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Extended Affine Root Systems

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Abstract. There are two notions of the extended affine root systems in the literature which both are introduced axiomatically. One, extended affine root system (SAERS for short), consists only of nonisotropic roots, while the other, extended affine root system (EARS for short), contains certain isotropic roots too. We show that there is a one to one correspondence between (reduced) SEARSs and EARSs. Namely the set of nonisotropic roots of any EARS is a (reduced) SEARS, and conversely, there is a unique way of adding certain isotropic elements to a SEARS to get an EARS. (It is known that some of extended affine root systems are not the root system of any extended affine Lie algebra.)

Introduction

In 1985 K. Saito [8] introduced the notion of an extended affine root system. An *extended affine root system* is a subset R of a finite dimensional real vector space $\mathcal{V}$, equipped with a positive semidefinite bilinear form, satisfying some axioms which are a very natural generalization of axioms for a finite and affine irreducible root system. Namely, R is discrete and spans $\mathcal{V}$, elements of R are nonisotropic, the reflections based on roots stabilize R, $2(\alpha,\beta)/(\alpha,\alpha)$ is an integer for any $\alpha,\beta\in R$, and finally R is irreducible. Finite and affine irreducible root systems are examples of extended affine root systems. In fact when the *nullity* of the root system (which is by definition the dimension of the radical of the form) is 1, these axioms imply Macdonald's axioms [7] for an affine root system. Saito achieved a complete classification of the root system when the nullity is 2 and the quotient of the root system modulo a certain 1-dimensional space, called a *marking*, is reduced (see Remark 1.2). Saito's motivation for defining such root systems were possible applications to the theory of singularities.

In 1990 R. Høegh-Krohn and B. Torresani [6] introduced a system of axioms for a new class of Lie algebras over the field of complex numbers. They call these algebras, quasi-simple Lie algebras. As part of their work, they extract a root system for the algebras and try to analyze the structure of such a root system by

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assigning a finite root system to it, and then describing the set of isotropic roots through its connection with the finite root system. Unfortunately, in their analysis of isotropic roots there was some inaccuracies. Namely, they assumed that the set of isotropic roots is always a lattice. This could have been avoided if they were aware of Saito's work.

In 1997 B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola in [1], provide a firm foundation for the class of quasi-simple Lie algebras and their root systems. With slight modification in axioms, they call these algebras *extended affine Lie algebras.* They show that the root system of an extended affine Lie algebra satisfies certain axioms which are a natural generalization of axioms for finite irreducible root systems. Then independently of Lie algebras, they introduce the axiomatic version of such root systems and call them *extended affine root systems.* Therefore, there are two notions of extended affine root systems in the literature, one from Saito and one from [1]. Unlike Saito's root systems, the root systems given in [1] contain isotropic roots. To distinguish these two class of roots in our work, we refer to a root system from [1] as an extended affine root system (EARS for short), and refer to a root system from Saito as Saito's extended affine root system (SEARS for short). In [1], the authors give a complete description of EARSs using the concept of a semilattice (instead of a lattice in [6]'s work).

The purpose of this work is to clarify the relation between SEARSs and EARSs. We show that there exists a one to one correspondence between the classes of EARSs and (reduced) SEARSs. Namely, the set of nonisotropic roots of any EARS is a reduced SEARS, and conversely given a reduced SEARS R, there is a (unique) way to extend R to an EARS, by adding some isotropic roots to R. The paper also contains some remarks regarding the notion of a marking showing that it reduces significantly the number of roots. (It is known that some of the extended affine root systems are not the root system of any extended affine Lie algebra, see [2].)

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Extended affine root systems

We start by a brief introduction of the two notions of extended affine root systems from [8] and [1]. We first recall Saito's definition for an extended affine root system.

