if i, j are conjugate indices with respect to u.

For example, the nested set $\{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7, 8\}, \{9, 10\}\}$ corresponds to the Dyck word $u = a\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}$.

We also recall that if we denote by dom S all the elements of N^* that belong to some pair of a nested set of pairs S, we say that two nested set S_1, S_2 are *matching* if dom S_1 = dom S_2 and dom A = dom B, $A \subseteq S_1$, $B \subseteq S_2$ imply that either $A = B = \emptyset$ or dom A = dom S_1.

Furthermore, we call $B \subseteq$ dom S *complete* if for every $a \in B$ with $\{a, b\} \in S$, we have $b \in B$. We write $S/B = \{\{a, b\} \in S : a \in B\}$. For every two nested sets S_1, S_2 with dom S_1 = dom S_2 that are not matching, there exists a partition $B_1, B_2, \ldots, B_k$ of dom S_1 with B_i complete, such that the sets $S_1/B_i, S_2/B_i$, $i \in [k]$ are matching; we then call S_1, S_2 *k-matching* [7].

Geometrically, if we draw two matching nested sets, one above and the other underneath the horizontal axis, they form a simple, closed curve, whereas two k-matching nested sets create k such curves; (see Figures 1 and 2).

In section 2 we define the m-Motzkin words. To each such word corresponds a pair of nested sets which are either matching or k-matching. The set of m-Motzkin words is partitioned into classes of either two or four elements.

In section 3 we prove that there exists a bijection between the set of closed meanders and the set of m-Motzkin words which correspond to matching nested sets. Using this bijection, we can generate closed meanders from m-Motzkin words.

In section 4 we extend the above results to systems of meanders and we present a recursive generation of these systems.

2 m-Motzkin words

For every pair $u = u_1u_2\cdots u_{2n}$, $u' = u'_1u'_2\cdots u'_{2n}$ of elements of $\mathcal{D}_{2n}$, we define $u \circ u'$ to be the word $w = w_1w_2\cdots w_{2n}$, with

$$w_i = \begin{cases} a, & \textit{if} \quad u_i = u'_i = a; \\ \bar{a}, & \textit{if} \quad u_i = u'_i = \bar{a}; \\ b, & \textit{if} \quad u_i = a,\ u'_i = \bar{a}; \\ \bar{b}, & \textit{if} \quad u_i = \bar{a},\ u'_i = a. \end{cases}$$

For example, from $u = a\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}$ and $u' = a\ \bar{a}\ a\ a\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}\ \bar{a}$ we obtain $u \circ u' = a\ b\ a\ \bar{b}\ \bar{a}\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}$.

We write $\widehat{W}_{2n} = \{w : w = u \circ u',\ \ u, u' \in \mathcal{D}_{2n}\}$.

Proposition 2.1 *If $w \in \widehat{W}_{2n}$ then w is a Motzkin word of $\{a, \bar{a}, b, \bar{b}\}^*$, with $|w|_b = |w|_{\bar{b}}$.*

Proof : Let $I_1 = \{i \in [2n] : u_i = u'_i = a\}$, $I_2 = \{i \in [2n] : u_i = a, u'_i = \bar{a}\}$, $I_3 = \{i \in [2n] : u_i = \bar{a}, u'_i = a\}$, $I_4 = \{i \in [2n] : u_i = u'_i = \bar{a}\}$.
Given that $u, u' \in \mathcal{D}_{2n}$ we have that: $|I_1|+|I_2| = |I_3|+|I_4|$ and $|I_1|+|I_3| = |I_2|+|I_4|$; so we get $|I_3| = |I_2|$ and $|I_1| = |I_4|$, i.e. $|w|_b = |w|_{\bar{b}}$ and $|w|_a = |w|_{\bar{a}}$.

Let now z be the word that we obtain by deleting every occurrence of $b, \bar{b}$ in w.

Obviously $|z|_a = |w|_a = |w|_{\bar{a}} = |z|_{\bar{a}}$. In order to show that z is a Dyck word, we must also have $|s|_a \geq |s|_{\bar{a}}$, for every factorization $z = st$. This is true, since if $|s|_a < |s|_{\bar{a}}$, for some such factorization, then for at least one of the words u, u' we would have a factorization pq with $|p|_a < |p|_{\bar{a}}$, contradicting the assumption that both u and u' are Dyck words. □

We call the elements of $\widehat{W}_{2n}$ *meandric Motzkin words* (or *m-Motzkin words*) of length $2n$.

