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Bijective Proofs of Parity Theorems for Partition Statistics

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Abstract

We give bijective proofs of parity theorems for four related statistics on partitions of finite sets. A consequence of our results is a combinatorial proof of a congruence between Stirling numbers and binomial coefficients.

1 Introduction

The notational conventions of this paper are as follows: $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{P} := \{1, 2, \dots\}$, $[0] := \varnothing$, and $[n] := \{1, \dots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. The binomial coefficient $\binom{n}{k}$ is equal to zero if k is a negative integer or if $0 \leqslant n < k$.

Let $\Pi(n, k)$ denote the set of all partitions of $[n]$ with k blocks and $\Pi(n)$ the set of all partitions of $[n]$. Associate to each $\pi \in \Pi(n, k)$ the ordered partition $(E_1, \dots, E_k)$ of $[n]$ comprising the same blocks as π, arranged in increasing order of their smallest elements, and define statistics $\tilde{w}$, $\hat{w}$, w^*, and w by

$$\tilde{w}(\pi) := \sum_{i=1}^{k} (i-1)(|E_i| - 1), \tag{1.1}$$

$$\hat{w}(\pi) := \sum_{i=1}^{k} i(|E_i| - 1) = \tilde{w}(\pi) + n - k, \tag{1.2}$$

$$w^*(\pi) := \sum_{i=1}^{k} i|E_i| = \tilde{w}(\pi) + n + \binom{k}{2}, \tag{1.3}$$

and

$$w(\pi) := \sum_{i=1}^{k}(i-1)|E_i| = \tilde{w}(\pi) + \binom{k}{2}. \tag{1.4}$$

Consider the generating functions (see [1], [3], [5], and [6])

$$\tilde{S}_q(n,k) := \sum_{\pi\in\Pi(n,k)} q^{\tilde{w}(\pi)}, \tag{1.5}$$

$$\hat{S}_q(n,k) := \sum_{\pi\in\Pi(n,k)} q^{\hat{w}(\pi)} = q^{n-k}\tilde{S}_q(n,k), \tag{1.6}$$

$$S^*_q(n,k) := \sum_{\pi\in\Pi(n,k)} q^{w^*(\pi)} = q^{\binom{k}{2}+n}\tilde{S}_q(n,k), \tag{1.7}$$

and

$$S_q(n,k) := \sum_{\pi\in\Pi(n,k)} q^{w(\pi)} = q^{\binom{k}{2}}\tilde{S}_q(n,k). \tag{1.8}$$

Summing the q-Stirling numbers $\tilde{S}_q(n,k)$, $\hat{S}_q(n,k)$, $S^*_q(n,k)$, and $S_q(n,k)$ over k yields the respective q-Bell numbers $\tilde{B}_q(n)$, $\hat{B}_q(n)$, $B^*_q(n)$, and $B_q(n)$. These polynomials reduce to the classical Stirling and Bell numbers when $q = 1$. Wagner [7] evaluates the foregoing polynomials when $q = -1$ using algebraic techniques and raises the question of finding bijective proofs.

We now describe a combinatorial method for evaluating these polynomials when $q = -1$. More generally, let Δ be a finite set of discrete structures and $I : \Delta \to \mathbb{N}$, with generating function

$$G(I,\Delta;q) := \sum_{\delta\in\Delta} q^{I(\delta)} = \sum_{k} |\{\delta \in \Delta :\ I(\delta) = k\}| q^k. \tag{1.9}$$

Of course, $G(I,\Delta;1) = |\Delta|$. If $\Delta_i := \{\delta \in \Delta :\ I(\delta) \equiv i \pmod 2\}$, then $G(I,\Delta;-1) = |\Delta_0| - |\Delta_1|$. Our strategy for finding $G(I,\Delta;-1)$ will be to identify a subset Δ^* of Δ contained completely within Δ_0 or Δ_1 and then to define an I-parity changing involution on $\Delta - \Delta^*$. The subset Δ^* thus captures both the sign and magnitude of $G(I,\Delta;-1)$. In the present setting, Δ will either be $\Pi(n)$ or $\Pi(n,k)$ and I, one of the aforementioned partition statistics.

