Journal of Lie Theory
Volume **10** (2000) 255–267

A proof of property (RD) for cocompact lattices of SL(3,R) and SL(3,C)

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Communicated by A. Valette

Abstract. We prove that cocompact lattices in $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$ have property (RD) of Jolissaint. This property asserts that functions on these groups which are l^2 with some polynomial decay belong to the reduced C^*-algebra. Ramagge, Robertson and Steger proved the same result for cocompact lattices in SL_3 of p-adic fields and we use the same method.

In this article we prove that discrete cocompact subgroups of $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$ satisfy property (RD) of Jolissaint (this property was introduced in [4, 5, 7]). The author originally proved the result for $SL_3(\mathbb{R})$ and the case of $SL_3(\mathbb{C})$ is due to the referee. The argument is a very close imitation of the argument of [9] : in this article Ramagge, Robertson and Steger prove a general result (stated below) implying property (RD) for finetely generated discrete subgroups of $SL_3(\mathbb{F})$, with $\mathbb{F}$ a finite extension of $\mathbb{Q}_p$. Our result is a special case of a conjecture of Valette which claims that any discrete group acting isometrically, properly and cocompactly either on a Riemannian symmetric space or on an affine building has property (RD) ([2] page 74).

Up to now property (RD) has been proved for free groups by Haagerup in [3], and then for hyperbolic groups by de la Harpe in [1], using [5]. Recently, in [9], Ramagge Robertson and Steger have proved property (RD) for any discrete group acting freely on the vertices of an $\tilde{A}^1 \times \tilde{A}^1$ or $\tilde{A}^2$ building by type-rotating automorphisms and this provided the first example of higher rank groups with property (RD). Our article is just an adaptation of [9] to $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$ and it doesn't bring any new idea. On the other hand a new idea is needed in order to prove property (RD) for cocompact lattices in Lie groups or p-adic groups of rank more than 1 and other than SL_3.

I thank Yves Benoist for many fruitful discussions.

1. General facts about property (RD)

Let Γ be a discrete group. A length l on Γ is a function $l : \Gamma \to \mathbb{R}_+$ such that $l(1) = 0$, $l(g^{-1}) = l(g)$ and $l(g_1 g_2) \leq l(g_1) + l(g_2)$. We write $\Gamma_r = \{g \in \Gamma, l(g) \leq r\}$.

ISSN 0949–5932 / $2.50

Definition 1.1. ([7]) We say that Γ satisfies (RD) with respect to a length l if there is a polynomial P such that, for any $r \in \mathbb{R}_+$, $f_1, f_2 \in \mathbb{C}\Gamma$ with $\mathrm{supp} f_1 \subset \Gamma_r$, we have $\|f_1 * f_2\|_{l^2(\Gamma)} \leq P(r)\|f_1\|_{l^2(\Gamma)}\|f_2\|_{l^2(\Gamma)}$.

It is enough to check the inequality for $f_1, f_2 \in \mathbb{R}_+\Gamma$.

Let us notice the following fact : if Γ satisfies (RD) with respect to a lenght l, there is a polynomial P such that, for any $r \in \mathbb{R}_+$ and for any $f \in \mathbb{C}\Gamma$ with $\mathrm{supp} f \in \Gamma_r$, one has $\|f\|_{l^2(\Gamma)} \leq \|f\|_{C_r^*(\Gamma)} \leq P(r)\|f\|_{l^2(\Gamma)}$. This very good estimate for the norm in $C_r^*(\Gamma)$ has an important consequence : $C_r^*(\Gamma)$ has the same K-theory as a much simpler algebra, that we introduce now.

Proposition 1.2. *If Γ satisfies (RD) w.r.t. l, if $s \in \mathbb{R}_+$ is big enough, the completion $H^s(\Gamma)$ of $\mathbb{C}\Gamma$ for the norm $\|f\| = \left(\sum_{g\in\Gamma} |f(g)|^2(1+l(g))^{2s}\right)^{1/2}$ is a Banach subalgebra of $C_r^*(\Gamma)$ which is dense and closed under holomorphic functional calculus.*

In [6] this is proved for the Jolissaint algebra $H^\infty(\Gamma) = \bigcap H^s(\Gamma)$. We give an adaptation of the proof to our case.

Proof. Let P be the polynomial in the definition. Take any $s \in \mathbb{R}_+$ such that $s > \deg(P)$.

a) We prove that $H^s(\Gamma)$ is a subspace of $C_r^*(\Gamma)$. We denote by χ_0 the characteristic function of $\{g, l(g) \in [0,1[\}$ and for any $n \in \mathbb{N}^*$ we denote by χ_n the characteristic function of $\{g, l(g) \in [2^{n-1}, 2^n[\}$. For any $f \in \mathbb{C}\Gamma$, $\|f\|_{C_r^*(\Gamma)} \leq \sum_{n=0}^\infty \|f\chi_n\|_{C_r^*(\Gamma)} \leq \sum_{n=0}^\infty P(2^n)\|f\chi_n\|_{l^2(\Gamma)} \leq C\|f\|_{H^s(\Gamma)}$ with $C = \left(P(1)^2 + \sum_{n=1}^\infty (P(2^n)(1+2^{n-1})^{-s})^2\right)^{\frac{1}{2}}$ by the Cauchy-Schwarz inequality.

b) We prove that $H^s(\Gamma)$ is an algebra. For any $f_1, f_2 \in \mathbb{C}\Gamma$ and for any $g \in \Gamma$ we have

$$\left|(f_1 * f_2)(g)\right|\left(1+l(g)\right)^s \leq \sum_{\substack{g_1,g_2,\ s.t.\\ g_1 g_2 = g}} 2^s \left|f_1(g_1)\right|\left|f_2(g_2)\right|\left((1+l(g_1))^s + (1+l(g_2))^s\right)$$

and therefore

$$\begin{aligned} &\left\| g \mapsto \left|(f_1 * f_2)(g)\right|\left(1+l(g)\right)^s \right\|_{l^2(\Gamma)} \\ \leq\ & \left\| g \mapsto \Big(\sum_{\substack{g_1,g_2,\ s.t.\\ g_1 g_2 = g}} 2^s \left|f_1(g_1)\right|\left|f_2(g_2)\right|\left(1+l(g_1)\right)^s \Big) \right\|_{l^2(\Gamma)} \\ +\ & \left\| g \mapsto \Big(\sum_{\substack{g_1,g_2,\ s.t.\\ g_1 g_2 = g}} 2^s \left|f_1(g_1)\right|\left|f_2(g_2)\right|\left(1+l(g_2)\right)^s \Big) \right\|_{l^2(\Gamma)}. \end{aligned}$$

