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On the Density of Languages Representing Finite Set Partitions [1]

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Abstract

We present a family of regular languages representing partitions of a set of n elements in less or equal c parts. The density of those languages is given by partial sums of Stirling numbers of second kind for which we obtain explicit formulas. We also determine the limit frequency of those languages. This work was motivated by computational representations of the configurations of some numerical games.

1 The languages L_c

Consider a game where natural numbers are to be placed, by increasing order, in a fixed number of columns, subject to some specific constraints. In these games column order is irrelevant. Numbering the columns, game configurations can be seen as sequences of column numbers where the successive integers are placed. For instance, the string

$$11213$$

stands for a configuration where 1, 2, 4 were placed in the first column, 3 was placed in the second and 5 was placed in the third. Because column order is irrelevant, and to have a unique representation for each configuration, it is not allowed to place an integer in the

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kth column if the $(k-1)$th is still empty, for any $k > 1$. Blanchard and al. [BHR04] and Reis and al. [RMP04] used this kind of representation to study the possible configurations of sum-free games.

Given c columns, let $\mathbb{N}_c = \{1, \dots, c\}$. We are interested in studying the set of game configurations as strings in $(\mathbb{N}_c)^\star$, i.e., in the set of finite sequences of elements of $\mathbb{N}_c$. Game configurations can be characterised by the following language $L_c \subset (\mathbb{N}_c)^\star$:

$$L_c = \{a_1 a_2 \cdots a_k \in (\mathbb{N}_c)^\star \mid \forall i \in \mathbb{N}_k, a_i \leq \max\{a_1, \dots, a_{i-1}\} + 1\}.$$

For $c = 4$, there are only 15 strings in L_4 of length 4, instead of the total possible 256 in $(\mathbb{N}_4)^4$:

1111	1112	1121	1122	1123
1211	1212	1213	1221	1222
1223	1231	1232	1233	1234

Given a finite set Σ, a regular expression (r.e.) α over Σ represents a (regular) language $L(\alpha) \subseteq \Sigma^\star$ and is inductively defined by: $\emptyset$ is a r.e and $L(\emptyset) = \emptyset$; ϵ (empty string) is a r.e and $L(\epsilon) = \{\epsilon\}$; $a \in \Sigma$ is a r.e and $L(a) = \{a\}$; if α_1 and α_2 are r.e., $\alpha_1 + \alpha_2$, $\alpha_1\alpha_2$ and $\alpha_1^\star$ are r.e., respectively with $L(\alpha_1 + \alpha_2) = L(\alpha_1) \cup L(\alpha_2)$, $L(\alpha_1\alpha_2) = L(\alpha_1)L(\alpha_2)$ and $L(\alpha_1{}^\star) = L(\alpha_1)^\star$, where we assume the usual precedence of the operators (see [HMU00]). A regular expression α is unambiguous if for each $w \in L(\alpha)$ there is only one path through α that matches w.

Theorem 1.1. *For all $c \geq 1$, L_c is a regular language.*

Proof. For $c = 1$, we have $L_1 = L(11^\star)$. We define by induction on c, a family of regular expressions:

$$\alpha_1 = 11^\star, \quad (1)$$

$$\alpha_c = \alpha_{c-1} + \prod_{j=1}^{c} j(1 + \cdots + j)^\star. \quad (2)$$

It is trivial to see that

$$\alpha_c = \sum_{i=1}^{c} \prod_{j=1}^{i} j(1 + \cdots + j)^\star. \quad (3)$$

For instance, α_4 is

$$11^\star + 11^\star 2(1+2)^\star + 11^\star 2(1+2)^\star 3(1+2+3)^\star + 11^\star 2(1+2)^\star 3(1+2+3)^\star 4(1+2+3+4)^\star.$$

It is also obvious that $L(\alpha_{c-1}) \subseteq L(\alpha_c)$, for $c > 1$. For any $c \geq 1$, we prove that

$$L_c = L(\alpha_c).$$

$L_c \supseteq L(\alpha_c)$: If $x \in L(\alpha_c)$ it is obvious that $x \in L_c$.

$L_c \subseteq L(\alpha_c)$: By induction on the length of $x \in L_c$: If $|x| = 1$ then $x \in L(\alpha_1) \subseteq L(\alpha_c)$. Suppose that for any string x of length $\leq n$, $x \in L(\alpha_c)$. Let $|x| = n+1$ and $x = ya$, where $a \in \mathbb{N}_c$ and $y \in L(\alpha_c)$. Let $c' = \max\{a_i \mid a_i \in y\}$. If $c' = c$, obviously $x \in L(\alpha_c)$. If $c' < c$, then $y \in L(\alpha_{c'})$, and $x \in L(\alpha_{c'+1}) \subseteq L(\alpha_c)$.

