Journal of Lie Theory
Volume **12** (2002) 69–79

Integral Structures on H-type Lie Algebras

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Communicated by M. Moskowitz

Abstract. In this paper we prove that every **H**-type Lie algebra possesses a basis with respect to which the structure constants are integers. Existence of such an integral basis implies via the Mal'cev criterion that all simply connected **H**-type Lie groups contain co-compact lattices. Since the Campbell-Hausdorff formula is very simple for two-step nilpotent Lie groups we can actually avoid invoking the Mal'cev criterion and exhibit our lattices in an explicit way. As an application, we calculate the isoperimetric dimensions of **H**-type groups.

1. Introduction

In this paper we prove that every **H**-type Lie algebra [7, 8, 9] possesses a basis with respect to which the structure constants are integers. We are going to call such a basis *an integral basis.* Existence of an integral basis implies via the Mal'cev criterion that all simply connected **H**-type Lie groups contain co-compact lattices. Since the Campbell-Hausdorff formula is very simple for two-step nilpotent Lie groups we can actually avoid invoking the Mal'cev criterion and exhibit our lattices in an explicit way.

The theory of **H**-type Lie algebras is related very closely to the theory of Clifford algebras and Clifford modules (cf. [4, 10]) and we are going to use the classification of Clifford modules in our construction.

We briefly recall the definition of **H**-type Lie algebras and establish notation and conventions for the sequel. Let $\mathcal{U}$ and $\mathcal{V}$ be two finite-dimensional inner product spaces over $\mathbb{R}$ of dimensions m and n respectively. Let $J : \mathcal{U} \longrightarrow \mathrm{End}(\mathcal{V})$, $z \mapsto J_z$, be a linear mapping satisfying

$$
\begin{aligned}
|J_z(v)| &= |z|\,|v| &(1)\\
J_z \circ J_z(v) &= -|z|^2\, v &(2)
\end{aligned}
$$

for all $z \in \mathcal{U}$ and $v \in \mathcal{V}$. Such a mapping J is called an orthogonal multiplication. Because of (2), by the universal property of Clifford algebras [10, Proposition 1.1, Chapter 1], J extends to an algebra homomorphism ϕ of the Clifford algebra

* The second author was supported in part by NSF Grant #DMS-9704743 and the PSC-CUNY Research Award Program

ISSN 0949–5932 / \$2.50

$C(\mathcal{U})$ into $\mathrm{End}(\mathcal{V})$ so that $\mathcal{V}$ acquires the structure of a module over $C(\mathcal{U})$. We will often write $\phi(\alpha)v = \alpha v$ and call αv the Clifford product of α and v.

After squaring, polarization of (1) yields

$$\begin{aligned}(J_{z_1}(v), J_{z_2}(v)) &= (z_1, z_2)\,|v|^2 && (3)\\ (J_z(v_1), J_z(v_2)) &= |z|^2(v_1, v_2) && (4)\end{aligned}$$

holding for all $z, z_1, z_2 \in \mathcal{U}$ and $v, v_1, v_2 \in \mathcal{V}$. In order not to complicate notation, we use the same notation for norms and inner products in $\mathcal{U}$ and $\mathcal{V}$. It follows immediately that

$$(J_z(v_1), v_2) = -(v_1, J_z(v_2)), \tag{5}$$

i.e. $J_z(v)$ is a skew-adjoint endomorphism of $\mathcal{V}$, $J_z^* = -J_z$. Now the Lie algebra structure is defined on $\mathcal{N} = \mathcal{U} \oplus \mathcal{V}$ by requiring that $\mathcal{U}$ be contained in the center and that the bracket of two elements of $\mathcal{V}$ belong to $\mathcal{U}$ and satisfy

$$(z, [v_1, v_2]) = (J_z(v_1), v_2).$$

In this way, $\mathcal{N}$ becomes a two-step nilpotent Lie algebra which is referred to as a Heisenberg-type or $\mathbf{H}$-type Lie algebra. Orthonormal bases $z_1, \ldots, z_m$ and $v_1, \ldots, v_n$ of $\mathcal{U}$ and $\mathcal{V}$ respectively give rise to the basis $z_1, \ldots, z_m, v_1 \ldots, v_n$ of $\mathcal{N}$ which is orthonormal. The only nonzero structure constants for $\mathcal{N}$ with respect to this basis occur among numbers $A_{i,j}^k$ defined by

$$[v_i, v_j] = A_{i,j}^k z_k. \tag{6}$$

We are now ready to state our main result.