The two terms are analoguous and for the first one we have

$$\left\| g \mapsto \Big(\sum_{\substack{g_1,g_2,\ s.t.\\ g_1 g_2 = g}} 2^s \left|f_1(g_1)\right|\left|f_2(g_2)\right|\left(1+l(g_1)\right)^s \Big) \right\|_{l^2(\Gamma)} \leq 2^s C \|f_1\|_{H^s(\Gamma)}\|f_2\|_{H^s(\Gamma)}$$

by part a).

c) Let $t \in]\deg(P), s[$. We first prove two intermediate results.

α) $H^t(\Gamma)$ is an algebra by part b) and $H^s(\Gamma)$ is stable under holomorphic functional calculus in $H^t(\Gamma)$. The proof is as follows. Since $H^s(\Gamma)$ is dense in $H^t(\Gamma)$, if $x \in H^s(\Gamma)$ has an invertible image in $H^t(\Gamma)$, there exists $y \in H^s(\Gamma)$ such that the norms in $H^t(\Gamma)$ of $1 - xy$ and $1 - yx$ are arbitrary small. Therefore we only have to prove that, for any $f \in H^s(\Gamma)$,

$$\lim_{n\to\infty} \|f^n\|_{H^s(\Gamma)}^{1/n} = \lim_{n\to\infty} \|f^n\|_{H^t(\Gamma)}^{1/n}.$$

For any $g \in \Gamma$ we have

$$|f^n(g)| \leq \sum_{\substack{g_1,\dots,g_n \in \Gamma,\\ g_1,\dots,g_n = g}} |f(g_1)| \dots |f(g_n)|$$

and if $g = g_1, \dots, g_n$, $(1 + l(g))^{s-t} \leq n^{s-t}\big((1 + l(g_1))^{s-t} + \cdots + (1 + l(g_n))^{s-t}\big)$. Therefore

$$\|f^n\|_{H^s(\Gamma)} = \|g \mapsto (1 + l(g))^{s-t}|f^n(g)|\|_{H^t(\Gamma)} \leq n^{s-t+1} C'^{n-1} \|f\|_{H^s(\Gamma)} \|f\|_{H^t(\Gamma)}^{n-1},$$

where C' is a constant such that $\|f_1 f_2\|_{H^t(\Gamma)} \leq C' \|f_1\|_{H^t(\Gamma)} \|f_2\|_{H^t(\Gamma)}$ for any $f_1, f_2 \in H^t(\Gamma)$. When n goes to infinity we get $\lim_{n\to\infty} \|f^n\|_{H^s(\Gamma)}^{1/n} \leq C' \|f\|_{H^t(\Gamma)}$ and the result easily follows by putting f^p instead of f in this inequality and making p go to infinity.

β) For any $f \in H^s(\Gamma)$ we have $\|f\|_{H^t(\Gamma)} \leq \|f\|_{H^s(\Gamma)}^{\frac{t}{s}} \|f\|_{l^2(\Gamma)}^{1-\frac{t}{s}}$ by Hlder's inequality.

Now let $f \in H^s(\Gamma)$. We have to prove that f has the same spectral radius in $H^s(\Gamma)$ and $C_r^*(\Gamma)$. If $\rho_{H^s(\Gamma)}(f) = 0$ this is obvious because $\rho_{C_r^*(\Gamma)}(f) \leq \rho_{H^s(\Gamma)}(f)$. Otherwise we have $\|f^n\|_{C_r^*(\Gamma)} \geq \|f^n\|_{l^2(\Gamma)} \geq \|f^n\|_{H^t(\Gamma)}^{\frac{s}{s-t}} \|f^n\|_{H^s(\Gamma)}^{-\frac{t}{s-t}}$ and the result follows from α). ∎

2. Analytical part of the proof

In this section we consider a discrete metric space (X, d) and a discrete group Γ acting freely and isometrically on X, and we introduce the groupoid $\mathcal{G} = X \times_\Gamma X$ such that $\mathcal{G}^{(0)} = \Gamma\backslash X$ and $\mathcal{G}^{(1)} = \Gamma\backslash X^2$ and we define $\mathcal{G}_r = \{[x,y] \in \mathcal{G}, d(x,y) \leq r\}$ for any $r \in \mathbb{R}_+$ and $\|f\|_{l^2(\mathcal{G})} = \big(\sum_{g\in\mathcal{G}} |f(g)|^2\big)^{\frac{1}{2}}$ for any $f \in \mathbb{C}\mathcal{G}$.

We say that X and Γ satisfy the property $P(X, \Gamma)$ if there is a polynomial P such that for any $r \in \mathbb{R}_+$, $f_1, f_2 \in \mathbb{R}_+\mathcal{G}$ with $\mathrm{supp} f_1 \in \mathcal{G}_r$, one has $\|f_1 * f_2\|_{l^2(\mathcal{G})} \leq P(r) \|f_1\|_{l^2(\mathcal{G})} \|f_2\|_{l^2(\mathcal{G})}$.