Definition 1. (Saito's Definition [8, §1]) Suppose $\mathcal{V}$ is a finite dimensional real vector space equipped with a symmetric bilinear form $(\cdot,\cdot)$. A nonempty subset R of $\mathcal{V}$ is called a *root system belonging to* $(\cdot,\cdot)$, if it satisfies the following axioms:

- (**SR1**) The additive subgroup of $\mathcal{V}$ generated by R is a lattice in $\mathcal{V}$. (That is, the additive subgroup of $\mathcal{V}$ generated by R is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis of $\mathcal{V}$.)
- (**SR2**) $(\alpha,\alpha)\neq 0$ for all $\alpha\in R$.
- (**SR3**) $\beta-(\beta,\frac{2\alpha}{(\alpha,\alpha)})\alpha\in R$ for all $\alpha,\beta\in R$.

- (**SR4**) $(\beta, \frac{2\alpha}{(\alpha,\alpha)}) \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

- (**SR5**) Irreducibility: R cannot be decomposed as a disjoint union $R_1 \uplus R_2$ where R_1 and R_2 are nonempty subsets of R satisfying $(R_1, R_2) = \{0\}$.

R is called *reduced* if whenever $\alpha, c\alpha \in R$ for some real number c, then $c \in \{\pm 1\}$. A root system belonging to $(\cdot,\cdot)$ is called an *extended affine root system* (SEARS for short) if $(\cdot,\cdot)$ is positive semidefinite.

For $\alpha \in \mathcal{V}$ with $(\alpha,\alpha) \neq 0$, define $w_\alpha \in \mathrm{GL}(\mathcal{V})$ by

$$w_\alpha(\lambda) = \lambda - (\lambda, \frac{2\alpha}{(\alpha,\alpha)})\alpha, \quad (\lambda \in \mathcal{V}). \tag{2}$$

Then (SR3) states that $w_\alpha(R) = R$ for all $\alpha \in R$.

Next, we recall the definition of an extended affine root system from [1].

Definition 3. [1, II.2.1] Assume that $\mathcal{V}$ is a finite dimensional real vector space with a positive semidefinite bilinear form $(\cdot,\cdot)$ and that R is a subset of $\mathcal{V}$. Let

$$R^\times = \{\alpha \in R \mid (\alpha,\alpha) \neq 0\} \quad \text{and} \quad R^0 = \{\alpha \in R \mid (\alpha,\alpha) = 0\}.$$

R is called an *extended affine root system* (EARS for short) in $\mathcal{V}$ if R satisfies the following 8 axioms:

- (**R1**) $0 \in R$.

- (**R2**) $-R = R$.

- (**R3**) R spans $\mathcal{V}$.

- (**R4**) $\alpha \in R^\times \Longrightarrow 2\alpha \notin R$.

- (**R5**) R is discrete in $\mathcal{V}$.

- (**R6**) If $\alpha \in R^\times$ and $\beta \in R$, then there exist nonnegative integers d and u such that
$$\{\beta + n\alpha \mid n \in \mathbb{Z}\} \cap R = \{-\beta - d\alpha, \ldots, \beta + u\alpha\}$$
and $d - u = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$.

- (**R7**) $R^\times$ cannot be decomposed as a disjoint union $R_1 \uplus R_2$ where R_1 and R_2 are nonempty subsets of $R^\times$ satisfying $(R_1, R_2) = \{0\}$.

- (**R8**) For any $\sigma \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

Elements of $R^\times (R^0)$ are called nonisotropic (isotropic) roots.

Lemma 4. *Suppose R is an EARS in $\mathcal{V}$. Then $R^\times$, the set of nonisotropic roots of R, is a reduced SEARS in $\mathcal{V}$.*

Proof. We must show that (SR1)-(SR5) hold for $R^\times$ and that $R^\times$ is reduced. Clearly $(\cdot,\cdot)$ is positive semidefinite on $\mathcal{V}$ and (SR2) holds. By [1, Chapter II], $R \subseteq \dot{R} + \Lambda \subseteq \mathcal{V}$ where $\dot{R}$ is a finite root system and Λ is a lattice in the radical of $(\cdot,\cdot)$. Thus $\langle R\rangle$ (the additive subgroup of $\mathcal{V}$ generated by R) is discrete. By (R8), $\langle R^\times\rangle = \langle R\rangle$ and so by (R3), $\langle R^\times\rangle$ is a lattice in $\mathcal{V}$. Hence (SR1) holds. (SR4) and (SR3) hold by (R6). (SR5) holds by (R7). Thus $R^\times$ is a SEARS. Moreover, if $\alpha, c\alpha \in R^\times$ for some real number c, then by (SR4), $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$ and so by (R4), $c \in \{\pm 1\}$. Thus $R^\times$ is reduced. ■

Next we want to show that corresponding to any reduced SEARS R, there exists a (unique) EARS $\tilde{R}$ which has R as the set of its nonisotropic roots. We proceed as follows.