Theorem 1.1. *For every $\mathbf{H}$-type Lie algebra $\mathcal{N} = \mathcal{U} \oplus \mathcal{V}$ as above there exist orthonormal bases $z_1, \ldots, z_m$ and $v_1, \ldots, v_n$ of $\mathcal{U}$ and $\mathcal{V}$ respectively so that the structure constants $A_{i,j}^k$ of the Lie algebra $\mathcal{N}$ with respect to the basis $z_1, \ldots, z_m, v_1 \ldots, v_n$ are integers and in fact take values $0, 1, -1$.*

The numbers $A_{i,j}^k$ are clearly equal to $(J_{z_k}(v_i), v_j) = (z_k v_i, v_j)$, i.e. depend only on the Clifford module structure and the inner product of $\mathcal{V}$. According to [10, Proposition 5.16], every Clifford module, i.e. a finite dimensional module over $C(\mathcal{U})$, admits an inner product such that the Clifford multiplication by elements of $\mathcal{U} \subset C(\mathcal{U})$ is an orthogonal multiplication. We remark that the Clifford multiplication is orthogonal if and only if elements of $\mathcal{U}$ act by skew-adjoint transformations, i.e. if and only if (5) holds.

Thus every Clifford module gives rise to an $\mathbf{H}$-type Lie algebra and Theorem 1.1 can be reformulated as a statement about Clifford modules as follows.

Theorem 1.2. *Given an inner product space $\mathcal{U}$ with an orthonormal basis $z_1, \ldots, z_m$ and a module V over the Clifford algebra $C(\mathcal{U})$ there exists an inner product $(\cdot,\cdot)$ on V and an orthonormal basis $v_1 \ldots, v_n$ of $\mathcal{V} = (V, (\cdot,\cdot))$ so that the Clifford multiplication $J_z(v) = zv$ by elements of $\mathcal{U}$ satisfies (1) and (2) and $(z_i v_p, v_q)$ is equal to $0, 1$ or -1 for all i, p, q.*

We will say that the choice of an inner product and a basis for V as above is an *integral structure* and that V with this additional structure is an *integral Clifford module.* Observe that the fact that the values of $(e_i v_p, v_q)$ are $0, 1, -1$ is equivalent to the assertion that each of the generators e_i acts on the basis $v_1, v_2, \ldots, v_n$ by a permutation and, possibly, some changes of sign.

¿From now on we will abandon the notational distinction between an inner product space $\mathcal{U}$ and the underlying vector space U.

2. Proof of Theorem 1.2

Since Clifford modules are completely reducible [10, p. 31], it suffices to prove Theorem 1.2 for irreducible Clifford modules. Suppose $\dim \mathcal{U} = k$ and let $e_1, \ldots, e_k$ be an orthonormal basis of $\mathcal{U}$. We are going to identify $\mathcal{U}$ with $\mathbb{R}^k$ and $C(\mathcal{U})$ with the algebra C_k generated over $\mathbb{R}$ by $e_1, \ldots, e_k$ subject to relations

$$e_i e_j = -e_j e_i, \qquad {e_i}^2 = -1. \tag{7}$$

According to the classification of irreducible Clifford modules, for every $k \neq 3 \pmod 4$ there exists only one isomorphism class of irreducible modules over C_k. If $k = 3 \pmod 4$ there are two such classes, but the dimensions as vector spaces over $\mathbb{R}$ of non-isomorphic modules are equal and the $\mathbf{H}$-type groups associated to them are isomorphic. We will denote an irreducible module over C_k by V_k.

Classification of irreducible Clifford modules proceeds by induction on k, cf. [10, p. 33], and we shall retrace this induction proving at every stage existence of an integral basis. It will be convenient to first classify $\mathbb{Z}_2$-graded Clifford modules. We briefly recall their definition. Denote by C_k^0 the subspace generated by products of even numbers of generators $e_1, \ldots, e_k$ and by C_k^1 the subspace generated by products of an odd number of generators. Then C_k^0 is a subalgebra, $C_k^0 \oplus C_k^1 = C_k$, and

$$C_k^i \cdot C_k^j \subset C_k^{i+j}$$

for $i, j \in \mathbb{Z}_2$. A finite dimensional space W over $\mathbb{R}$ is a $\mathbb{Z}_2$-graded Clifford module if $W = W^0 \oplus W^1$ and

$$C_k^i \cdot W^j \subset W^{i+j}$$

with $i, j \in \mathbb{Z}_2$.