In § 2, we give bijective proofs establishing $\tilde{B}_q(n)$ and $\hat{B}_q(n)$ as well as the four q-Stirling numbers when $q = -1$. In § 3, a bijection yielding $B^*_{-1}(n)$ and $B_{-1}(n)$ is given. A consequence of our results is a combinatorial proof requested by Stanley of the congruence [4, p. 46]

$$S(n,k) \equiv \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \pmod 2, \quad 0 \leqslant k \leqslant n, \tag{1.10}$$

where $S(n,k) = |\pi(n,k)|$ denotes the Stirling number of the second kind.

2 The First Bijection

Throughout, we'll represent $\pi \in \Pi(n)$ by $(E_1, E_2, \dots)$, the unique ordered partition of $[n]$ comprising the same blocks as π, arranged in increasing order of their smallest elements. Let $F_0 = F_1 = 1$, with $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 2$.

Theorem 2.1. *For all* $n \in \mathbb{N}$,

$$\tilde{B}_{-1}(n) := \sum_{k=0}^{n} \tilde{S}_{-1}(n,k) = F_n. \tag{2.1}$$

Proof. Let $\Pi_i(n) := \{\pi \in \Pi(n) : \ \tilde{w}(\pi) \equiv i \pmod 2\}$ so that $\tilde{B}_{-1}(n) = |\Pi_0(n)| - |\Pi_1(n)|$. To prove (2.1), we'll identify a subset $\tilde{\Pi}(n)$ of $\Pi_0(n)$ such that $|\tilde{\Pi}(n)| = F_n$ along with a $\tilde{w}$-parity changing involution of $\Pi(n) - \tilde{\Pi}(n)$.

The set $\tilde{\Pi}(n)$ consists of those partitions $\pi = (E_1, E_2, \dots)$ whose blocks satisfy the two conditions:

each block of odd index comprises a set of consecutive integers; (2.2a)

each block of even index is a singleton. (2.2b)

Now $|\tilde{\Pi}(n)| = F_n$, as $|\tilde{\Pi}(n)|$ is seen to satisfy the Fibonacci recurrence, upon considering whether or not $\{n\}$ is a block. For if $\{n\}$ is not a block and $n-2$ belongs to an odd-numbered (respectively, even-numbered) block of $\pi \in \tilde{\Pi}(n)$, then $\{n-1, n\}$ constitutes a proper subset of (respectively, all of) the last block of π.

Suppose now that $\pi = (E_1, E_2, \dots)$ belongs to $\Pi(n) - \tilde{\Pi}(n)$ and that i_0 is the smallest of the integers i for which E_{2i-1} fails to satisfy (2.2a) or E_{2i} fails to satisfy (2.2b). Let M be the largest member of $E_{2i_0-1} \cup E_{2i_0}$. If M belongs to E_{2i_0-1}, move it to E_{2i_0}, while if M belongs to E_{2i_0}, move it to E_{2i_0-1} (note that if $|E_{2i_0}| = 1$, then necessarily $M \in E_{2i_0-1}$). The resulting map is a parity changing involution of $\Pi(n) - \tilde{\Pi}(n)$. □

Below, we illustrate the fixed point set $\tilde{\Pi}(n)$ and the pairings of $\Pi(n) - \tilde{\Pi}(n)$ when $n = 4$, wherein the first two members of each row are paired.