□

2 Counting the strings of L_c

The density of a language L over a finite set Σ, $\rho_L(n)$, is the number of strings of length n that are in L, i.e.,

$$\rho_L(n) = |L \cap \Sigma^n|.$$

In particular, the density of L_c is

$$\rho_{L_c}(n) = |L_c \cap \mathbb{N}_c^n|.$$

Using generating functions we can determine a closed form for $\rho_{L_c}(n)$. Recall that, a (ordinary) generating function for a sequence $\{a_n\}$ is a formal series (see [GKP94])

$$G(z) = \sum_{i=0}^{\infty} a_n z^n.$$

If $A(z)$ and $B(z)$ are generating functions for the density functions of the languages represented by unambiguous regular expressions A and B, and $A + B$, AB and $A^\star$ are also unambiguous r.e., we have that $A(z) + B(z)$, $A(z)B(z)$ and $\dfrac{1}{1 - A(z)}$, are the generating functions for the density functions of the corresponding languages (see [SF96], page 378).

As α_c are unambiguous regular expressions, from (3), we obtain the following generating function for $\{\rho_{L_c}(n)\}$:

$$T_c(z) = \sum_{i=1}^{c} \prod_{j=1}^{i} \frac{z}{(1 - jz)} = \sum_{i=1}^{c} \frac{z^i}{\prod_{j=1}^{i}(1 - jz)}.$$

Notice that

$$S_i(z) = \frac{z^i}{\prod_{j=1}^{i}(1 - jz)}$$

are the generating functions for the Stirling numbers of second kind

$$S(n, i) = \frac{1}{i!} \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} (i - j)^n,$$

which are, for each n, the number of ways of partitioning a set of n elements into i nonempty sets (see [GKP94] and A008277).

Then, a closed form for the density of L_c, $\rho_{L_c}(n)$, is given by

$$\rho_{L_c}(n) = \sum_{i=1}^{c} S(n, i), \tag{4}$$

i.e., a partial sum of Stirling numbers of second kind.

In Table 1 we present the values of $\rho_{L_c}(n)$, for $c = 1..8$ and $n = 1..13$. For some sequences, we also indicate the corresponding number in Sloane's On-Line Encyclopedia of Integer Sequences [Slo03]. The closed forms were calculated using the Maple computer algebra system [Hec03].

From expression (4), it is also easy to see that

Theorem 2.1.

$$\lim_{c\to\infty} \rho_{L_c}(n) = B_n,$$

where B_n are the Bell numbers, i.e., for each n, the number of ways a set of n elements can be partitioned into nonempty subsets.

Proof. Bell numbers, B_n, can be defined by the sum

$$B_n = \sum_{i=1}^{n} S(n,i).$$

And, as $S(n,i) = 0$ for $i > n$, we have

$$\lim_{c\to\infty} \rho_{L_c}(n) \;=\; \sum_{i=1}^{n} S(n,i) + \lim_{c\to\infty} \sum_{i=n+1}^{c} S(n,i) \;=\; B_n.$$

□

In Table 1, for each $c \geq 1$, the subsequence for $n \leq c$ coincides with the first c elements of B_n (A000110).

Moreover, we can express $\rho_{L_c}(n)$, and then the partial sums of Stirling numbers of second kind, as a generic linear combination of nth powers of k, for $k \in \mathbb{N}_c$. Let $S^j(n,i)$ denote the jth term in the summation of a Stirling number $S(n,i)$, i.e.,

$$S^j(n,i) = \frac{1}{i!}(-1)^j \binom{i}{j} (i-j)^n.$$

Lemma 2.1. *For all n and $0 \leq i \leq n$,*

$$S^0(n,i) = -S^1(n,i+1). \tag{5}$$

Proof.

$$S^1(n,i+1) \;=\; \frac{1}{(i+1)!}(-1)\binom{i+1}{1} i^n \;=\; (-1)\frac{1}{i!} i^n \;=\; -S^0(n,i).$$