Proposition 2.1. *If $P(X,\Gamma)$ holds then Γ satisfies (RD) w.r.t. the lenght $l(g) = d(x_0, gx_0)$ for any $x_0 \in X$.*

Proof. For any $g_1, g_2 \in \Gamma$, $[x_0, g_1x_0] \circ [x_0, g_2x_0] = [x_0, g_1g_2x_0]$. Let us define $T : \mathbb{C}\Gamma \to \mathbb{C}\mathcal{G}$ by $T(f)([x,y]) = 0$ if $x \notin \Gamma x_0$ or $y \notin \Gamma x_0$ and $T(f)([x_0, gx_0]) = f(g)$. For any $f \in \mathbb{C}\Gamma$, $\|f\|_{l^2(\Gamma)} = \|T(f)\|_{l^2(\mathcal{G})}$ and for $f_1, f_2 \in \mathbb{C}\Gamma$, $T(f_1 *_\Gamma f_2) = T(f_1) *_\mathcal{G} T(f_2)$. ∎

Definition 2.2. Let (Z,d) be a metric space and $\delta>0$. For any points $x_i\in Z$ we say that $x_1\dots x_n$ is a δ-path if $d(x_1,x_2)+\dots+d(x_{n-1},x_n)\le d(x_1,x_n)+\delta$ and that $x_1x_2x_3$ is a δ-thin triangle if there exists $t\in Z$ such thatx_1tx_2, x_2tx_3 and x_3tx_1 are δ-paths. We say that (Z,d) satisfies (H_δ) if there exists a polynomial P such thatfor any $r\in\mathbb{R}_+$, $x,y\in Z$, one has

$$\#\{t\in Z, xty\ \delta\text{-path}, d(x,t)\le r\}\le P(r).$$

Proposition 2.3. *Let $\delta>0$. If (X,d) satisfies (H_δ), there exists a polynomial P such that for any $r\in\mathbb{R}_+$, $f_1,f_2,f_3\in\mathbb{R}_+\mathcal{G}$, with* $\operatorname{supp}(f_1)\in\mathcal{G}_r$, *one has*

$$\sum_{\substack{(x_1,x_2,x_3)\in\Gamma\backslash X^3,\\ x_1x_2x_3\ \delta-thin}} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2])\le P(r)\|f_1\|_{l^2(\mathcal{G})}\|f_2\|_{l^2(\mathcal{G})}\|f_3\|_{l^2(\mathcal{G})}.$$

This proposition implies the result of [1] : hyperbolic groups satisfy property (RD).

The following lemma is obvious.

Lemma 2.4. *If H_1,H_2,H_3 are Hilbert spaces, and $T_1\in\mathcal{L}(H_3,H_2),T_2\in\mathcal{L}(H_1,H_3),T_3\in\mathcal{L}(H_2,H_1)$ have finite Hilbert-Schmidt norms, $|Tr(T_1T_2T_3)|\le\|T_1\|_{HS}\|T_2\|_{HS}\|T_3\|_{HS}$.*

Proof of the proposition 2.3. We have

$$\sum_{\substack{(x_1,x_2,x_3)\in\Gamma\backslash X^3,\\ x_1x_2x_3\ \delta-\text{thin}}} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2])$$

$$\le\sum_{\substack{(x_1,x_2,x_3,t)\in\Gamma\backslash X^4,\\ x_1tx_2,x_2tx_3,x_3tx_1\ \delta-\text{paths}}} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2]).$$

Note that if x_2tx_3 is a δ-path and $d(x_2,x_3)\le r$, then $d(x_2,t)\le r+\delta$ and $d(x_3,t)\le r+\delta$.

Let $H_1,H_2,H_3\subset l^2(\Gamma\backslash X^2)$ be defined by $H_1=l^2(\Gamma\backslash X^2)$, $H_2=l^2(\{(t,x_2)\in\Gamma\backslash X^2, d(x_2,t)\le r+\delta\})$ and $H_3=l^2(\{(t,x_3)\in\Gamma\backslash X^2, d(x_3,t)\le r+\delta\})$, and let $T_1\in\mathcal{L}(H_3,H_2)$ be the operator defined as a matrix by

- $T_{1,[t,x_2],[t',x_3]}=f_1([x_2,x_3])$ if t,t' are in the same Γ-orbit (in this case we suppose $t=t'$) and if x_2tx_3 is a δ-path
- and otherwise the coefficient is 0,

and let T_2 and T_3 be defined in the same way. We have

$$\sum_{\substack{(x_1,x_2,x_3,t)\in\Gamma\backslash X^4,\\ x_1tx_2,x_2tx_3,x_3tx_1\ \delta-\text{paths}}} f_1([x_2,x_3])f_2([x_3,x_1])f_3([x_1,x_2])=Tr(T_1T_2T_3),$$

but

$$\|T_1\|^2_{HS}\le\sum_{\substack{(x_2,t,x_3)\in\Gamma\backslash X^3,\\ x_2tx_3\ \delta-\text{path},\\ d(x_2,t)\le r+\delta}}\left|f_1([x_2,x_3])\right|^2\le P(r+\delta)\|f_1\|^2_{l^2(\mathcal{G})}$$

by (H_δ) and in the same way $\|T_2\|_{HS}\le\sqrt{P(r+\delta)}\|f_2\|_{l^2(\mathcal{G})}$ and $\|T_3\|_{HS}\le\sqrt{P(r+\delta)}\|f_3\|_{l^2(\mathcal{G})}$. The proposition follows. ∎

Let $\delta > 0$. We say that X and Γ satisfy (K_δ) if there exist $k \in \mathbb{N}$ and Γ-invariant subsets $\mathcal{T}_1, \ldots, \mathcal{T}_k$ of X^3 such that

$(K_\delta a)$ there exists $C_1 \in \mathbb{R}_+$ such that for any $(x_1, x_2, x_3) \in X^3$, there exist $i \in \{1, \ldots, k\}$ and $(t_1, t_2, t_3) \in \mathcal{T}_i$ such that $\max\big(d(t_1,t_2), d(t_2,t_3), d(t_3,t_1)\big) \leq C_1\big(\min\big(d(x_1,x_2), d(x_2,x_3), d(x_3,x_1)\big) + \delta\big)$, and $x_1t_1t_2x_2$, $x_2t_2t_3x_3$, $x_3t_3t_1x_1$ are δ-paths,

$(K_\delta b)$ for any $i \in \{1, \ldots, k\}$ and $t_1, t_2, t_3, t_3' \in X$, if (t_1, t_2, t_3) and (t_1, t_2, t_3') are in $\mathcal{T}_i$ then the triangles $t_1t_3t_3'$ and $t_2t_3t_3'$ are δ-thin.

Theorem 2.5. *If, for some $\delta > 0$, X and Γ satisfy (H_δ) and (K_δ) then $P(X,\Gamma)$ holds and therefore Γ satisfies property (RD).*

Proof. Let $\mathcal{G} = X \times_\Gamma X$, and $f_1, f_2, f_3 \in \mathbb{R}_+\mathcal{G}$ with $\mathrm{supp}(f_1) \subset \mathcal{G}_r$.