Let R be a SEARS in a finite dimensional real vector space $\mathcal{V}$. Let $\mathcal{V}^0$ be the radical of $(\cdot,\cdot)$ on $\mathcal{V}$. Since $(\cdot,\cdot)$ is positive semidefinite, we have

$$\mathcal{V}^0 = \{\alpha \in \mathcal{V} \mid (\alpha,\alpha) = 0\}.$$

Let $^- : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0$ be the canonical map. By (SR2), $\mathcal{V} \neq \mathcal{V}^0$ and so $\bar{\mathcal{V}} \neq \{0\}$.

Define a unique symmetric bilinear form on $\bar{\mathcal{V}}$ by

$$(\bar{\alpha}, \bar{\beta}) = (\alpha, \beta) \quad \text{for } \alpha, \beta \in \mathcal{V}.$$

Then $(\cdot,\cdot)$ is positive definite on $\bar{\mathcal{V}}$. Put

$$\bar{R} = \{\bar{\alpha} \mid \alpha \in R\}.$$

For $\alpha \in \mathcal{V} \setminus \mathcal{V}^0$, we set $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Let $w_\alpha \in \mathrm{GL}(\mathcal{V})$ be defined as in (2). Define the isometry $w_{\bar{\alpha}}$ on $\bar{\mathcal{V}}$ by

$$w_{\bar{\alpha}}(\bar{\lambda}) = \bar{\lambda} - (\bar{\lambda}, \bar{\alpha}^\vee)\bar{\alpha} \qquad (\lambda \in \mathcal{V}).$$

Clearly

$$w_{\bar{\alpha}}(\bar{\lambda}) = \overline{w_\alpha(\lambda)}. \tag{5}$$

Let $\mathcal{W}$ be the subgroup of $\mathrm{GL}(\mathcal{V})$ generated by $w_\alpha, \alpha \in R$, and $\bar{\mathcal{W}}$ the subgroup of $\mathrm{GL}(\bar{\mathcal{V}})$ generated by $w_{\bar{\alpha}}, \alpha \in R$. By (SR3), $\mathcal{W}$ stabilizes R. Now $\bar{\mathcal{V}}$ is a finite dimensional real vector space with a positive definite bilinear form, and by (SR4), $(\bar{\alpha}, \bar{\beta}^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$. Thus $\bar{R}$ is a finite set (see [1, II. 2.9]). Now we have,

(a) $\bar{R}$ is a finite set in $\bar{\mathcal{V}}$ which spans $\bar{\mathcal{V}}$ (since R spans $\mathcal{V}$).

(b) $w_{\bar{\alpha}}(\bar{R}) \subseteq \bar{R}$ for any $\bar{\alpha} \in \bar{R}$ (by (SR3) and (5)).

(c) $(\bar{\alpha}, \bar{\beta}) \in \mathbb{Z}$ for all $\bar{\alpha}, \bar{\beta} \in \bar{R}$ (by (SR4)).

(d) $\bar{R}$ cannot be decomposed as a disjoint union $\bar{R}_1 \uplus \bar{R}_2$ where $\bar{R}_1$ and $\bar{R}_2$ are nonempty subsets of $\bar{R}$ satisfying $(\bar{R}_1, \bar{R}_2) = \{0\}$ (by (SR5)).

Thus $\bar{R}$ is an irreducible finite root system in $\bar{\mathcal{V}}$.

Now fix a choice of a fundamental system $\bar{\Pi} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_\ell\}$ for $\bar{R}$. Fix a preimage $\dot{\alpha}_i$ in R of $\bar{\alpha}_i$ under $^-$. Let

$$\dot{\mathcal{V}} = \mathrm{span}_{\mathbb{R}}\{\dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\alpha}_\ell\}.$$

Then $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$ and $^-$ restricted to $\dot{\mathcal{V}}$ is an isometry of $\dot{\mathcal{V}}$ onto $\bar{\mathcal{V}}$.