□

Applying (5) in the summation (4) of $\rho_{L_c}(n)$, each term $S(n,i)$ simplifies the subterm $S^1(n,i)$ with the subterm $S^0(n,i-1)$, for $i \geq 2$. We obtain

$$\begin{aligned}
\rho_{L_1}(n) &= S^0(n,1), \\
\rho_{L_2}(n) &= S^0(n,2), \\
\rho_{L_c}(n) &= S^0(n,c) + \sum_{i=3}^{c}\sum_{j=2}^{i-1} S^j(n,i), \qquad \text{for } c > 2; \\
&= \frac{c^n}{c!} + \sum_{i=3}^{c}\sum_{j=2}^{i-1} S^j(n,i), \qquad \text{for } c > 2.
\end{aligned}$$

c	$\rho_{L_c}(n)$	OEIS
1	1 $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$	
2	$\frac{1}{2}2^n$ $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, \ldots$	
3	$\frac{1}{6}3^n + \frac{1}{2}$ $1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842, 29525, 88574, 265721, \ldots$	A007051
4	$\frac{1}{24}4^n + \frac{1}{4}2^n + \frac{1}{3}$ $1, 2, 5, 15, 51, 187, 715, 2795, 11051, 43947, 175275, 700075, 2798251, \ldots$	A007581
5	$\frac{1}{120}5^n + \frac{1}{12}3^n + \frac{1}{6}2^n + \frac{3}{8}$ $1, 2, 5, 15, 52, 202, 855, 3845, 18002, 86472, 422005, 2079475, 10306752, \ldots$	A056272
6	$\frac{1}{720}6^n + \frac{1}{48}4^n + \frac{1}{18}3^n + \frac{3}{16}2^n + \frac{11}{30}$ $1, 2, 5, 15, 52, 203, 876, 4111, 20648, 109299, 601492, 3403127, 19628064, \ldots$	A056273
7	$\frac{1}{5040}7^n + \frac{1}{240}5^n + \frac{1}{72}4^n + \frac{1}{16}3^n + \frac{11}{60}2^n + \frac{53}{144}$ $1, 2, 5, 15, 52, 203, 877, 4139, 21110, 115179, 665479, 4030523, 25343488, \ldots$	A099262
8	$\frac{1}{40320}8^n + \frac{1}{1440}6^n + \frac{1}{360}5^n + \frac{1}{64}4^n + \frac{11}{180}3^n + \frac{53}{288}2^n + \frac{103}{280}$ $1, 2, 5, 15, 52, 203, 877, 4140, 21146, 115929, 677359, 4189550, 27243100, \ldots$	A099263

Table 1: Density functions of L_c, for $c = 1..8$.

If the sums are rearranged such that $i = k + j$, we have

$$\rho_{L_c}(n) = \frac{c^n}{c!} + \sum_{k=1}^{c-2} \sum_{j=2}^{c-k} S^j(n, k+j), \tag{6}$$

where

$$\begin{aligned} S^j(n, k+j) &= \frac{k^n}{(k+j)!}(-1)^j \binom{k+j}{j} \qquad (7) \\ &= \frac{k^n}{k!j!}(-1)^j. \qquad (8) \end{aligned}$$

Replacing (8) into equation (6) we get

$$\rho_{L_c}(n) = \frac{c^n}{c!} + \sum_{k=1}^{c-2} \frac{k^n}{k!} \left(\sum_{j=2}^{c-k} \frac{(-1)^j}{j!} \right). \tag{9}$$

In equation (9), the coefficients of k^n, $1 \le k \le c$, can be calculated using the following recurrence relation:

$$\begin{aligned} \gamma_1^1 &= 1; \\ \gamma_1^c &= \gamma_1^{c-1} + \frac{(-1)^{c-1}}{(c-1)!}, \text{ for } c > 1; \\ \gamma_k^c &= \frac{\gamma_{k-1}^{c-1}}{k}, \text{ for } c > 1 \text{ and } 2 \le k \le c. \end{aligned}$$

And, we have

Theorem 2.2. *For all $c \ge 1$,*

$$\rho_{L_c}(n) = \sum_{k=1}^{c} \gamma_k^c k^n. \tag{10}$$

From the expression (10), the closed forms in Table 1 are easily derived.

Finally, we can obtain the limit frequency of L_c in $(\mathbb{N}_c)^\star$. Since

$$\rho_{(\mathbb{N}_c)^\star}(n) = c^n,$$

and $\lim_{n\to\infty} \left(\frac{k}{c}\right)^n = 0$, for $1 \le k \le c-2$, we have

Theorem 2.3. *For all $c \ge 1$,*

$$\lim_{n\to\infty} \frac{\rho_{L_c}(n)}{\rho_{(\mathbb{N}_c)^\star}(n)} = \frac{1}{c!}.$$

3 A bijection between strings of L_c and partitions of finite sets

The connection between the density of L_c and Stirling numbers of second kind is not accidental. Each string of L_c with length n corresponds to a partition of $\mathbb{N}_n$ with no more than c parts. This correspondence can be made explicit as follows.