Let now $w = w_1w_2\cdots w_{2n}$, with $w \in \widehat{W}_{2n}$. From w we obtain two words $r = r_1r_2\cdots r_{2n}$, $r' = r'_1r'_2\cdots r'_{2n}$ of $\{a, \bar{a}\}^*$, with

$$r_i = \begin{cases} a, & \text{if } w_i = a \text{ or } b; \\ \bar{a}, & \text{if } w_i = \bar{a} \text{ or } \bar{b}, \end{cases} \qquad r'_i = \begin{cases} a, & \text{if } w_i = a \text{ or } \bar{b}; \\ \bar{a}, & \text{if } w_i = \bar{a} \text{ or } b. \end{cases}$$

We call r and r' *relatives* of w.

Practically, in order to obtain r we change each occurrence of $b, \bar{b}$ of w into $a, \bar{a}$ respectively, whereas in order to obtain r' we change $b, \bar{b}$ into $\bar{a}, a$ respectively.

Proposition 2.2 *Let $w \in \widehat{W}_{2n}$ with $w = u \circ u'$, $u, u' \in \mathcal{D}_{2n}$ and let r, r' be its relatives. Then $r = u$ and $r' = u'$.*

Proof : Let $w_i = a$ (resp. $\bar{a}$). Then $r_i = a = u_i$ and $r'_i = a = u'_i$ (resp. $r_i = \bar{a} = u_i$ and $r'_i = \bar{a} = u'_i$). Let now $w_i = b$ (resp. $\bar{b}$). Then $r_i = a = u_i$ and $r'_i = \bar{a} = u'_i$ (resp. $r_i = \bar{a} = u_i$ and $r'_i = a = u'_i$).

So, we realize that in every case the elements of r and u as well as the elements of r' and u' coincide, giving the required result. $\square$

From the bijection between the sets $\mathcal{D}_{2n} \times \mathcal{D}_{2n}$ and $\widehat{W}_{2n}$ that we have established, we obviously get the following relation :

$$|\widehat{W}_{2n}| = (C_n)^2.$$

Notice that from the word $u \circ u'$ we immediately obtain the word $u' \circ u$, by interchanging the letters b and $\bar{b}$. So, in order to generate the set $\widehat{W}_{2n}$ it is actually enough to construct half of its elements.

So, by the above procedure we also create for each $w \in \widehat{W}_{2n}$ two nested sets S_w, S'_w on $[2n]$ corresponding to the words $r, r' \in \mathcal{D}_{2n}$.

We denote with W_{2n} (resp W^k_{2n}) the set of al the words $w \in \widehat{W}_{2n}$ for which S_w, S'_w are matching (resp. k-matching).

For example, the word $w = a\ b\ a\ \bar{b}\ \bar{a}\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}$ is an m-Motzkin word, for which we have $r = a\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}$ and $r' = a\ \bar{a}\ a\ a\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}\ \bar{a}$.

The corresponding nested sets $S_w = \{\{1,6\},\{2,5\},\{3,4\},\{7,8\},\{9,10\}\}$ and $S'_w = \{\{1,2\},\{3,10\}, \{4,5\},\{6,7\},\{8,9\}\}$ are matching.

Similarly, the word $w = a\ a\ b\ \bar{b}\ \bar{a}\ \bar{b}\ a\ \bar{a}\ b\ \bar{a}$ is a m-Motzkin word, for which we have $r = a\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}\ a\ \bar{a}\ a\ \bar{a}$ and $r' = a\ a\ \bar{a}\ a\ \bar{a}\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}$. The corresponding nested sets

$$S_w = \{\{1,6\},\{2,5\},\{3,4\},\{7,8\},\{9,10\}\}$$

and

$$S'_w = \{\{1,10\},\{2,3\},\{4,5\},\{6,9\},\{7,8\}\}$$

are 3-matching, with $B_1 = \{1,6,9,10\}$, $B_2 = \{2,3,4,5\}$ and $B_3 = \{7,8\}$, thus determining the matching nested sets :
$S_w/B_1 = \{\{1,6\},\{9,10\}\}$, $S_w/B_2 = \{\{2,5\},\{3,4\}\}$, $S_w/B_3 = \{\{7,8\}\}$
$S'_w/B_1 = \{\{1,10\},\{6,9\}\}$, $S'_w/B_2 = \{\{2,3\},\{4,5\}\}$, $S'_w/B_3 = \{\{7,8\}\}$.