Let $\dot{R}$ be the image of R under the projection $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0 \to \dot{\mathcal{V}}$. Then $^-$ maps $\dot{R}$ bijectively onto $\bar{R}$. Hence $\dot{R}$ is a finite root system in $\dot{\mathcal{V}}$ which is isomorphic to $\bar{R}$ under $^-$. Note that

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \dot{\alpha} + \sigma \in R \quad \text{for some } \sigma \in \mathcal{V}^0\}. \tag{6}$$

We can write $\dot{R} = \dot{R}_{sh} \uplus \dot{R}_{\ell g} \uplus \dot{R}_{ex}$ where $\dot{R}_{sh}, \dot{R}_{lg}$ and $\dot{R}_{ex}$ denote the sets of short, long and extra long roots of $\dot{R}$, respectively. We make the convention that if $\dot{R}$ is simply laced, then every root is a short root. (Note that $\dot{R}_{lg}$ and $\dot{R}_{ex}$ might be empty).

Definition 7. *The nullity of R is defined to be the dimension ν of $\mathcal{V}^0$. The rank and type of R is defined to be the rank and type of the finite root system $\bar{R}$ (which is the same as the rank and type of $\dot{R}$).*

Now set

$$S = \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R \text{ for some } \dot{\alpha} \in \dot{R}_{sh}\} \tag{8}$$

$$L = \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R \text{ for some } \dot{\alpha} \in \dot{R}_{lg}\} \quad (\text{ if } \dot{R}_{lg} \neq \emptyset) \tag{9}$$

$$E = \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R \text{ for some } \dot{\alpha} \in \dot{R}_{ex}\} \quad (\text{ if } \dot{R}_{ex} \neq \emptyset). \tag{10}$$

Set

$$\tilde{R} = (S + S) \cup R.$$

We call $\tilde{R}$ the *isotropic-extension* of R.

We want to show that the isotropic-extension $\tilde{R}$ of R is an EARS. In order to show this result, we need to establish some lemmas first. Even though some of the techniques which we will use here seem similar to those in [1, Chapter II], we must however be careful because we are only allowed to use Saito's axioms which are different from those of [1]. Recall that a root $\dot{\alpha}$ in a finite root system is called *reduced* if $\frac{1}{2}\dot{\alpha}$ is not a root.

Lemma 11. *If $\dot{\alpha} \in \dot{R}$ is reduced, then $\dot{\alpha} \in R$. That is $\dot{R}_{sh} \cup \dot{R}_{lg} \subseteq R$.*

Proof. Let $\dot{\alpha} \in \dot{R}$. We have $\bar{\dot{\alpha}} \in \bar{R}$, and so $\bar{\dot{\alpha}} = r\bar{\beta}$ where $\beta \in R, \bar{\beta}$ is reduced in $\bar{R}$ and $r \in \{1, 2\}$. Then

$$\bar{\beta} = w_{\bar{\dot{\alpha}}_{i_1}} \cdots w_{\bar{\dot{\alpha}}_{i_k}} \bar{\dot{\alpha}}_{i_{k+1}} = \overline{w_{\dot{\alpha}_{i_1}} \cdots w_{\dot{\alpha}_{i_k}} \dot{\alpha}_{i_{k+1}}}$$

for some $i_1, \ldots, i_{k+1} \in \{1, 2, \ldots, \ell\}$. Let $\dot{\beta} := w_{\dot{\alpha}_{i_1}} \cdots w_{\dot{\alpha}_{i_k}} \dot{\alpha}_{i_{k+1}}$. Then $\dot{\beta} \in R \cap \dot{R}$ by (SR3), since $\dot{\alpha}_{i_j} \in R \cap \dot{R}$, for $1 \leq j \leq k+1$. Also $\bar{\beta} = \bar{\dot{\beta}}$ and so $\bar{\dot{\alpha}} = r\bar{\dot{\beta}}$. Since $^- : \dot{\mathcal{V}} \to \bar{\mathcal{V}}$ is an isometry, we get $\dot{\alpha} = r\dot{\beta}, r \in \{1, 2\}$. If $\dot{\alpha}$ is reduced, then $r = 1$ and so $\dot{\alpha} = \dot{\beta} \in R$, as required. ∎