We need to define an analog of integral structure on $\mathbb{Z}_2$-graded Clifford modules (we use definitions and notation of [4, Chapter 11, Sections 4,6] regarding $\mathbb{Z}_2$-graded tensor products of Clifford algebras and Clifford modules).

Definition 2.1. Suppose W^0 is a module over C_k^0. A choice of inner product $(\cdot, \cdot)$ on W^0 and an orthonormal basis $w_1, \ldots w_m$ with respect to this inner product is called *integral* if the basis elements are permuted with a possible change in sign by Clifford multiplication by all double products $e_i e_j$ and

$$(ze_k w, ze_k w) = |z|^2 |w|^2$$

for all $w \in W^0$ and $z \in \mathbb{R}^k$. We will call W^0 with the inner product and a basis satisfying the conditions above *integral.*

If W^0 is the 0-component of a $\mathbb{Z}_2$-graded Clifford module W then multiplication by e_k is an isomorphism of W^0 onto W^1. Therefore we can transfer the inner product on W^0 to W^1 and define the inner product on $W^0 \oplus W^1 = W$ by requiring the two summands to be orthogonal. A simple calculation then shows that the Clifford multiplication by elements of $\mathbb{R}^k$ is an orthogonal multiplication and the elements of the basis $w_1, \ldots, w_m, e_k w_1, \ldots e_k w_m$ are permuted by Clifford multiplication by every e_i with a possible change in sign. In particular, Theorem 1.2 holds for W. In the sequel we are only going to use inner products such that $W^0 \perp W^1$ and the multiplication by e_k maps W^0 onto W^1 isometrically.

Our proof of Theorem 1.2 will be carried out by showing that for every k there exists an irreducible $\mathbb{Z}_2$-graded Clifford module W_k with an integral structure. This will imply that *every* (ungraded) irreducible Clifford module is integral. It will follow *a posteriori* that all $\mathbb{Z}_2$-graded Clifford modules are integral. The main fact in the classification of Clifford modules is that if W_k and W_8 are irreducible $\mathbb{Z}_2$-graded modules over C_k and C_8 respectively, then $W_k \hat{\otimes} W_8$ is an irreducible $\mathbb{Z}_2$-graded module over $C_k \hat{\otimes} C_8 \simeq C_{k+8}$. Our proof of Theorem 1.2 will consist of exhibiting an integral irreducible $\mathbb{Z}_2$-graded Clifford module W_k for $k = 1, \ldots, 8$ and then showing that $W_k \hat{\otimes} W_8$ is integral if W_k carried an integral structure. To handle the low dimensional cases we need the following general lemma.

Lemma 2.2. *Suppose V is an integral module over the algebra C_k^0. Then $W = C_k \otimes_{C_k^0} V$ is a $\mathbb{Z}_2$-graded module over C_k with the $\mathbb{Z}_2$-grading given by $W^0 = 1 \otimes V$ and $W^1 = e_k \otimes V$. Thus W^0 has integral structure transfered from V via the isomorphism $v \mapsto 1 \otimes v$. In addition, if V is irreducible as a C_k^0-module then W is irreducible as a $\mathbb{Z}_2$-graded module over C_k.*

Proof. This is obvious (see [4, Chapter 11, Proposition 6.3]). ∎

Our next task is to describe integral structures on Clifford modules over C_k for $k \leq 8$. For $k \leq 7$, we refer to the description of (ungraded) Clifford modules given in [1].

Case $\mathbf{k} = 1$. $\mathbb{R} \simeq i\mathbb{R}$ is acting on $\mathbb{C}$ by complex multiplication. The real and imaginary parts are 0- and 1-components of $\mathbb{Z}_2$-grading respectively and the standard inner product together with 1 as the basis of the 0-component give the integral structure.