It is easy to obtain the following result.

Proposition 2.3 *If $w \in W^k_{2n}$ then there exist k subwords $w^j \in W_{2s_j}$, $j = 1,2,\ldots,k$ with $s_1 + s_2 + \cdots + s_k = n$ which can be recognized in w.*

For example, in the word $w = a\ a\ b\ \bar{b}\ \bar{a}\ \bar{b}\ a\ \bar{a}\ b\ \bar{a} \in W^3_{10}$, we recognize the subwords $w^1 = w_1w_6w_9w_{10} = a\ \bar{b}\ b\ \bar{a} \in W_4$, $w^2 = w_2w_3w_4w_5 = a\ b\ \bar{b}\ \bar{a} \in W_4$ and $w^3 = w_7w_8 = a\ \bar{a} \in W_2$.

We continue by introducing three internal operations in the set $\widehat{W}_{2n}$:
For $w \in \widehat{W}_{2n}$, we define the words $w^{\shortmid}$, w^{-} and w^{+} as follows:

$$w^{\shortmid}_i = \bar{w}_{2n+1-i} \quad (\textit{where } \bar{\bar{w}}_j = w_j)$$

$$w^{-}_i = \begin{cases} w_i, & \text{if } w_i \in \{a, \bar{a}\}; \\ b, & \text{if } w_i = \bar{b}; \\ \bar{b}, & \text{if } w_i = b, \end{cases} \qquad w^{+}_i = \begin{cases} \bar{a}, & \text{if } w_{2n+1-i} = a; \\ a, & \text{if } w_{2n+1-i} = \bar{a}; \\ w_{2n+1-i}, & \text{if } w_{2n+1-i} \in \{b, \bar{b}\}, \end{cases}$$

for every $i \in [2n]$. We may call these operations *mirror*, *overturn* and *mirror-overturn* respectively.

For example, if $w = a\ b\ a\ \bar{b}\ \bar{a}\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}$ then $w^{\shortmid} = a\ \bar{b}\ b\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{b}\ \bar{a}$, $w^{-} = a\ \bar{b}\ a\ b\ \bar{a}\ b\ \bar{b}\ b\ \bar{b}\ \bar{a}$ and $w^{+} = a\ b\ \bar{b}\ b\ \bar{b}\ a\ \bar{b}\ \bar{a}\ b\ \bar{a}$.

It is obvious that for any $w \in \widehat{W}_{2n}$, the words $w^{\shortmid}$, w^{-} and w^{+} also belong to $\widehat{W}_{2n}$. We have that $w^{\shortmid} = w$ (resp. $w^{+} = w$) iff $w^{+} = w^{-}$ (resp. $w^{-} = w^{\shortmid}$) as well as that $w^{-} \neq w$ and $w^{+} \neq w^{\shortmid}$. We thus obtain the following result.

Proposition 2.4 *The set $\widehat{W}_{2n}$ can be partitioned into classes of either two or four elements.*

Let $A_{2n} = \{w \in \widehat{W}_{2n} : w_2 = a, w_{2n-1} = \bar{a}\}$ and $B_{2n} = \{w \in \widehat{W}_{2n} : w_2 = \bar{b}\}$. By the previous properties of $w^{\shortmid}$, w^{-} and w^{+} we have the following proposition.

Proposition 2.5 *i) If $w \in A_{2n}$, then $w^{\shortmid}, w^{-}, w^{+} \in A_{2n}$.*
ii) If $w \notin A_{2n}$, then at least one of the words w, $w^{\shortmid}, w^{-}, w^{+}$ belongs to B_{2n}.