Let $\dot{W}$ be the subgroup of $\mathrm{GL}(\mathcal{V})$ generated by $w_{\dot{\alpha}}, \dot{\alpha} \in \dot{R}$. If $\dot{\alpha} \in \dot{R}$, then $\dot{\alpha} = r\dot{\beta}$ for some reduced $\dot{\beta} \in \dot{R}, r \in \{1, 2\}$. By Lemma 11, $\dot{\beta} \in R$ and so by (SR3),

$$w_{\dot{\alpha}}(R) = w_{r\dot{\beta}}(R) = w_{\dot{\beta}}(R) \subseteq R \quad (\dot{\alpha} \in \dot{R}). \tag{12}$$

Lemma 13. $R = (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E)$.

Proof. Let $\alpha \in R \subseteq \mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$. So $\alpha = \dot{\alpha} + \sigma$ for some $\dot{\alpha} \in \dot{\mathcal{V}}$ and $\sigma \in \mathcal{V}^0$. By (6), $\dot{\alpha} \in \dot{R}$. Now by the way S, L and E are defined, we have $\alpha \in (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{lg} + E)$.

For the reverse inclusion, first let $\dot{\alpha} + \sigma \in \dot{R}_{sh} + S$, where $\dot{\alpha} \in \dot{R}_{sh}$ and $\sigma \in S$. By the way S is defined, there exists $\dot{\beta} \in \dot{R}_{sh}$ such that $\dot{\beta} + \sigma \in R$. Since all roots of a given length in $\dot{R}$ are conjugate under $\dot{\mathcal{W}}$, there exists $\dot{w} \in \dot{\mathcal{W}}$ such that $\dot{w}\dot{\beta} = \dot{\alpha}$. Now by (12), $\dot{\alpha} + \sigma = \dot{w}\dot{\beta} + \sigma = \dot{w}(\dot{\beta} + \sigma) \in \dot{w}(R) \subseteq R$. This shows that $\dot{R}_{sh} + S \subseteq R$. A similar argument shows that $(\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E) \subseteq R$. This completes the proof of the Lemma. ■

Next, we normalize the bilinear form so that

$$(\dot{\alpha}, \dot{\alpha}) = \begin{cases} 2 & \text{if } \dot{\alpha} \in \dot{R}_{sh} \\ 2k & \text{if } \dot{\alpha} \in \dot{R}_{lg}, \qquad (\text{if } \dot{R}_{lg} \neq \emptyset) \end{cases} \tag{14}$$

where $k = 2$ for types $X = B_\ell, C_\ell, F_4, BC_\ell$; and $k = 3$ for type $X = G_2$. Then if $\dot{\alpha}$ is an extra long root, that is $\dot{\alpha}$ is two times a short root, we have $(\dot{\alpha}, \dot{\alpha}) = 8$.

Note that, up to this point, we have just assumed that R is a SEARS. *From now on we assume that R is a reduced SEARS.*

Lemma 15. *(i) If $\dot{R}_{lg} \neq \emptyset$, then $kS + L \subseteq L$ and $S + L \subseteq S$.*

(ii) If $\dot{R}_{ex} \neq \emptyset$ and $X = BC_\ell (\ell \geq 2)$, then

$$L + 2S \subseteq L, \quad S + L \subseteq S, \quad E + 2L \subseteq E, \quad L + E \subseteq L \text{ and } E \cap 2S = \emptyset.$$

(iii) If $\dot{R}_{ex} \neq \emptyset$ and $X = BC_1$, then

$$S + E \subseteq S, \quad 4S + E \subseteq E \text{ and } 2S \cap E = \emptyset.$$