$\Pi_0(n) - \tilde{\Pi}(n)$	$\Pi_1(n)$	$\tilde{\Pi}(n)$
$\{1,2,4\},\ \{3\}$	$\{1,2\},\ \{3,4\}$	$\{1,2,3,4\}$
$\{1,3,4\},\ \{2\}$	$\{1,3\},\ \{2,4\}$	$\{1,2,3\},\ \{4\}$
$\{1\},\ \{2,3,4\}$	$\{1,4\},\ \{2,3\}$	$\{1\},\ \{2\},\ \{3,4\}$
$\{1,3\},\ \{2\},\ \{4\}$	$\{1\},\ \{2,3\},\ \{4\}$	$\{1,2\},\ \{3\},\ \{4\}$
$\{1,4\},\ \{2\},\ \{3\}$	$\{1\},\ \{2,4\},\ \{3\}$	$\{1\},\ \{2\},\ \{3\},\ \{4\}$

Note that the above bijection preserves the number of blocks of $\pi \in \Pi(n)$. We'll use its restriction to $\Pi(n,k)$ to prove

Theorem 2.2. *For all* $n \in \mathbb{N}$,

$$\tilde{S}_{-1}(n,k) = \binom{n - \lfloor k/2 \rfloor - 1}{n-k}, \qquad 0 \leqslant k \leqslant n. \tag{2.3}$$

Proof. Let $\Pi_i(n,k) := \Pi_i(n) \cap \Pi(n,k)$ for $i = 0, 1$, $\tilde{\Pi}(n,k) := \tilde{\Pi}(n) \cap \Pi(n,k)$, and $\pi = (E_1, \ldots, E_k) \in \tilde{\Pi}(n,k)$. If k is even, identify each pair of blocks (E_{2i-1}, E_{2i}), $1 \leqslant i \leqslant k/2$, with summands x_i in a composition $x_1 + \cdots + x_{k/2} = n$, where each $x_i \geqslant 2$. If k is odd, identify $(E_1, E_2), \ldots, (E_{k-2}, E_{k-1}), (E_k)$ with summands x_i in $x_1 + \cdots + x_{(k+1)/2} = n$ where $x_i \geqslant 2$ for $1 \leqslant i \leqslant \frac{k-1}{2}$ and $x_{(k+1)/2} \geqslant 1$. The cardinality of $\tilde{\Pi}(n,k)$ is then given by the right hand side of (2.3), and the restriction of the prior bijection to $\Pi(n,k) - \tilde{\Pi}(n,k)$ is again an involution, and inherits the parity changing property, which proves (2.3). □

From (2.3) along with (1.6), (1.7), and (1.8), we have

$$\hat{S}_{-1}(n,k) = (-1)^{n-k} \binom{n - \lfloor k/2 \rfloor - 1}{n-k}, \qquad 0 \leqslant k \leqslant n, \tag{2.4}$$

$$S^*_{-1}(n,k) = (-1)^{\binom{k}{2}+n} \binom{n - \lfloor k/2 \rfloor - 1}{n-k}, \qquad 0 \leqslant k \leqslant n, \tag{2.5}$$

and

$$S_{-1}(n,k) = (-1)^{\binom{k}{2}} \binom{n - \lfloor k/2 \rfloor - 1}{n-k}, \qquad 0 \leqslant k \leqslant n. \tag{2.6}$$

The bijection establishing (2.3) clearly applies to (2.4)–(2.6) as well.

Let $S(n,k) = |\Pi(n,k)|$ denote the Stirling number of the second kind. The bijection of Theorem 2.2 also proves combinatorially that

$$S(n,k) \equiv \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \pmod 2, \qquad 0 \leqslant k \leqslant n, \tag{2.7}$$

since off of a set of cardinality $\binom{n-\lfloor k/2 \rfloor - 1}{n-k}$, each partition $\pi \in \Pi(n,k)$ is paired with another of opposite $\tilde{w}$-parity. This furnishes an answer to a question raised by Stanley [4, p. 46].

Let $F_{-3} = -1$, $F_{-2} = 1$, and $F_{-1} = 0$. We conclude this section by proving

Theorem 2.3. *For all* $n \in \mathbb{N}$,

$$\hat{B}_{-1}(n) := \sum_{k=0}^{n} \hat{S}_{-1}(n,k) = (-1)^{n-1} F_{n-3}. \tag{2.8}$$