Case $\mathbf{k} = 2$. Consider the space of quaternions $\mathbb{H}$ equipped with the standard inner product. Identify $\mathbb{R}^2$ with the span of i, j and let it act on $\mathbb{H}$ by quaternion multiplication. This makes $\mathbb{H}$ into an irreducible module over C_2. In addition, the decomposition $\mathbb{H} = \operatorname{span}\{1, k\} \oplus \operatorname{span}\{i, j\}$ is a $\mathbb{Z}_2$-grading and an integral structure is given by the basis $1, k, i, j$.

Case $\mathbf{k} = 3$. In this case the space of quaternions becomes a module over the Clifford algebra C_3 represented in the algebra of endomorphisms of $\mathbb{H}$ as the subalgebra generated by quaternion multiplications by i, j, and k. Clearly, the standard basis $1, i, j, k$ is permuted with possibly a change in sign by these endomorphisms so that $\mathbb{H}$ as a module over C_3 carries an integral structure. However, this module does not have a natural $\mathbb{Z}_2$-graded structure. We regard $\mathbb{H}$ as a module over $C_3^0 \subset C_3$ and apply Lemma 2.2 to create an integral $\mathbb{Z}_2$-graded irreducible C_3-module structure on $C_3 \otimes_{C_3^0} \mathbb{H}$.

To cover the remaining low-dimensional cases we will use the algebra of octonions $\mathbb{O}$ with its standard generators $1, i_1, \dots i_7$ and multiplication table given in [5, Page 448] with $c = -1$.

Case $\mathbf{k} = 4$. This is analogous to the case $k = 2$. $\mathbb{R}^4$ is identified with $\operatorname{span}\{i_1, i_2, i_3, i_4\}$ which acts on $\mathbb{O}$ by octonion multiplication. The resulting Clifford module is $\mathbb{Z}_2$-graded with the 0-component equal to $\operatorname{span}\{1, i_5, i_6, i_7\}$ and the 1-component $\operatorname{span}\{i_1, i_2, i_3, i_4\}$. These bases are orthonormal with respect to the standard inner product and give rise to an integral structure. This follows from inspection of the multiplication table.

Case $\mathbf{k} = 5, 6, 7$. We treat these three cases simultaneously. The Clifford algebra C_k can be represented as an algebra of endomorphisms of $\mathbb{O}$ generated by transformations of octonion multiplications by i_j, $1 \le j \le k$. The resulting Clifford module V_k is irreducible but not $\mathbb{Z}_2$-graded. As above, we regard it as a module over C_k^0 which allows us to create an irreducible, $\mathbb{Z}_2$-graded, integral Clifford module $W_k = C_k \otimes_{C_k^0} \mathbb{O}$ using Lemma 2.2.

Case $\mathbf{k} = 8$. This uses the isomorphism, for every $k \ge 2$, $\phi : C_{k-1} \longrightarrow C_k^0$ defined as follows [4, Chapter 11, Section 6]. Let $x = x_0 + x_1$ be the decomposition of $x \in C_k$ into its 0 and 1 components. Then $\phi(x) = x_0 + e_k x_1$ where, given standard generators $e_1, \dots, e_k$ of C_k, we regard C_{k-1} as the subalgebra generated by $e_1, \dots, e_{k-1}$. The octonions are a module over C_7 as above (we relabel i_j as e_j) which allows us to regard $\mathbb{O}$ as a module over C_8^0 by defining the multiplication xo as $\phi^{-1}(x)o$. Since $\phi^{-1}(e_j) = e_j$ and $\phi^{-1}(e_8 e_j) = e_j$ for $j \le 7$ and $\mathbb{O}$ was integral as a module over C_7, the resulting module over C_8^0 is integral. Applying Lemma 2.2 we obtain an integral, irreducible, $\mathbb{Z}_2$-graded module $W_8 = C_8 \otimes_{C_8^0} \mathbb{O}$. We remark that W_8 is irreducible as an ungraded Clifford module as well. This is because W_8^0 is irreducible as a module over C_8^0 (since the pair (W_8^0, C_8^0) is isomorphic to (V_7, C_7) and V_7 is irreducible as a module over C_7).