Proof. (i) Let $\delta \in S$ and $\lambda \in L$. There exist $\dot{\alpha} \in \dot{R}_{sh}$ and $\dot{\beta} \in \dot{R}_{lg}$ such that $(\dot{\alpha}, \dot{\beta}^\vee) = -1$ and $(\dot{\beta}, \dot{\alpha}^\vee) = -k$. By Lemma 13, $\dot{\alpha} + \delta, \dot{\beta} + \lambda \in R$. Thus by (SR3),

$$\dot{\alpha} + \dot{\beta} + \delta + \lambda = w_{\dot{\beta}}(\dot{\alpha}) + \delta + \lambda = w_{\dot{\beta}+\lambda}(\dot{\alpha} + \delta) \in R,$$

and

$$\dot{\beta} + k\dot{\alpha} + k\delta + \lambda = w_{\dot{\alpha}}(\dot{\beta}) + k\delta + \lambda = w_{\dot{\alpha}+\delta}(\dot{\beta} + \lambda) \in R.$$

Now $\dot{\alpha} + \dot{\beta} = w_{\dot{\beta}}(\dot{\alpha}) \in \dot{R}_{sh}$ and $\dot{\beta} + k\dot{\alpha} = w_{\dot{\alpha}}(\dot{\beta}) \in \dot{R}_{lg}$. Therefore by the way S and L are defined we have $\delta + \lambda \in S$ and $k\delta + \lambda \in L$. This proves (i).

(ii) The proof of the first inclusion is similar to that of part (i). The proofs of the third and the fourth parts are also similar to that of part (i), replacing the roles of $\dot{R}_{sh}$ and $\dot{R}_{lg}$ with $\dot{R}_{lg}$ and $\dot{R}_{ex}$, respectively. To prove $E \cap 2S = \emptyset$, suppose on the contrary that $\delta \in E \cap 2S$. Then there exists $\lambda \in S$ such that $\delta = 2\lambda$. Let $\dot{\alpha} \in \dot{R}_{sh}$, then $2\dot{\alpha} \in \dot{R}_{ex}$ and so by Lemma 13, $\dot{\alpha} + \lambda \in R$ and $2\dot{\alpha} + 2\lambda \in R$. But this contradicts the fact that R is reduced.

(iii) In this case $\dot{R}$ has the form $\dot{R} = \{\pm\dot{\alpha}, \pm 2\dot{\alpha}\}$. Now let $\delta \in S$ and $\lambda \in E$. Then $\pm\dot{\alpha} + \delta \in R$ and $\pm 2\dot{\alpha} + \lambda \in R$. So

$$w_{-\dot{\alpha}+\delta}(2\dot{\alpha} + \lambda) = -2\dot{\alpha} + \lambda + 4\delta.$$

Thus $\lambda + 4\delta \in E$ and hence $S + 4E \subseteq E$. Also

$$w_{-2\dot{\alpha}+\lambda}(\dot{\alpha} + \delta) = -\dot{\alpha} + \delta + \lambda.$$

Thus $\delta + \lambda \in S$, so $E + S \subseteq S$. The proof of $2S \cap E = \emptyset$ is similar to that of part (ii). ■

We recall from [1, II.§1] that a *semilattice* in a finite dimensional real vector space $\mathcal{U}$ is a subset S of $\mathcal{U}$ satisfying:
(**S1**) $0 \in S$,
(**S2**) $S \pm 2S \subseteq S$,
(**S3**) S spans $\mathcal{U}$,
(**S4**) S is discrete in $\mathcal{U}$.
A nonempty subset S of $\mathcal{U}$ which satisfies (S2), (S3) and (S4) is called a *translated semilattice.*

Lemma 16. *S and L are semilattices in $\mathcal{V}^0$ and E is a translated semilattice in $\mathcal{V}^0$.*

Proof. By Lemma 11, $\dot{R}_{sh} \cup \dot{R}_{lg} \subseteq R$. So by the way S and L are defined, we have $0 \in S$ and $0 \in L$. Thus (S1) holds for S and L. For (S2), let $\delta \in S$ and $\dot{\alpha} \in \dot{R}_{sh}$. Then by Lemma 13, $\dot{\alpha} + \delta \in R$. By (SR3), $R = -R$, so $-\dot{\alpha} - \delta \in R$. Thus $-\delta \in S$, and $S = -S$. Now let $\delta, \lambda \in S$ and $\dot{\alpha} \in \dot{R}_{sh}$. Then $\dot{\alpha} + \lambda, -\dot{\alpha} + \delta \in R$ and

$$w_{\dot{\alpha}+\lambda}(-\dot{\alpha} + \delta) = \dot{\alpha} + \delta + 2\lambda.$$