We shall abbreviate (x_1, x_2, x_3) by x. For any $i \in \{1, ..., k\}$, let J_i denote the set of all $t = (t_1, t_2, t_3) \in \mathcal{T}_i$ satisfying $\max(d(t_1,t_2), d(t_2,t_3), d(t_3,t_1)) \leq C_1(r+\delta)$, and for any $t \in X^3$ let $K(t)$ be the set of all $(x_1, x_2, x_3) \in X^3$ for which $d(x_2,t_2) \leq r+\delta$, $d(x_3,t_3) \leq r+\delta$ holds, and $x_1t_1t_2x_2$, $x_2t_2t_3x_3$, $x_3t_3t_1x_1$ are δ-paths. For any $x \in X^3$ such $d(x_2,x_3) \leq r$, there exist $i \in \{1, ..., k\}$ and $t \in J_i$, such that $x \in K(t)$. Therefore we have

$$\begin{aligned}
&\sum_{x \in \Gamma\backslash X^3} f_1([x_2,x_3]) f_2([x_3,x_1]) f_3([x_1,x_2]) \\
\leq\ &\sum_{i=1}^{k} \sum_{t \in \Gamma\backslash J_i} \sum_{x \in K(t)} f_1([x_2,x_3]) f_2([x_3,x_1]) f_3([x_1,x_2]) \\
\leq\ &\sum_{i=1}^{k} \sum_{t \in \Gamma\backslash J_i} h_1([t_2,t_3]) h_2([t_3,t_1]) h_3([t_1,t_2]),
\end{aligned}$$

where $h_1 \in \mathbb{R}_+\mathcal{G}$ is defined by

$$h_1([t_2,t_3]) = \Big(\sum_{\substack{(x_2,x_3) \in X^2, \\ x_2t_2t_3x_3\ \delta-\mathrm{path}, \\ d(x_2,t_2) \leq r+\delta}} f_1([x_2,x_3])^2 \Big)^{1/2}$$

if $d(t_2,t_3) \leq C_1(r+\delta)$ and $h_1([t_2,t_3]) = 0$ otherwise and $h_2, h_3 \in \mathbb{R}_+\mathcal{G}$ are defined by similar expressions. The last inequality comes from lemma 2.4 with $H_1 = H_2 = H_3 = l^2(X)$ and T_1 with coefficient $T_{1,x_2,x_3} = f_1([x_2,x_3])$ if $x_2t_2t_3x_3$ is a δ-path and $d(x_2,t_2) \leq r+\delta$ and 0 otherwise and with T_2 and T_3 defined in the same way. But

$$\|h_1\|^2_{l^2(\mathcal{G})} \leq \sum_{\substack{(x_2,t_2,t_3,x_3) \in \Gamma\backslash X^4, \\ x_2t_2t_3x_3\ \delta-\mathrm{path}, \\ d(x_2,t_2) \leq r+\delta \\ d(t_2,t_3) \leq C_1(r+\delta)}} f_1([x_2,x_3])^2 \leq P(r+\delta) P(C_1(r+\delta)) \|f_1\|^2_{l^2(\mathcal{G})}$$

and the same inequality holds for h_2 and h_3.

Fix $i \in \{1, \dots, k\}$ and replace $C_1(r+\delta)$ by r. It remains to show that there is a polynomial P such thatfor any $r \in \mathbb{R}_+$ and $h_1, h_2, h_3 \in \mathbb{R}_+\mathcal{G}$, with support in $\mathcal{G}_r$, we have

$$\sum_{(t_1,t_2,t_3)\in\Gamma\backslash\mathcal{T}_i} h_1([t_2,t_3])h_2([t_3,t_1])h_3([t_1,t_2]) \leq P(r)\|h_1\|_{l^2(\mathcal{G})}\|h_2\|_{l^2(\mathcal{G})}\|h_3\|_{l^2(\mathcal{G})}.$$

But the sum is equal to $\langle h_1 *_{\mathcal{T}_i} h_2, \tilde{h}_3\rangle_{l^2(\mathcal{G})}$ for some partial convolution along $\mathcal{T}_i$, where $\tilde{h}_3([x,y]) = \overline{h_3([y,x])}$. We compute

$$\langle h_1 *_{\mathcal{T}_i} h_2, h_1 *_{\mathcal{T}_i} h_2\rangle_{l^2(\mathcal{G})} = \sum_{\substack{(t_1,t_2,t_3,t_3')\in\Gamma\backslash X^4,\\ (t_1,t_2,t_3)\in\mathcal{T}_i,\,(t_1,t_2,t_3')\in\mathcal{T}_i}} h_1([t_2,t_3])h_2([t_3,t_1])\,\overline{h_1([t_2,t_3'])h_2([t_3',t_1])}.$$

By $(K_\delta b)$ the triangle $t_1t_3t_3'$ and $t_2t_3t_3'$ are δ-thin. By proposition 2.3 there is a polynomial P with

$$\Big\|[t_3,t_3'] \mapsto \sum_{\substack{t_1\in X,\\ t_1t_3t_3'\ \delta-\text{thin}}} h_2([t_3,t_1])\overline{h_2([t_3',t_1])}\Big\|_{l^2(\mathcal{G})} \leq P(r)\|h_2\|^2_{l^2(\mathcal{G})},$$

$$\Big\|[t_3,t_3'] \mapsto \sum_{\substack{t_2\in X,\\ t_2t_3t_3',\ \delta-\text{thin}}} h_1([t_2,t_3])\overline{h_1([t_2,t_3'])}\Big\|_{l^2(\mathcal{G})} \leq P(r)\|h_1\|^2_{l^2(\mathcal{G})},$$

and the theorem follows. ■

3. Geometrical part of the proof

3.1. The case of $\tilde{A}^2$-buildings.

The following theorem is easily deducible from the arguments of [9].