Let $a_1a_2\cdots a_n$ be a string of L_c. This string corresponds to the partition $\{A_j\}_{j\in\mathbb{N}_{c'}}$ of $\mathbb{N}_n$ with $c' = \max\{a_1,\ldots,a_n\}$, such that for each $i \in \mathbb{N}_n$, $i \in A_{a_i}$. For example, the string 1123 corresponds to the partition $\{\{1,2\},\{3\},\{4\}\}$ of $\mathbb{N}_4$ into 3 parts.

This defines a bijection. That each string corresponds to a unique partition is obvious. Given a partition $\{A_j\}_{j\in\mathbb{N}_{c'}}$ of $\mathbb{N}_n$ with $c' \leq c$, we can construct the string $b_1\cdots b_n$, such that for $i \in \mathbb{N}_n$, $b_i = j$ if $i \in A_j$. For the partition $\{\{1,2\},\{3\},\{4\}\}$, we obtain 1123.

4 Counting the strings of L_c of length equal or less than a certain value

Although strings of L_c of arbitrary length represent game configurations, for computational reasons[2] we consider all game configurations with the same length, padding with zeros the positions of integers not yet in one of the c columns. In this way, we obtain the languages $L_c^0 = L_c\{0^\star\}$. So, determining the number of strings of length equal or less than n that are in L_c is tantamount to determining the density of L_c^0, i.e.,

$$\rho_{L_c^0}(n) = |L_c^0 \cap (\{0\} \cup \mathbb{N}_c)^n|.$$

As seen in Section 2, and because $L_c\{0^\star\} = L(\alpha_c 0^\star)$, the generating function $T_c'(z)$ of $\rho_{L_c^0}(n)$ can be obtained as the product of a generating function for $\rho_{L_c}(n)$, $T_c(z)$, by a generating function for $\rho_{\{0\}^\star}(n)$, e.g., $\dfrac{1}{1-z}$. Thus, the generating function for $\{\rho_{L_c^0}(n)\}$ is

$$T_c'(z) = T_c(z)\frac{1}{1-z} = \sum_{i=1}^{c} \frac{z^i}{(1-z)\prod_{j=1}^{i}(1-jz)}$$

and a closed form for $\rho_{L_c^0}(n)$ is (as expected)

$$\rho_{L_c^0}(n) = \sum_{m=1}^{n}\sum_{i=1}^{c} S(m,i), \tag{11}$$

where m starts at 1 because $S(m,i) = 0$, for $i > m$.

[2]The data structures used in the programs are arrays of fixed length.

Using expression (9) in (11) we have

$$\begin{aligned}
\rho_{L_1^0}(n) &= n; \\
\rho_{L_2^0}(n) &= 2^n - 1; \\
\rho_{L_c^0}(n) &= \sum_{m=1}^{n}\left(\frac{c^m}{c!} + \sum_{k=1}^{c-2}\frac{k^m}{k!}\sum_{j=2}^{c-k}\frac{(-1)^j}{j!}\right), \qquad \text{for } c > 2; \\
&= \frac{c^{n+1}-c}{(c-1)c!} + n\sum_{j=2}^{c-1}\frac{(-1)^j}{j!} + \sum_{k=2}^{c-2}\frac{k^{n+1}-k}{(k-1)k!}\left(\sum_{j=2}^{c-k}\frac{(-1)^j}{j!}\right) \\
&= \frac{c^n-1}{(c-1)(c-1)!} + n\sum_{j=2}^{c-1}\frac{(-1)^j}{j!} + \sum_{k=2}^{c-2}\frac{k^n-1}{(k-1)(k-1)!}\left(\sum_{j=2}^{c-k}\frac{(-1)^j}{j!}\right).
\end{aligned}$$

If we use the equation (10), we have

Theorem 4.1. *For all $c \geq 1$,*

$$\rho_{L_c^0}(n) = n\gamma_1^c + \sum_{k=2}^{c}\frac{\gamma_k^c(k^{n+1}-k)}{k-1}.$$

Proof.

$$\rho_{L_c^0}(n) = \sum_{m=1}^{n}\sum_{k=1}^{c}\gamma_k^c k^m = \sum_{k=1}^{c}\gamma_k^c\sum_{m=1}^{n}k^m = n\gamma_1^c + \sum_{k=2}^{c}\frac{\gamma_k^c(k^{n+1}-k)}{k-1}.$$

□

In the Table 2 we present the values of $\rho_{L_c^0}(n)$, for $c = 1..8$ and $n = 1..13$. As before, the limiting sequence as $c \to \infty$ is the sequence of partial sums of Bell numbers.