We are now ready for the inductive step in the argument. Suppose W_k and W_l are integral, $\mathbb{Z}_2$-graded Clifford modules over C_k and C_l respectively. Let $e_1, \dots, e_k$ and $f_1, \dots, f_l$ be the standard generators of C_k and C_l. Suppose W_k^0 and W_l^0 are equipped with inner products and integral bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ respectively. Recall that as a vector space $W_k \hat{\otimes} W_l$ is isomorphic to $W_k \otimes W_l$ so that we can equip $W_k \hat{\otimes} W_l$ with an inner product by requiring that

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2) \cdot (y_1, y_2).$$

In particular, different components $W_k^i \otimes W_l^j$ for $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ are orthogonal. Under the isomorphism $C_k \hat{\otimes} C_l \simeq C_{k+l}$, cf. [4, Corollary 4.8] (note that our notation differs from the notation in [4] since we use the symbol $\hat{\otimes}$ instead of $\otimes$ to denote the tensor product of $\mathbb{Z}_2$-graded algebras), the standard generators of C_{k+l} correspond to $e_1 \otimes 1, \ldots, e_k \otimes 1, 1 \otimes f_1, \ldots 1 \otimes f_l$. To exhibit an integral structure we need a basis of $(W_k \hat{\otimes} W_l)^0 = (W_k^0 \otimes W_l^0) \oplus (W_k^1 \otimes W_l^1)$ in addition. The basis that we are going to use is

$$\{v_i \otimes w_j, e_k v_i \otimes f_l w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}. \tag{8}$$

Lemma 2.3. *The bases described above give rise to an integral structure on* $(W_k \hat{\otimes} W_l)^0$ *and, consequently, on* $W_k \hat{\otimes} W_l$.

Proof. We first verify that the Clifford multiplication by elements of the space U spanned by the generators of $C_k \hat{\otimes} C_l$ is orthogonal. Recall that it suffices to verify that each element $z \in U$ acts on $W_k \hat{\otimes} W_l$ as a skew-symmetric endomorphism. We abuse the notation and write z for an endomorphism associated with z. Clearly,

$$\begin{aligned} (e_i \otimes 1)^* &= e_i^* \otimes 1 = -e_i \otimes 1 \\ (1 \otimes f_j)^* &= 1 \otimes f_j^* = -1 \otimes f_j \end{aligned}$$

since Clifford multiplications on W_k and W_l are orthogonal. Since every element of U is a linear combination of such products, our assertion follows.

To verify properties of the integral structure, it suffices to show that elements of the basis (8) are permuted up to sign by the double products of generators. There are three cases to consider $(e_i \otimes 1) \cdot (e_j \otimes 1) = e_i e_j \otimes 1$, $(1 \otimes f_i) \cdot (1 \otimes f_j) = 1 \otimes f_i f_j$, and $(e_i \otimes 1) \cdot (1 \otimes f_j) = (e_i \otimes f_j)$. In the first two cases the action is as desired since the multiplication by e_p's permutes $v_1, \ldots, v_m$ up to sign and the multiplication by f_q's acts the same way on $w_1, \ldots, w_n$. To treat the third case, note that $e_p = \pm e_k e_p e_k$ and $f_q = \pm f_l f_q f_l$. Thus, up to signs,

$$(e_p \otimes f_q) \cdot (v_i \otimes w_j) = e_k e_p e_k v_i \otimes f_l f_q f_l w_j = e_k v_{i'} \otimes f_l w_{j'}$$

since double products of generators of C_k and C_l permute up to sign the distinguished bases of W_k^0 and W_l^0 respectively. Similarly,

$$(e_p \otimes f_q) \cdot (e_k v_i \otimes f_l w_j) = e_p e_k v_i \otimes f_q f_l w_j = e_k v_{i'} \otimes f_l w_{j'}.$$

This proves the lemma. ■

Corollary 2.4. *Suppose that* W_k *is an irreducible, integral,* $\mathbb{Z}_2$*-graded module over* C_k. *Then* $W_k \hat{\otimes} W_8$ *is an irreducible, integral,* $\mathbb{Z}_2$*-graded module over* $C_k \hat{\otimes} C_8 \simeq C_{k+8}$ *with the integral structure described above.*