From the previous results it is clear that in order to construct $\widehat{W}_{2n}$ it is enough to have A_{2n} and B_{2n}. In order now to generate each element $w = u \circ u'$ of A_{2n} (resp. B_{2n}), it is enough to consider only the words $u = u_1 u_2 \cdots u_{2n}$, $u' = u'_1 u'_2 \cdots u'_{2n}$ of $\mathcal{D}_{2n}$ with $u_2 = u'_2 = a$ and $u_{2n-1} = u'_{2n-1} = \bar{a}$ (resp. $u_2 = \bar{a}$, $u'_2 = a$).

3 Meanders

We recall that a *closed meander of order n* is a closed self avoiding curve, crossing an infinite horizontal line $2n$ times (A005315).

Let M_{2n} be the set of all closed meanders of order n.

As opposed to previous papers [1], [8], the study of meanders will follow here the order of the crossings of the horizontal line rather than the meandric curve itself.

It is clear that if $\mu \in M_{2n}$, the lines above (resp. underneath) the horizontal line uniquely define a nested set U_μ (resp. L_μ) on $[2n]$ with U_μ, L_μ being matching and

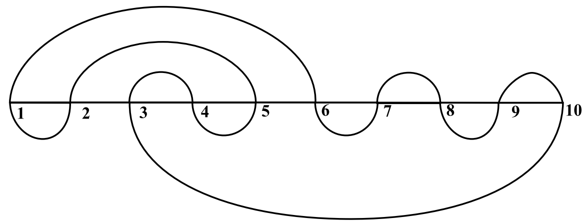

Figure 1: A closed meander of order 5

conversely two matching nested sets U_μ and L_μ uniquely define the meander μ. This allows us to actually identify a meander $\mu \in M_{2n}$ to a pair (U_μ, L_μ) of nested sets of $[2n]$.

For example, for the closed meander μ of Figure 1 we have:
$U_\mu = \{\{1,6\},\{2,5\},\{3,4\},\{7,8\},\{9,10\}\}$,
$L_\mu = \{\{1,2\},\{3,10\},\{4,5\},\{6,7\},\{8,9\}\}$.

To each meander of M_{2n} corresponds a unique word of W_{2n}. Intuitively, this correspondence becomes obvious when we assign the letters $a, \bar{a}, b, \bar{b}$ to the various kinds of intersection *opening*, *closing*, *proceeding upwards*, *proceeding downwards* respectively, occurring along the horizontal line.

So, the word $w = a\ b\ a\ \bar{b}\ \bar{a}\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}$ corresponds to the closed meander of Figure 1.

In order to develop formally these ideas we need the following result, obtained by considering all the possible orderings for the elements i, j, h of the pairs $\{i, j\} \in U_\mu$ and $\{i, h\} \in L_\mu$.

To every $\mu \in M_{2n}$ corresponds a unique word $w \in W_{2n}$, with

$$w_i = \begin{cases} a, & \textit{if} \quad i < j, h; \\ \bar{a}, & \textit{if} \quad h, j < i; \\ b, & \textit{if} \quad h < i < j; \\ \bar{b}, & \textit{if} \quad j < i < h, \end{cases}$$

where $\{i, j\} \in U_\mu$, $\{i, h\} \in L_\mu$.

So, from the nested sets U_μ, L_μ of the previous example we create again the word $w = a\ b\ a\ \bar{b}\ \bar{a}\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}$.

Conversely, to every word $w \in W_{2n}$ with S_w, S'_w matching, corresponds a unique meander $\mu \in M_{2n}$ with $U_\mu = S_w$, $L_\mu = S'_w$.

From the above, we have the following result.

Proposition 3.1 *There exists a bijection between the sets M_{2n} and W_{2n}.*

In order to determine U_μ and L_μ (and hence construct the meander $\mu \in M_{2n}$) we use the notion of conjugate indices of a Dyck word. So, given a word $w \in W_{2n}$, we create its relatives r, r' and we find the conjugate indices of these Dyck words, which indicate the pairs of U_μ and L_μ respectively.

We recall that a pair $\{a, b\}$ of a nested set S is called *short pair* if there is no $c \in \text{dom } S$ with either $a < c < b$ or $b < c < a$, [7]. We have the following result.