Proof. Let $n \geqslant 3$, $\tilde{\Pi}(n)$ be as in the proof of Theorem 2.1, and $\hat{\Pi}(n) \subseteq \tilde{\Pi}(n)$ consist of those partitions with an odd number of blocks and whose last block is a singleton. First, $|\hat{\Pi}(n)| = |\tilde{\Pi}(n-3)| = F_{n-3}$ as the removal of $n-2$, $n-1$, and n from $\pi \in \hat{\Pi}(n)$ is seen to be a bijection between $\hat{\Pi}(n)$ and $\tilde{\Pi}(n-3)$. Since $\hat{w}(\pi) = \tilde{w}(\pi) + n - k$ and since every $\pi \in \hat{\Pi}(n)$ has an even $\tilde{w}(\pi)$ value and an odd number of blocks, the $\hat{w}$-parity of each $\pi \in \hat{\Pi}(n)$ is opposite the parity of n. Thus, $\hat{\Pi}(n)$ agrees with the right hand side of (2.8) in both sign and magnitude.

The $\tilde{w}$-parity changing involution of Theorem 2.1 defined on $\Pi(n) - \tilde{\Pi}(n)$ also changes the $\hat{w}$-parity. We now extend this involution to $\Pi(n) - \hat{\Pi}(n)$ as follows: if the last block of $\pi \in \tilde{\Pi}(n) - \hat{\Pi}(n)$ is $\{n\}$, merge it with the penultimate block; if the last block is not a singleton, take n from this block and form the singleton $\{n\}$. The resulting extension is a $\hat{w}$-parity changing involution of $\Pi(n) - \hat{\Pi}(n)$. □

3 A Second Bijection

The Bell numbers $B^*_{-1}(n)$ are quite different from the numbers $\tilde{B}_{-1}(n)$ and $\hat{B}_{-1}(n)$, as demonstrated by the following theorem.

Theorem 3.1. *For all* $n \in \mathbb{N}$,

$$B^*_{-1}(n) := \sum_{k=0}^{n} S^*_{-1}(n,k) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3; \\ -1, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{3.1}$$

Proof. Let $\Pi_i(n) := \{\pi \in \Pi(n) : \ w^*(\pi) \equiv i \pmod 2\}$ and $\Pi^*(n)$ consist of those partitions $\pi = (E_1, E_2, \dots)$ whose blocks satisfy

$$E_{2i-1} = \{3i-2, 3i-1\}, \quad E_{2i} = \{3i\} \quad \text{for } 1 \leqslant i \leqslant \lfloor n/3 \rfloor. \tag{3.2}$$

Then $\Pi^*(n)$ is a singleton contained in $\Pi_0(n)$ if $n \equiv 0 \pmod 3$ or contained in $\Pi_1(n)$ if $n \equiv 1 \pmod 3$. If $n \equiv 2 \pmod 3$, $\Pi^*(n)$ is a doubleton containing two partitions of opposite w^*-parity, which we pair.

Suppose now that $\pi = (E_1, E_2, \dots) \in \Pi(n) - \Pi^*(n)$ and that i_0 is the smallest index for which condition (3.2) fails to hold. Let $n_1 = 3i_0 - 2$, $n_2 = 3i_0 - 1$, $n_3 = 3i_0$ and $V_1 = E_{2i_0-1}$, $V_2 = E_{2i_0}$, $V_3 = E_{2i_0+1}$ (the latter two if they occur). Consider the following four disjoint cases concerning the relative positions of the n_i within the V_i:

- (I) $n_2 \in V_2$, $n_3 \in V_3$, and $|V_2 \cup V_3| \geqslant 3$;
- (II) Either (a) or (b) holds where (a) $V_2 = \{n_2\}$ and $V_3 = \{n_3\}$,
  (b) $n_2, n_3 \in V_1$;
- (III) $n_2 \in V_2$ and $n_3 \in V_1 \cup V_2$;
- (IV) $n_2 \in V_1$, $n_3 \in V_2$, and $|V_1 \cup V_2| \geqslant 4$.