Proof. All assertions except irreducibility are contained in the lemma above. By [4, Chapter 11, 6.5] $\dim_{\mathbb{R}} W_k \hat{\otimes} W_8 = \dim_{\mathbb{R}} W_k \cdot \dim_{\mathbb{R}} W_8 = 16 \dim_{\mathbb{R}} W_k$ is equal to the dimension of an irreducible, $\mathbb{Z}_2$-graded Clifford module over C_{k+8}. It follows immediately that $W_k \hat{\otimes} W_8$ is irreducible. ■

k	a_k	b_k
1	2	2
2	4	4
3	4	8
4	8	8
5	8	16
6	8	16
7	8	16
8	16	16

Table 1: Dimensions of irreducible Clifford modules

An easy induction using the explicit description of irreducible, $\mathbb{Z}_2$-graded Clifford modules over $C_k, 1 \le k \le 8$ and the lemma above yields existence of an irreducible, $\mathbb{Z}_2$-graded Clifford module W_k over C_k for every k.

Some additional work is required to show that every (ungraded) Clifford module has an integral structure. Let a_k be the dimension of an irreducible Clifford module over C_k, and let b_k denote the dimension of an irreducible, $\mathbb{Z}_2$-graded module over C_k. It follows from [4, Propositions 6.2 and 6.3, Chapter 11] that $a_k = b_{k+1}/2$. From basic periodicity [4, 6.5, Chapter 11], $a_{k+8} = 16a_k$ and $b_{k+8} = 16b_k$. All values of a_k and b_k can now be computed from Table 1, which summarizes some of the information about Clifford modules of low dimensions.

Recall now (cf. [4, Chapter 6]) that if $k = 1, 2, 4, 8 \pmod 8$ then the dimensions over $\mathbb{R}$ of an irreducible Clifford module and an irreducible $\mathbb{Z}_2$-graded Clifford module are equal. Thus $\mathbb{Z}_2$-graded Clifford modules W_k constructed above are irreducible as (ungraded) Clifford modules. The classification also says that for these values of k there exists exactly one isomorphism class of Clifford modules over C_k which proves Theorem 1.2 for $k = 1, 2, 4, 8 \pmod 8$.

Next consider the case $k = 3 \pmod 4$. According to the classification there are two non-isomorphic Clifford modules over C_k. We are going to show that their direct sum is isomorphic as an ungraded Clifford module to W_k and that the integral structure of W_k induces integral structures on the two modules. The key role in the proof of this fact is played by the "volume element" $\omega = e_1 e_2 \ldots e_k$. ω belongs to the center of C_k and satisfies $\omega^2 = 1$, cf. [10, Proposition 3.3, Chapter 1]. In addition, the multiplication by ω is a symmetric operator on every orthogonal Clifford module. This can be seen as follows. If $k = 4l + 3$, then

$$\omega^* = e_k^* e_{k-1}^* \ldots e_1^* = (-1)^{4l+3} e_k e_{k-1} \ldots e_1 = -(-1)^{\frac{(4l+4)(4l+3)}{2}} e_1 e_2 \ldots e_k = \omega,$$

since multiplications by e_p's are skew-symmetric and e_p's anti-commute. Now define ϕ_+ and ϕ_- to be multiplications by central elements $(1+\omega)/2$ and $(1-\omega)/2$ respectively. $\phi_\pm$ are self-adjoint and satisfy $\phi_\pm^2 = \phi_\pm$, i.e. they are orthogonal projections onto their images. Since $(1+\omega)(1-\omega) = 0$ and $\phi_+ + \phi_- = 1$ the ranges of these projections are perpendicular. Since $1 \pm \omega$ are central, they are in fact (ungraded) Clifford submodules V_+ and V_- of W_k. Since $\omega W_k^0 = W_k^1$, $\phi_\pm$ is injective on W_k^0. A count of dimensions shows that $W_k = \phi_+ W_k^0 \oplus \phi_- W_k^0$ is an orthogonal direct decomposition of W_k into two Clifford submodules. ω acts on $V_\pm$ by multiplication by ± 1 so that the two modules are *non-isomorphic*. To conclude our analysis we exhibit integral bases of the two modules. We do the argument for V_+ since the argument for V_- is identical after $1+\omega$ is replaced by $1-\omega$. Let $w_1, \dots, w_m$ be an integral basis of W_0, i.e. an orthonormal basis which is permuted up to sign by multiplications by double products e_ie_j. Let