Thus $\delta + 2\lambda \in S$ and so $S \pm 2S \subseteq S$. A similar argument shows

$$L \pm 2L \subseteq L, \quad \text{and} \quad E \pm 2E \subseteq E.$$

So (S2) holds for S, L and E.

Next, suppose $\dim \mathcal{V}^0 = \nu$. By (SR1), R spans $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$ and by (SR2), elements of R are nonisotropic. Since the image of R under the projection $\mathcal{V} \to \mathcal{V}^0$ spans $\mathcal{V}^0$, there exists an $\mathbb{R}$-basis $\{\sigma_1, \sigma_2, \ldots, \sigma_\nu\}$ of $\mathcal{V}^0$ such that $\dot{\beta}_i + \sigma_i \in R$, for some $\dot{\beta}_i \in \dot{\mathcal{V}}$. By the way $\dot{R}, S, L$ and E are defined we have $\dot{\beta}_i \in \dot{R}$ and $\sigma_i \in S \cup L \cup E$ for $1 \leq i \leq \nu$. But by Lemma 15 and the fact that $0 \in S$, we have $S \cup L \cup E = S$, so $\sigma_i \in S$ for $1 \leq i \leq \nu$. Thus S spans $\mathcal{V}^0$. Again by Lemma 15 and the fact that $0 \in L$, we have $kS \subseteq L$. Thus L spans $\mathcal{V}^0$. Finally from Lemma 15 and the fact that $E = -E$, it follows that $8S \subseteq E + E$ and so E spans $\mathcal{V}^0$. Thus (S3) holds for S, L and E. Next we show (S4). Note that by Lemma 13, $\dot{R}_{sh} + S \subseteq R$. Since $\dot{R}_{sh} = -\dot{R}_{sh}$, we have $2S \subseteq \langle R \rangle$. But by (SR1), $\langle R \rangle$ is a lattice in $\mathcal{V}$, so S is a subset of a discrete set and therefore is discrete. By Lemma 15, $L \subseteq S$ and $E \subseteq S$, so L and E are also discrete. ∎

Lemma 17. *(i) If $X = A_\ell (\ell \geq 2), D_\ell, E_\ell, F_4, G_2, C_\ell (\ell \geq 3)$, then S is a lattice.*
(ii) If $X = B_\ell (\ell \geq 3), F_4, G_2, BC_\ell (\ell \geq 3)$, then L is a lattice.

Proof. (i) In this case there exist roots $\dot{\alpha}, \dot{\beta} \in \dot{R}_{sh}$ such that $(\dot{\alpha}, \dot{\beta}^\vee) = -1$. Now if $\delta, \lambda \in S$, then $\dot{\alpha} + \delta, \dot{\beta} + \lambda \in R$, so by (SR3),

$$w_{\dot{\alpha}+\delta}(\dot{\beta} + \lambda) = w_{\dot{\alpha}}(\dot{\beta}) + (\delta + \lambda) \in R.$$

Since $w_{\dot{\alpha}}(\dot{\beta}) \in \dot{R}_{sh}$, we have $\delta + \lambda \in S$. Thus $S + S \subseteq S$ and so S is a lattice.

(ii) In this case there exist roots $\dot{\alpha}, \dot{\beta} \in \dot{R}_{lg}$ such that $(\dot{\alpha}, \dot{\beta}^\vee) = -1$. So using a similar argument as in part (i), we see that L is a lattice. ∎

We are now ready to state the main result of this work.