Theorem 3.1. (extracted from [9]) *If X is a free Γ-space and Z is the set of vertices of some $\tilde{A}^2$-building on which Γ acts by type rotating automorphisms, and d is the graph-theoretic distance on the 1-squeletton, and $\theta : X \to Z$ is a surjective Γ-equivariant map such that $\sup_{z\in Z}\#(\theta^{-1}(z)) < +\infty$, and X is equipped with the distance $\theta^*(d)$, then X and Γ satisfy (H_0) and (K_0) with $k=2$, $\mathcal{T}_1\cup\mathcal{T}_2$ the set of $(t_1,t_2,t_3)\in X^3$ such that $\theta(t_1)\theta(t_2)\theta(t_3)$ is an equilateral triangle in some apartment, and $(t_1,t_2,t_3)\in\mathcal{T}_1$ if $(\theta(t_1),\theta(t_2))$ is of shape $(p,0)$, $p\in\mathbb{N}$, and $(t_1,t_2,t_3)\in\mathcal{T}_2$ if $(\theta(t_1),\theta(t_2))$ is of shape $(0,p)$, $p\in\mathbb{N}^*$. Consequently Γ satisfies property (RD).*

We thus obtain a very slight improvement of the result of [9].

Corollary 3.2. *Any discrete group Γ acting on the set X of vertices of an $\tilde{A}^2$-building by type-rotating automorphisms satisfies property (RD), provided that $\sup_{x\in X}\#\{g\in\Gamma, gx=x\} < +\infty$.*

3.2. The case of $SL_3(R)$.

It is helpful to write 3×3-diagonal matrices with diagonal entries a, b, and c as $D(a,b,c)$. We now consider $G=SL_3(\mathbb{R})$, $K=SO_3(\mathbb{R})$,

$$\begin{aligned} A &= \{D(e^{\alpha_1},e^{\alpha_2},e^{\alpha_3}), \quad \alpha_1+\alpha_2+\alpha_3=0\} \quad \text{and} \\ \overline{A^+} &= \{D(e^{\alpha_1},e^{\alpha_2},e^{\alpha_3}), \quad \alpha_1\geq\alpha_2\geq\alpha_3, \alpha_1+\alpha_2+\alpha_3=0\}. \end{aligned}$$

We equip G/K with the distance $d(x,y)=\log\|x^{-1}y\|+\log\|y^{-1}x\|$. We remark that $d(x,y)=\rho\log a$ if $x^{-1}y\in KaK$ with a in $\overline{A^+}$, and ρ is defined by $\rho(\alpha_1,\alpha_2,\alpha_3)=\alpha_1-\alpha_3$.

For any $t\in\mathbb{R}, x,y,z\in G/K$, we say that (x,y) is of shape $(t,0)$ if $x^{-1}y\in Ke^{-\frac{t}{3}}D(e^t,1,1)K$ and that (x,y,z) is an *equilateral triangle* of oriented size t if there exists $g\in G$ such that

$$x=gK,\ y=ge^{-\frac{t}{3}}D(e^t,1,1)K \text{ and } z=ge^{-\frac{2t}{3}}D(e^t,e^t,1)K.$$

If $t\in\mathbb{R}_-$ and (x,y) is of shape $(t,0)$ we say also that (x,y) is of shape $(0,-t)$: in this way our terminology completely agrees with [9]. For any $\delta_0>0$ and for $j=1,2$ we let $\mathcal{T}_{j,\delta_0}$ denote the set of all $(t_1,t_2,t_3)\in(G/K)^3$ for which there exists a t in $\mathbb{R}_+$ if $j=1$, and in $\mathbb{R}_-$ if $j=2$, and for which there is a triple $(s_1,s_2,s_3)\in(G/K)^3$ such that $s_1s_2s_3$ is an equilateral triangle of oriented size t, and that $d(s_1,t_1)\leq\delta_0$, $d(s_2,t_2)\leq\delta_0$, and $d(s_3,t_3)\leq\delta_0$.

Theorem 3.3. *Let Γ be a discrete subgroup of $G=SL_3(\mathbb{R})$, Z a Γ-invariant discrete subspace of G/K, and $r\in\mathbb{R}_+$ such that the two following conditions are fulfilled :*

(I1) $\bigcup_{x\in Z}B(x,r)=G/K$

(I2) for any $R\in\mathbb{R}_+$ $\sup_{x\in G/K}\#(B(x,R)\cap Z)$ *is finite.*

Let X be a free Γ-space and $\theta:X\to Z$ a Γ-equivariant map such that $\sup_{z\in Z}\#(\theta^{-1}(z))<+\infty$*, and equip X with the distance $\theta^*(d)$.*

Then X and Γ satisfy (H_δ) and (K_δ) for some $\delta>0$ and with $k=2$ and $\mathcal{T}_1=\theta^{-1}(\mathcal{T}_{1,r})=\{(t_1,t_2,t_3)\in X^3,(\theta(t_1),\theta(t_2),\theta(t_3))\in\mathcal{T}_{1,r}\}$ *and* $\mathcal{T}_2=\theta^{-1}(\mathcal{T}_{2,r})$. *Consequently Γ satisfies property (RD).*

If Γ is a discrete cocompact subgroup of $SL_3(\mathbb{R})$, every Γ-orbit Z in G/K satisfies (I1) and (I2), and we can choose $X=\Gamma$ and θ obvious, and therefore Γ satisfies (RD).

Proof of the theorem. This proof is everything until part 4.

a) We first prove that X satisfies property (H_δ) for any $\delta>0$. This part of the argument works for any linear connected semi-simple Lie group.

We recall some notations from chapter 5 of [8]. Let G be a linear connected semi-simple Lie group, K a maximal compact subgroup, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the decomposition associated to the Cartan involution, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, Σ the set of restricted roots, $\mathfrak{g}_\lambda$ the root space associated to $\lambda\in\Sigma$, Σ^+ a choice of a set of positive roots, $\mathfrak{a}^+=\{H\in\mathfrak{a},,\lambda(H)>0 \text{ for all } \lambda\in\Sigma^+\}$, $A=\exp(\mathfrak{a})$, $A^+=\exp(\mathfrak{a}^+)$, and $\rho=\frac{1}{2}\sum_{\lambda\in\Sigma^+}(\dim\mathfrak{g}_\lambda)\lambda$. We have $G=K\overline{A^+}K$ and

$$\int_G f(g)dg=\int_{K\times A^+\times K}f(kak')\prod_{\lambda\in\Sigma+}(\sinh\lambda(\log a))^{\dim\mathfrak{g}_\lambda}dkdadk' \qquad (1)$$