$$v_p = \frac{1}{\sqrt{2}}(1+\omega)w_p \in V_+$$

for $p = 1, \dots, m$. Since $\omega W_k^0 = W_k^1$ is perpendicular to W_k^0, the Pythagorean theorem insures that this is an orthonormal basis. For a generator e_i of C_k we calculate

$$(e_iv_p, v_q) = (e_iw_p + e_i\omega w_p, w_q + \omega w_q)/2.$$

Observe that $e_i\omega w_p$ and w_q belong to W_k^0 while e_iw_p and ωw_q are in W_k^1. Therefore the inner product above simplifies to

$$((e_iw_p, \omega w_q) + (e_i\omega w_p, w_q))/2 = (e_iw_p, \omega w_q),$$

since ω is central and self-adjoint. Now ωw_q can be expressed as a product of e_i times a product of an even number of generators of C_k. It therefore follows from the definition of an integral basis that $e_i\omega w_q$ is up to sign equal to $e_iw_{q'}$ so that, finally

$$(e_iv_p, e_iv_{q'}) = (v_p, v_{q'}) = \delta_{p,q'}$$

which proves that the basis $v_1, \dots, v_m$ is integral.

The remaining two cases of Theorem 1.2 are $k = 5, 6 \pmod 8$ say $k = 8l + r$, with r equal to 5 or 6. By the classification, in either case there is only one isomorphism class of irreducible modules over C_k and the dimension over $\mathbb{R}$ of an irreducible module in this class is equal to the dimension of the Clifford module V_+ over C_{4l+7} constructed above. Clearly, C_k can be regarded as the subalgebra of C_{4l+7} generated by the first k generators so that $V_+ = V_k$ becomes a module over C_k. It is irreducible since its dimension is that of an irreducible module and, trivially, integral. Theorem 1.2 is proved.

We conclude this section with a very simple proof, available now, of the fact that two non-isomorphic Clifford modules over C_k, $k = 3 \pmod 4$ give rise to isomorphic **H**-type algebras. This is very well known, cf. [11], but we give the proof for completeness. The structure constants of the **H**-type Lie algebra $U \oplus V_+$ associated to the Clifford module V_+ have been computed above and are equal to $(e_iv_p, v_q) = (e_iw_p, \omega w_q)$ (here $U = \operatorname{span}\{e_1, \dots, e_k\}$). In the analogous calculation for V_- ω is replaced by $-\omega$ so that the corresponding structure constants for the Lie algebra $U \oplus V_-$ (with respect to the basis $e_1, \dots, e_k, v_1', \dots, v_m'$ where $v_p' = (1/\sqrt{2})(1-\omega)w_p$ are negatives of the structure constants for $U \oplus V_+$ if we choose the obvious correspondence between bases of V_+ and V_-. However, if we deploy the set $-e_1, \dots, -e_k, v_1', \dots, v_1', \dots, v_m'$ as the basis for $U \oplus V_-$ instead, the structure constants for the two Lie algebras are equal so that the algebras are isomorphic.

3. Examples and applications

In this section we show that every simply connected Lie group of Heisenberg type contains a co-compact lattice. We are also going to calculate the isoperimetric dimension of such groups.

We choose and fix an integral basis as in Theorem 1.1. Since **H**-type Lie algebras are two-step nilpotent (in particular nilpotent) the exponential mapping is a global diffeomorphism of the Lie algebra and the group. We will therefore use it to identify the two. In addition the Campbell-Hausdorff formula that expresses the group multiplication in terms of the Lie bracket takes a particularly simple form

$$X \cdot Y = X + Y + \tfrac{1}{2}[X, Y] \tag{9}$$

where $X, Y \in N = U \oplus V$ as in the Introduction. Let $U_{\mathbb{Z}}$ be the abelian subgroup of U consisting of linear combination with integer coefficients of the generators $e_1, \ldots, e_k$ of $C(U) \simeq C_k$. Let $V_{\mathbb{Z}}$ be the lattice in V generated by elements of an integral basis. Since $[V_{\mathbb{Z}}, V_{\mathbb{Z}}] \subset U_{\mathbb{Z}}$ our candidate for the lattice L in the group $(N, \cdot)$ is the subset