Proposition 3.2 *Each digram $a\bar{a}, a\bar{b}, b\bar{a}, b\bar{b}$ (resp. $a\bar{a}, ab, \bar{b}\bar{a}, \bar{b}b$) of a word $w \in W_{2n}$ corresponds to a short pair of U_μ (resp. L_μ) in the associated meander μ.*

So, we can also determine the meander $\mu \in M_{2n}$ by repetitively contracting the given word $w \in W_{2n}$, using each time propositions 3.1 and 3.2.

So for $w = a\ b\ a\ \bar{b}\ \bar{a}\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}$, we have:

$$w = \begin{array}{cc|cc|cc|cc||cc|} a & b & a & \bar{b} & \bar{a} & \bar{b} & b & \bar{b} & b & \bar{a} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{c|cc|c} a & b & \bar{a} & \bar{b} \\ 1 & 2 & 5 & 6 \end{array}$$

$$\begin{array}{|cc|} a & \bar{b} \\ 1 & 6 \end{array}$$

giving $U_\mu = \{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7, 8\}, \{9, 10\}\}$.

Similarly, we have

$$w = \begin{array}{|cc|c|cc||cc||cc|c} a & b & a & \bar{b} & \bar{a} & \bar{b} & b & \bar{b} & b & \bar{a} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{|cc|} a & \bar{a} \\ 3 & 10 \end{array}$$

giving $L_\mu = \{\{1, 2\}, \{3, 10\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\}$.

We thus finally get the meander μ of Figure 1 again.

Let $\mu \in M_{2n}$ and $w \in W_{2n}$ its corresponding word. If we draw the meanders $\mu^{\mid}$, μ^-, μ^+ that correspond to the words $w^{\mid}$, w^-, w^+ we realize that the above operations define meanders symmetric to the meander μ that corresponds to w, with respect to a vertical axis, to the horizontal line and to their intersection respectively.

It is easy to check that:

$\{i, j\} \in U_{\mu^{\mid}}$ iff $\{2n + 1 - i, 2n + 1 - j\} \in U_\mu$,
$\{i, j\} \in L_{\mu^{\mid}}$, iff $\{2n + 1 - i, 2n + 1 - j\} \in L_\mu$,
$U_{\mu^-} = L_\mu$, $L_{\mu^-} = U_\mu$,
$U_{\mu^+} = L_{\mu^{\mid}}$, $L_{\mu^+} = U_{\mu^{\mid}}$.

Hence, according to Proposition 2.4 we have the following result.

Proposition 3.3 *The set M_{2n} can be partitioned into classes of either two or four elements.*

4 Systems of meanders

We can extend the definition of closed meanders to *systems of closed meanders with k components* (or *k-meanders*) by allowing configurations with k disconnected meanders [5]. We will denote the set of all k-meanders of order n with $M_{2n}^k, k \in \{2, 3, \ldots, n\}$.

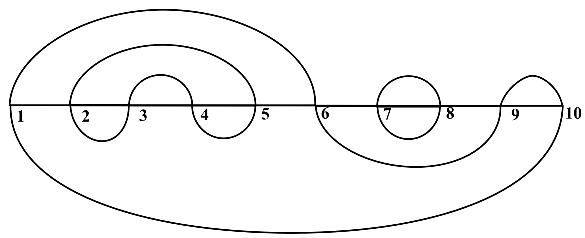

Figure 2: A 3-meander of order 5

Obviously, like in the case of meanders, a k-meander ν also determines the corresponding nested sets U_ν, L_ν that are now k-matching.

For example, for the 3-meander ν of Figure 2 we have:
$U_\nu = \{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7, 8\}, \{9, 10\}\}$
$L_\nu = \{\{1, 10\}, \{2, 3\}, \{4, 5\}, \{6, 9\}, \{7, 8\}\}$.

We can still assign the letters $a, \bar{a}, b, \bar{b}$ to the various kinds of intersection, thus creating the corresponding word of W_{2n}^k.

So, the word $w = a\ a\ b\ \bar{b}\ \bar{a}\ \bar{b}\ a\ \bar{a}\ b\ \bar{a}$ corresponds to the 3-meander of Figure 2.

It is easy to check that if we refer to meanders of M_{2n}^k instead of M_{2n}, to W_{2n}^k instead of W_{2n} and to k-matching instead of matching nested sets, we can apply propositions 3.1, 3.2 and 3.3 to k-meanders.