is the integral formula corresponding to this decomposition. If $\omega_1, \ldots, \omega_k$ are the fundamental weights, we have $\rho = n_1\omega_1 + \cdots + n_k\omega_k$ for some positive integers $n_1, \ldots, n_k$ depending on the multiplicities of the roots. We introduce the non-symmetric fonction d_i on $(G/K)^2$: if $x, y \in G/K$ and $x^{-1}y = KaK$ with $a \in \overline{A^+}$, $d_i(x,y) = \omega_i(\log a)$. Since G admits a representation of highest weight a multiple of ω_i, if we choose an hermitian metric on this representation compatible with the Cartan involution on G, for any $x, y \in G/K$, $d_i(x,y)$ is a fraction of the log of the norm of the image by this representation of any antecedent of $x^{-1}y$ in G. Therefore, for any $x, y, z \in G/K$, $d_i(x,z) \leq d_i(x,y) + d_i(y,z)$. Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+^*$, and consider the non-symmetric function $d_\alpha(x,y) = \sum_{i=1}^k \alpha_i d_i(x,y)$. Up to a constant there is a unique G-invariant element of volume on G/K.

Lemma 3.4. *For any $\delta > 0$ there is a polynomial P such that for any $r \in \mathbb{R}_+$ and $x, y \in G/K$,*

$$\mathrm{Vol}\{t \in G/K, d_\alpha(x,t) + d_\alpha(t,y) \leq d_\alpha(x,y) + \delta \text{ and } d_\alpha(x,t) \leq r\} \leq P(r).$$

The lemma is false if some α_i is 0 : in this case the best estimate for the volume grows exponentially in r.

Proof. We denote by d the following distance on G/K :
$d(x,y) = \sum_{i=1}^k n_i d_i(x,y) = \rho \log(a)$ if $x^{-1}y = KaK$ with $a \in \overline{A^+}$.

Denote by 1 the origin in G/K. We may assume $x = 1$. There exists some constant C_0 depending on α such that the conditions $d_\alpha(1,t) + d_\alpha(t,y) \leq d_\alpha(1,y) + \delta$ and $d_\alpha(1,t) \leq r$ imply $d_i(1,t) + d_i(t,y) \leq d_i(1,y) + C_0\delta$ for any i and $d(1,t) \leq C_0 r$. Because of (1) there exists some constant $C_1 \in \mathbb{R}_+^*$ such that

$$\mathrm{Vol}\{z \in G/K, \exists k \in K, d(y,kz) \leq 1\} \geq C_1 e^{2d(1,y)}.$$

Therefore

$$\mathrm{Vol}\Big\{(t,z) \in (G/K)^2, \forall i, d_i(1,t) + d_i(t,z) \leq d_i(1,y) + C_0\delta + 1, d(1,t) \leq C_0 r\Big\}$$
$$\geq C_1 e^{2d(1,y)} \mathrm{Vol}\Big\{t \in G/K, \forall i, d_i(1,t) + d_i(t,y) \leq d_i(1,y) + C_0\delta, d(1,t) \leq C_0 r\Big\}$$

because $\Big\{t \in (G/K), \forall i, d_i(1,t) + d_i(t,z) \leq d_i(1,y) + C_0\delta + 1, d(1,t) \leq C_0 r\Big\} \supset \Big\{t \in (G/K), \forall i, d_i(1,t) + d_i(t,y) \leq d_i(1,y) + C_0\delta, d(1,t) \leq C_0 r\Big\}$ if $d(y,z) \leq 1$ and $\mathrm{Vol}\Big\{t \in (G/K), \forall i, d_i(1,t) + d_i(t,z) \leq d_i(1,y) + C_0\delta + 1, d(1,t) \leq C_0 r\Big\}$ depends only on Kz in $K\backslash G/K$. The following fact comes from (1) : there exists a constant C_2 such that

$$\text{for any } a_1, \ldots, a_k \in \mathbb{R}_+, \ \mathrm{Vol}\{u \in G/K, \forall i, d_i(1,u) \leq a_i\} \leq C_2 e^{2\sum n_i a_i}.$$

Now fix $t \in G/K$ such that $d_i(1,t) \leq C_0 r$. We have

$$\mathrm{Vol}\Big\{z \in (G/K), \forall i, d_i(1,t) + d_i(t,z) \leq d_i(1,y) + C_0\delta + 1\Big\}$$
$$= \mathrm{Vol}\Big\{z \in (G/K), \forall i, d_i(t,z) \leq d_i(1,y) + C_0\delta + 1 - d_i(1,t)\Big\}$$
$$\leq C_2 \exp\big(2d(1,y) + 2(C_0\delta + 1)(\textstyle\sum_{i=1}^k n_i) - 2d(1,t)\big),$$

and $\int_{t \in G/K, d(1,t) \leq C_0 r} e^{-2d(1,t)} dt \leq P(r)$ for some polynomial P, because of (1). The lemma now results from Fubini's lemma. ∎

We now come back to the notations of the theorem 3.3. We fix $\delta > 0$ and we explain why X satisfies (H_δ). Of course it is enough to prove that (Z,d) satisfies (H_δ). By (I2) there exists $N \in \mathbb{N}$ such that $\#(B(x,1) \cap Z) \leq N$ for any $x \in G/K$. For any $r \in \mathbb{R}_+, x, y \in G/K$,

$$\#\{t \in Z, xty\ \delta\text{-path}, d(x,t) \leq r\} \leq$$
$$\tfrac{N}{V}\mathrm{Vol}\{t' \in G/K, xt'y\ (\delta+2)\text{-path}, d(x,t') \leq r+1\},$$

where $V = \mathrm{Vol}\{z \in G/K, d(1,z) \leq 1\}$, because if $d(t,t') \leq 1$, $d(x,t) \leq r$ and xty is a δ-path, then $xt'y$ is a $(\delta+2)$-path and $d(x,t') \leq r+1$. It remains to apply lemma 3.4 with $d_\alpha = d$.

b) We now prove that X and Γ satisfy (K_δ) with $k=2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ as in the theorem 3.3 and if δ is big enough. This part of the proof really use fact that the flats in G/K are of type A^2.