Finally, we determine the limit frequency of L_c^0 in $(\mathbb{N}_c)^\star\{0\}^\star$. Notice that

$$\rho_{(\mathbb{N}_c)^\star\{0\}^\star}(n) = \frac{c^{n+1}-1}{c-1},$$

as it is a sum of the first n terms of a geometric progression of ratio c.

We have,

Theorem 4.2. *For all $c \geq 1$,*

$$\lim_{n\to\infty}\frac{\rho_{L_c^0}(n)}{\rho_{(\mathbb{N}_c)^\star\{0\}^\star}(n)} = \frac{1}{c!}.$$

c	$\rho_{L_c^0}(n)$	OEIS
1	n 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, ...	
2	$2^n - 1$ 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, ...	A000225
3	$\frac{1}{4}3^n + \frac{1}{2}n - \frac{1}{4}$ 1, 3, 8, 22, 63, 185, 550, 1644, 4925, 14767, 44292, 132866, 398587, ...	A047926
4	$\frac{1}{18}4^n + \frac{1}{2}2^n + \frac{1}{3}n - \frac{5}{9}$ 1, 3, 8, 23, 74, 261, 976, 3771, 14822, 58769, 234044, 934119, 3732370, ...	
5	$\frac{1}{96}5^n + \frac{1}{8}3^n + \frac{1}{3}2^n + \frac{3}{8}n - \frac{15}{32}$ 1, 3, 8, 23, 75, 277, 1132, 4977, 22979, 109451, 531456, 2610931, 12917683, ...	A099265
6	$\frac{1}{600}6^n + \frac{1}{36}4^n + \frac{1}{12}3^n + \frac{3}{8}2^n + \frac{11}{30}n - \frac{439}{900}$ 1, 3, 8, 23, 75, 278, 1154, 5265, 25913, 135212, 736704, 4139831, 23767895, ...	A099266
7	$\frac{1}{4320}7^n + \frac{1}{192}5^n + \frac{1}{54}4^n + \frac{3}{32}3^n + \frac{11}{30}2^n + \frac{53}{144}n - \frac{31}{64}$ 1, 3, 8, 23, 75, 278, 1155, 5294, 26404, 141583, 807062, 4837585, 30181073, ...	
8	$\frac{1}{35280}8^n + \frac{1}{1200}6^n + \frac{1}{288}5^n + \frac{1}{48}4^n + \frac{11}{120}3^n + \frac{53}{144}2^n + \frac{103}{280}n - \frac{57023}{117600}$ 1, 3, 8, 23, 75, 278, 1155, 5295, 26441, 142370, 819729, 5009279, 32252379, ...	

Table 2: Density functions of L_c^0, for $c = 1..8$.

Proof.

$$\begin{aligned}
\lim_{n\to\infty}\frac{\rho_{L_c^0}(n)}{\rho_{(\mathbb{N}_c)^\star\{0\}^\star}(n)} &= \lim_{n\to\infty}\left(\frac{(c^{n+1}-c)(c-1)}{(c-1)c!(c^{n+1}-1)}+\frac{n(c-1)}{c^{n+1}-1}\sum_{j=2}^{c-1}\frac{(-1)^j}{j!}\right)\\
&\quad+\lim_{n\to\infty}\left(\sum_{k=2}^{c-2}\frac{(k^{n+1}-k)(c-1)}{(k-1)k!(c^{n+1}-1)}\left(\sum_{j=2}^{c-k}\frac{(-1)^j}{j!}\right)\right)\\
&= \frac{1}{c!}\left(\lim_{n\to\infty}\frac{1}{1-\frac{1}{c^{n+1}}}-\lim_{n\to\infty}\frac{c}{c^{n+1}-1}\right)\\
&\quad+\sum_{j=2}^{c-1}\frac{(-1)^j}{j!}\lim_{n\to\infty}\left(\frac{n(c-1)}{c^{n+1}-1}\right)\\
&\quad+\sum_{k=2}^{c-2}\frac{(c-1)}{(k-1)(k-1)!}\sum_{j=2}^{c-k}\frac{(-1)^j}{j!}\lim_{n\to\infty}\left(\frac{(\frac{k}{c})^n}{c-\frac{1}{c^n}}-\frac{1}{c^{n+1}-1}\right)\\
&= \frac{1}{c!}.
\end{aligned}$$

□

5 Conclusion

In this note we presented a family of regular languages representing finite set partitions and studied their densities. Although it is well-known that the number of partitions of a set of n elements into no more than c nonempty sets is given by partial sums of Stirling numbers of second kind, we determined explicit formulas for their closed forms, as linear combinations of k^n, for $k \in \mathbb{N}_c$. We also determined the limit frequency of those languages, which gives an estimate of the space saved with those representations.

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