So, similarly to proposition 3.1, there exists a bijection between the sets M_{2n}^k and W_{2n}^k, i.e., to every $\nu \in M_{2n}^k$ corresponds a unique word $w \in W_{2n}^k$ obtained by the formula for w_i.

Conversely, to every $w \in W_{2n}^k$ with S_w, S'_w k-matching, corresponds a unique system of meanders $\nu \in M_{2n}^k$ with $U_\nu = S_w$, $L_\nu = S'_w$.

P. Di Francesco et al. [2] have given formulae for the cardinality of M_{2n}^k, for $k = n-3, n-2, n-1$, whereas for $k = n$ we have $|M_{2n}^n| = C_{2n}$, given that $W_{2n}^n = \mathcal{D}_{2n}$.

Similarly to proposition 3.2, we can now determine the system of meanders $\nu \in M_{2n}^k$ from the word $w \in W_{2n}^k$.

Let now S be a member of the set N_{2n} of the nested sets of pairs on $[2n]$; let $\{a,d\},\{b,c\} \in S$ with $a < b < c < d$ and such that for every $\{e,f\} \in S$ with $e < b < f$ we have $e \leq a$; then $\{a,d\}$ (resp. $\{b,c\}$) is called *father* (resp. *child*) of $\{b,c\}$ (resp. $\{a,d\}$). We call two elements $\{i,j\},\{k,l\}$ of S *brothers* if the have the same father, or if they have no father.

We define two operations in N_{2n} as follows:

If $\{b,c\}$ and its father $\{a,d\}$ belong to $S \in N_{2n}$ with $a < b < c < d$, then $\sigma(S;a,b)$ is the set obtained if we replace the pairs $\{a,d\}$ and $\{b,c\}$ with the pairs $\{a,b\}$ and $\{c,d\}$. It is obvious that $\sigma(S;a,b) \in N_{2n}$ and that $\{a,b\}$ and $\{c,d\}$ are brothers in $\sigma(S;a,b)$.

If $\{a,b\},\{c,d\}$ are brothers in $S \in N_{2n}$, with $a < b < c < d$, then $\tau(S;a,c)$ is the set obtained if we replace the pairs $\{a,b\}$ and $\{c,d\}$ with the pair $\{a,d\}$ and $\{b,c\}$. It is obvious that $\tau(S;a,c) \in N_{2n}$ and that $\{a,d\}$ is the father of $\{b,c\}$ in $\tau(S;a,c)$.

The above definitions imply that if $\{a,d\}$ is the father of $\{b,c\}$ in the set $S \in N_{2n}$, then $\tau(\sigma(S;a,b);a,c) = S$, whereas if $\{a,b\},\{c,d\}$ are brothers, then

$$\sigma(\tau(S;a,c);a,b) = S.$$

We also have the following result.

Proposition 4.1 *Let $\nu \in M_{2n}^k$. If the father $\{a,d\}$ and the child $\{b,c\}$ (resp. the brothers $\{a,b\},\{c,d\}$) of U_ν belong to the same component of ν then the set $U = \sigma(U_\nu;a,b)$ (resp. $U = \tau(U_\nu;a,c)$) and L_ν are $(k+1)$-matching, thus defining a meander $\xi \in M_{2n}^{k+1}$, whereas if they belong to different components of ν, then ξ belongs to M_{2n}^{k-1}.*

It is obvious that the above result still holds if we interchange L with U.

Proposition 4.1 is important since it enables us to recursively construct the sets $M_{2n}^k, k = j+1, j+2, \ldots, n$ if the set M_{2n}^j is known for some $j \in [n-1]$.

For example, if $\mu \in M_{10}$ is the meander of Figure 1, we have that the sets $U_\nu = U_\mu$ and $L_\nu = \tau(L_\mu;6,8) = \{\{1,2\},\{3,10\},\{4,5\},\{6,9\},\{7,8\}\}$ determine a meander $\nu \in M_{10}^2$; a second application of proposition 4.1 gives $U_\xi = U_\nu$ and

$$L_\xi = \tau(L_\nu;1,3) = \{\{1,10\},\{2,3\},\{4,5\},\{6,9\},\{7,8\}\}$$

which determine the meander $\xi \in M_{10}^3$ of Figure 2.

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