The following lemma is analoguous to the study of the foldings in [9]. We call a flat in G/K any subset of G/K equal to gAK, for some $g \in G$. We now study the distance of some fixed point of G/K to the points of a fixed flat in G/K. Up to left translation by an element of G, we may suppose that the flat is AK. We prove the following result : there is some $\delta > 0$ such that for any $x \in G/K$ there exist y, y_2, y_3 in G/K such that, for any $a \in A$, $xy(aK)$ is a δ-path, and such that yy_2y_3 is an equilateral triangle, y_2 and y_3 are at distance less than δ from AK and for any $a \in A$ there exists some point z on the side y_2y_3 of the triangle yy_2y_3 such that $yz(aK)$ is a δ-path.

Denote by W_0 the subgroup of G whose elements are

$$\mathrm{Id}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In the following lemma we shall again use the abbreviation for diagonal matrices:

$$D(a,b,c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Lemma 3.5. *For some $\delta > 0$ the following is true. For any $x \in G/K$ there exist $y \in G/K, t \in \mathbb{R}_+, m \in W_0A$ such that*

(i) *$xy(aK)$ is a δ-path for any $a \in A$,*

(ii) *$|d(y, mD(e^{s_1}, e^{s_2}, e^{s_3})K) - t - \max(|s_2 - s_3|, s_1 - s_2 - t, s_1 - s_3 - t, s_2 - s_1, s_3 - s_1)| \leq \delta$ for any $s_1, s_2, s_3 \in \mathbb{R}, s_1 + s_2 + s_3 = 0$,*

(iii) *there exists $h_2 \in G$ such that $d(h_2D(e^{s_1}, e^{s_2}, e^{s_3})K, mD(e^{s_1}, e^{s_2}, e^{s_3})K) \leq \delta$ if $s_1 + s_2 + s_3 = 0$ and $s_2 \geq s_3$ and $d(h_2 e^{-\frac{2t}{3}} D(e^t, 1, e^t)K, y) \leq \delta$,*

(iv) *there exists $h_3 \in G$ such that $d(h_3D(e^{s_1}, e^{s_2}, e^{s_3})K, mD(e^{s_1}, e^{s_2}, e^{s_3})K) \leq \delta$ if $s_1 + s_2 + s_3 = 0$ and $s_2 \leq s_3$ and $d(h_3 e^{-\frac{2t}{3}} D(e^t, e^t, 1)K, y) \leq \delta$.*

We notice that the second condition is in fact a consequence of the two last ones (for a different δ).

Proof. Up to left translation by some element $m \in W_0 A$ we may suppose that $x = \lambda \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} K$, with $\lambda \in \mathbb{R}_+^*$, $e_1, e_2, e_3 \in \mathbb{R}^3$, $\|e_1\| = \|e_2\| = \|e_3\| = 1$ and $\|e_2 \wedge e_3\| \leq \min(\|e_1 \wedge e_2\|, \|e_1 \wedge e_3\|)$. We have

$$\begin{aligned} \big|\|e_1 \wedge e_2\| - \|e_1 \wedge e_3\|\big| &\leq \min_{\mu = \pm 1} \|e_1 \wedge (e_3 - \mu e_2)\| \\ &\leq \min_{\mu=\pm 1} \|e_3 - \mu e_2\| \leq \sqrt{2}\|e_2 \wedge e_3\|. \end{aligned}$$

Denote by ν an element of $\{-1, 1\}$ where the minimum of $\|e_3 - \mu e_2\|$ is reached. Take $t = \log(\frac{\|e_1 \wedge e_2\|}{\|e_2 \wedge e_3\|}) \in \mathbb{R}_+$. By the last inequalities $|t - \log(\frac{\|e_1 \wedge e_3\|}{\|e_2 \wedge e_3\|})| \leq \log(1+\sqrt{2})$. For $a = D(e^{s_1}, e^{s_2}, e^{s_3})$, with $s_1 + s_2 + s_3 = 0$, we have

$$d(aK, x) = \log \left\| a^{-1} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \right\| + \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}^{-1} a \right\|$$

and

$$\left| \log \left\| a^{-1} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \right\| - \max(-s_1, -s_2, -s_3) \right| \leq \log 3$$

and since

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}^{-1} = \det(e_1, e_2, e_3)^{-1} \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix}^t$$

we have

$$\begin{aligned} \Bigg| \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}^{-1} a \right\| &+ \log \|e_1 \wedge e_2 \wedge e_3\| \\ - \max(s_1 + \log \|e_2 \wedge e_3\|, s_2 + \log \|e_1 \wedge e_3\|, &s_3 + \log \|e_1 \wedge e_2\|) \Bigg| \leq \log 3. \end{aligned}$$

Therefore

$$\begin{aligned} \big| d(aK, x) - \max(-s_1, -s_2, -s_3) &- \max(s_1 - t, s_2, s_3) \\ &+ \log \|e_1 \wedge e_2 \wedge e_3\| - \log \|e_1 \wedge e_2\| \big| \leq \delta_0 \end{aligned}$$

for some numerical constant δ_0.

Now consider

$$y = e^{\frac{t}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \nu & e^{-t} \end{pmatrix} K.$$

By applying the last argument to y instead of x, we obtain

$$\big| d(aK, y) - \max(-s_1, -s_2, -s_3) - \max(s_1 - t, s_2, s_3) - t \big| \leq \delta_1$$

for some numerical constant δ_1 (different from δ_0 because one has to normalize $(0, \nu, e^{-t})$). But

$$d(x,y) = \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e^t(e_3 - \nu e_2) \end{pmatrix} \right\| + \log \left\| \begin{pmatrix} e_1 \\ e_2 \\ e^t(e_3 - \nu e_2) \end{pmatrix}^{-1} \right\|$$

since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \nu & e^{-t} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\nu e^t & e^t \end{pmatrix}.$$

Therefore

$$\left| d(x,y) - \log \|e_1 \wedge e_2\| + t + \log \|e_1 \wedge e_2 \wedge e_3\| \right| \leq \delta_2$$

for some numerical constant δ_2 because $\|e^t(e_3 - \nu e_2)\| \leq \sqrt{2}\|e_1 \wedge e_2\|$ and $\|e_2 \wedge (e^t(e_3 - e_2))\| = \|e_1 \wedge e_2\|$ and $\|e_1 \wedge e^t(e_3 - \nu e_2)\| \leq \|e^t(e_3 - \nu e_2)\|$.