Within each case, we pair partitions of opposite parity as shown below, leaving the other blocks undisturbed:

- (i) $V_2 = \{n_2, \dots, M\}$, $V_3 = \{n_3, \dots\} \leftrightarrow V_2 = \{n_2, \dots\}$, $V_3 = \{n_3, \dots, M\}$, where M is the largest member of $V_2 \cup V_3$;
- (ii) $V_1 = \{n_1, \dots\}$, $V_2 = \{n_2\}$, $V_3 = \{n_3\} \leftrightarrow V_1 = \{n_1, n_2, n_3, \dots\}$;
- (iii) $V_1 = \{n_1, n_3, \dots\}$, $V_2 = \{n_2, \dots\} \leftrightarrow V_1 = \{n_1, \dots\}$, $V_2 = \{n_2, n_3, \dots\}$;
- (iv) $V_1 = \{n_1, n_2, \dots, N\}$, $V_2 = \{n_3, \dots\} \leftrightarrow V_1 = \{n_1, n_2, \dots\}$, $V_2 = \{n_3, \dots, N\}$, where N is the largest member of $V_1 \cup V_2$.

The resulting map is a parity changing involution of $\Pi(n) - \Pi^*(n)$, which implies (3.1). □

Below, we illustrate the fixed point set $\Pi^*(n)$ along with the pairings of $\Pi(n) - \Pi^*(n)$ when $n = 4$.

$\Pi_0(n)$ $\qquad$ $\Pi_1(n) - \Pi^*(n)$ $\qquad$ $\Pi^*(n)$

$\{1,2,3,4\}$	$\{1,4\},\ \{2\},\ \{3\}$	$\{1,2\},\ \{3\},\ \{4\}$
$\{1,2\},\ \{3,4\}$	$\{1,2,4\},\ \{3\}$	
$\{1,3\},\ \{2,4\}$	$\{1\},\ \{2,3,4\}$	
$\{1,4\},\ \{2,3\}$	$\{1,3,4\},\ \{2\}$	
$\{1\},\ \{2,3\},\ \{4\}$	$\{1,3\},\ \{2\},\ \{4\}$	
$\{1\},\ \{2,4\},\ \{3\}$	$\{1\},\ \{2\},\ \{3,4\}$	
$\{1\},\ \{2\},\ \{3\},\ \{4\}$	$\{1,2,3\},\ \{4\}$	

Note that the bijection above, like the one used for Theorem 2.3, does not always preserve the number of blocks and hence has no meaningful restriction to $\Pi(n,k)$, unlike the bijection of Theorem 2.1.

Remark. In [2], Ehrlich evaluates $\sigma(n) := -\sum_{\pi\in\Pi(n)}(-1)^{\alpha(\pi)}$, where $\alpha(\pi) := \sum_{i \text{ odd}} |E_i|$ for $\pi = (E_1, E_2, \dots) \in \Pi(n)$. The bijection of Theorem 3.1 establishing $B^*_{-1}(n)$ also provides an alternative to Ehrlich's iterative argument establishing his $\sigma(n)$ since

$$\begin{aligned}\sigma(n) &= -\sum_{\pi=(E_1,E_2,\dots)\in\Pi(n)} (-1)^{|E_1|+|E_3|+|E_5|+\cdots} \\ &= -\sum_{\pi=(E_1,E_2,\dots)\in\Pi(n)} (-1)^{|E_1|+2|E_2|+3|E_3|+\cdots} \\ &= -B^*_{-1}(n).\end{aligned}$$

Since $S_q(n,k) = q^{-n} S^*_q(n,k)$,

$$B_{-1}(n) := \sum_{k=0}^{n} S_{-1}(n,k) = (-1)^n B^*_{-1}(n),$$

and so by (3.1),

$$B_{-1}(n) = \begin{cases} (-1)^n, & \text{if } n \equiv 0 \pmod 3; \\ (-1)^{n+1}, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{3.3}$$

with the above bijection clearly showing this. The preceding also supplies a combinatorial proof that $B(n)$, the n^{th} Bell number, is even if and only if $n \equiv 2 \pmod 3$ since every partition of $[n]$ is paired with another of opposite w^*-parity when $n \equiv 2 \pmod 3$ and since all partitions are so paired except for one otherwise (cf. Ehrlich [2, p. 512]).

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