$$L = \tfrac{1}{2}U_{\mathbb{Z}} \oplus V_{\mathbb{Z}}.$$

Verification of this claim is straightforward after first checking that $-X$ is the inverse of X with respect to the group operation. Thus L is a subgroup of N and it is obviously discreet. The product of an arbitrary element $u + v$ of N with $w \in V$ is equal to $u + (1/2)[v, w] + v + w$ and has $u + (1/2)[v, w]$ as its U component and $v + w$ as the component in V. Given v we can choose $w \in V_{\mathbb{Z}}$ so that $v + w$ has all coefficients in the interval $[0, 1]$ when expanded with respect to the distinguished basis of V. We can then act on $u + (1/2)[v, w]$ by elements u_1 of $(1/2)U_{\mathbb{Z}}$. Since this action is by ordinary translation ($U_{\mathbb{Z}}$ is contained in the center of N) we can easily achieve that $u + v + (1/2)[v, w] + u_1$ has its coefficients with respect to the basis $e_1, \ldots, e_k$ in the interval $[0, 1/2]$. Hence all coefficients in the expansion of

$$(u + v) \cdot (u_1 + w) = u + u_1 + \frac{1}{2}[v, w] + v + w$$

with respect to the integral basis chosen lie in $[0, 1]$, which proves that L is co-compact.

To calculate the isoperimetric dimension of N we will need to calculate the ranks of groups in the lower central series of L. Using our explicit description of the product in N one verifies very easily that the bracket operation of the Lie algebra *coincides* with the commutator operation in the group. We therefore have $[L, L] = [V_{\mathbb{Z}}, V_{\mathbb{Z}}] = U_{\mathbb{Z}}$. The second equality requires a proof. Observe first that

$$|(z, [v, w])| = |(zv, w)| \leq |z| \cdot |v| \cdot |w|$$

if $z \in U,\ v, w \in V$. It follows that

$$|[v, w]| \leq |v| \cdot |w|.$$

Now let e_i be one of the generators of C_k. Then e_i induces a permutation of the basis vectors of V with a possible change in sign. Thus given v_p, $e_k v_p = \pm v_q$ for some q. It follows that $|(e_i, [v_p, v_q])| = |(e_1 v_p, v_q)| = 1$ so that $e_i = \pm[v_p, v_q]$. Thus each of the generators e_i is a commutator which proves the claim. We see therefore that

$$L/[L, L] \simeq (\tfrac{1}{2}U_{\mathbb{Z}}/U_{\mathbb{Z}}) \oplus V_{\mathbb{Z}} \simeq \mathbb{Z}_2 \oplus V_{\mathbb{Z}}.$$

We remark that there is no ambiguity in the interpretation of the quotient above since $[L,L] \subset U$ and the group multiplication by $u \in U$ amounts to the the translation by u because of (9).

A theorem of Bass [2] gives the estimate of growth of L in the word metric with respect to a finite set of generators. In our case we can take for example the generating set S given by the vectors of an integral basis and their negatives. The lower central series of L reduces to $L_0 = L \supset L_1 = [L,L] \supset L_2 = 0$. The ranks of successive quotients are therefore equal to $\dim V$ and $\dim U$. If $d = \dim V + 2\dim U$ then by Bass' theorem L has polynomial growth of degree d, i.e. the number $g(R)$ of distinct elements of L in the metric ball of radius R in L satisfies

$$g(R) \geq cR^d.$$

where the constant c depends on the choice of the generating set. A result of Coulhon and Saloff-Coste [3] asserts that the Cayley graph of a group of polynomial growth of degree d satisfies a d-dimensional isoperimetric inequality. Since every Lie group has bounded geometry we can invoke Kanai's theorem [6] to conclude that the **H**-type group under consideration satisfies an isoperimetric inequality of the same kind. More precisely we have the following.

Theorem 3.1. *Suppose $N = U \oplus V$ is an **H**-type group equipped with a left-invariant metric. There exists a positive constant c such that for every relatively compact subset F of N with smooth boundary ∂F we have*

$$\frac{A(\partial F)}{V(F)^{1-1/d}} \geq c,$$

where $A(\partial F)$ and $v(F)$ denote the $n-1$-dimensional volume of the boundary of F and $V(F)$ stands for the volume of F.

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Received September 12, 2000
and in final form February 7, 2001