Thus the first two assertions are proved. Now we take

$$h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \nu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad h_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \nu & 1 \end{pmatrix}.$$

We check that

$$h_3 e^{-\frac{2t}{3}} D(e^t, e^t, 1)K = y \quad \text{and that} \quad d(h_2 e^{-\frac{2t}{3}} D(e^t, 1, e^t)K, y)$$

is bounded by a numerical constant, because

$$\begin{aligned} y^{-1} h_2 e^{-\frac{2t}{3}} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix} K &= K \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\nu e^t & e^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & \nu^{-1} \\ 0 & 0 & 1 \end{pmatrix} K \\ &= K \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & \nu^{-1} \\ 0 & -\nu & 0 \end{pmatrix} K. \end{aligned}$$

Finally it is obvious that

$$d\big(h_2 D(e^{s_1}, e^{s_2}, e^{s_3})K, \quad D(e^{s_1}, e^{s_2}, e^{s_3})K\big) \leq \delta$$

if $s_1 + s_2 + s_3 = 0$ and $s_2 \geq s_3$, and if δ is big enough. The last condition for h_3 is similar. ∎

Now consider $x_1, x_2, x_3 \in G/K$. Take $x = x_1$ and choose any flat containing x_2 and x_3. After a small discussion of the possible positions of x_2 and x_3 in this flat (a similar discussion occurs in [9]) we obtain the following lemma which immediately implies that property (K_δ) holds with $k = 2$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ as in the theorem 3.3 if δ is big enough.

Lemma 3.6. *For some $\delta > 0$ the following is true. For any $x_1, x_2, x_3 \in G/K$ there exist $t_1, t_2, t_3 \in G/K$ such that $x_1 t_1 t_2 x_2$, $x_2 t_2 t_3 x_3$ and $x_3 t_3 t_1 x_1$ are δ-paths and $t_1 t_2 t_3$ is an equilateral triangle.*

c) Now we prove that property $(K_\delta b)$ holds with $k = 2$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ as in the theorem if δ is big enough.

This results from the following lemma, applied to $\delta_0 = 2r$.

Lemma 3.7. *For any $\delta_0 > 0$ there exists $\delta > 0$ such that the following is true. For any $s, t \in \mathbb{R}$ of the same sign, and $x_1, y_1, x_2, z_2 \in G/K$ with $d(x_1, x_2) \leq \delta_0$, (x_1, y_1) of shape $(s, 0)$ and (x_2, z_2) of shape $(t, 0)$, then $x_1 y_1 z_2$ is a δ-thin triangle.*

Proof. By symmetry with respect to 1 in G/K we may suppose $s, t \in \mathbb{R}_+$. Up to left translation by an element of G we may suppose $x_1 = 1$ and $y_1 = e^{-\frac{s}{3}} D(e^s, 1, 1)K$. We have $z_2 = h e^{-\frac{t}{3}} D(e^t, 1, 1)K$ with $h \in G$ such that $d(1, hK) \leq \delta_0$. Write

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \qquad \text{and} \qquad h^{-1} = \begin{pmatrix} h'_{11} & h'_{12} & h'_{13} \\ h'_{21} & h'_{22} & h'_{23} \\ h'_{31} & h'_{32} & h'_{33} \end{pmatrix}.$$

By Cramer's formula there exists a constant δ_1 depending only on δ_0 such that $\big| \log(\max(|h_{21}|, |h_{31}|)) - \log(\max(|h'_{21}|, |h'_{31}|)) \big| \leq \delta_1$.
Take $r = \min(s, t, -\log(\max(|h_{21}|, |h_{31}|))$ and $u = e^{-\frac{r}{3}} D(e^r, 1, 1)K$. Then $x_1 u y_1$, $y_1 u z_2$ and $z_2 u x_1$ are δ-paths for some δ depending only on δ_0. Indeed $d(x_1, u) = r$, $d(u, y_1) = s - r$, and since

$$\begin{aligned} u^{-1} z_2 &= K e^{\frac{r-t}{3}} D(e^{-r}, 1, 1) h D(e^t, 1, 1) K \\ &= K e^{\frac{r-t}{3}} \begin{pmatrix} e^{t-r} h_{11} & e^{-r} h_{12} & e^{-r} h_{13} \\ e^t h_{21} & h_{22} & h_{23} \\ e^t h_{31} & h_{32} & h_{33} \end{pmatrix} K \end{aligned}$$

$$\begin{aligned} \text{and } z_2^{-1} u &= K e^{\frac{t-r}{3}} D(e^{-t}, 1, 1) h^{-1} D(e^r, 1, 1) K \\ &= K e^{\frac{t-r}{3}} \begin{pmatrix} e^{r-t} h'_{11} & e^{-t} h'_{12} & e^{-t} h'_{13} \\ e^r h'_{21} & h'_{22} & h'_{23} \\ e^r h'_{31} & h'_{32} & h'_{33} \end{pmatrix} K, \end{aligned}$$

we see that $|d(u, z_2) - (t - r)| \leq \delta_2$ and $|d(y_1, z_2) - (t - r) - (s - r)| \leq \delta_2$ by the same computation, where δ_2 depends only on δ_0. ■

The lemmas 3.5 and 3.7 are more intuitive if one considers quadratic forms on $\mathbb{R}^3$ instead of elements of $SL_3(\mathbb{R})$ but it is more difficult to write correct proofs in this way.

4. The case of $SL_3(\mathbb{C})$

Very few things need to be changed. We put $G = SL_3(\mathbb{C})$ and $K = SU_3(\mathbb{C})$ instead of $SL_3(\mathbb{R})$ and $SO_3(\mathbb{R})$ but A is the same. The part a) of the proof of theorem 3.3 is true for any linear connected semi-simple group. In the part b), W_0 remains the same and the lemma 3.5 is still true. In the proof of lemma 3.5 we have to take μ and ν in $\{z \in \mathbb{C}, |z| = 1\}$ instead of $\{-1, 1\}$. Lemma 3.7 and its proof remain the same.

It would be interesting to know wether the lemmas 3.5 and 3.7 are still true for the groups $SL(3, \mathbb{H})$ and $E_{6(-26)}$, whose associated symetric spaces have also flats of type A^2.

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Received September 25, 1998
and in final form May 1, 1999