Theorem 18. *Any reduced SEARS is the set of nonisotropic roots of a unique EARS. Also the set of nonisotropic roots of any EARS is a reduced SEARS. In other words, there is a one to one correspondence between EARSs and reduced SEARSs.*

Proof. Let R be a reduced SEARS in a finite dimensional real vector space $\mathcal{V}$. Let $\dot{R}, S, L$ and E be defined by (6), (8), (9) and (10), respectively. As before set $\tilde{R} = (S+S) \cup R$. By Lemmas 13 and 16,

$$\tilde{R} = (S+S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E), \tag{19}$$

where S and L are semilattices in $\mathcal{V}^0$ and E is a translated semilattice in $\mathcal{V}^0$. By Lemmas 15 and 16 and [1, II.2.36], $\tilde{R}$ is an *EARS* in $\mathcal{V}$. (We remark here that the finite root system $\dot{R}$ defined by (6) does not contain zero, while the finite root system $\dot{R}$ which appears in [1, II.§2] contains zero. However, the proof of [1, II.2.36] does not depend on this minor difference and so we can use this theorem here.) Clearly $\tilde{R}^\times = R$, that is R is the set of nonisotropic roots of $\tilde{R}$. To show the uniqueness, let $\tilde{R}_1$ be any *EARS* in a vector space $\mathcal{V}'$ containing $\mathcal{V}$ as a subspace such that $\tilde{R}_1^\times = R$. By (R3) and (R8), $\tilde{R}^\times = R = \tilde{R}_1^\times$ spans $\mathcal{V}$ and $\mathcal{V}'$, so $\mathcal{V} = \mathcal{V}'$. Now let $\sigma \in \tilde{R}_1^0$, the set of isotropic roots of $\tilde{R}_1$. By (R8), there exists $\alpha \in \tilde{R}_1^\times$ such that $\alpha + \sigma \in \tilde{R}_1^\times$. Since

$$\tilde{R}_1^\times = R = (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E),$$

there exist $\dot{\alpha} \in \dot{R}$ and $\delta \in S$ such that $\alpha = \dot{\alpha} + \delta$ (note that $E \subseteq L \subseteq S$). Therefore, $\sigma + \delta \in S$ and $\sigma \in S + S = \tilde{R}^0$. Thus $\tilde{R}_1^0 \subseteq \tilde{R}^0$. A similar argument shows $\tilde{R}^0 \subseteq \tilde{R}_1^0$. This finishes the proof of the first statement. The proof of the second statement is given in Lemma 3. ∎

We close this paper with a few remarks:

Remark 1.1. (i) The type, the rank and the nullity of an EARS are defined exactly as they are defined for a SEARS (see Definition 7 and [1, II. §2]).

(ii) The axioms for a reduced SEARS are equivalent to axioms for an EARS, in the following sense. Axioms (R1)-(R8) for an EARS $\tilde{R}$ implies axioms (SR1)-(SR5) for the SEARS $\tilde{R}^\times$ (by Theorem 18). Also it follows that $\tilde{R}^\times$ is reduced. Conversely, axioms (SR1)-(SR5) for a reduced SEARS R implies axioms (R1)-(R8) for the corresponding EARS $\tilde{R}$.

(iii) When $\nu = 1$, it is shown in [8, §1] that a SEARS satisfies axioms of [7] for an affine root system. Therefore by (ii) an EARS of nullity 1 is the root system of an affine Kac–Moody Lie algebra. (This is also shown in [ABGP] and [3].)

Remark 1.2. (i) Let R be a Saito's extended affine root system of nullity 2. According to K. Saito [8], a 1-dimensional subspace G of the radical $(\cdot, \cdot)$ is called a *marking* if $G \cap \langle R \rangle$ is a lattice in G. The pair (R, G) is called a *marked* SEARS. It turns out that the image R/G of R in $\mathcal{V}/G$ under the natural map is an affine root system (see [8, §1]). In [8], Saito has classified all marked SEARS (R, G) for which the affine root system R/G is reduced. It is clear that the reduceness of

R/G implies the reduceness of R. Thus by Theorem 18, the isotropic-extension of any marked SEARS (R,G) with R/G reduced, is an EARS. However, the converse is not true. For example, suppose $\dot{R}$ is a finite root system of type BC_1. Consider a semilattice S and a translated semilattice E as follows. Let

$$S=\mathbb{Z}\delta_1\oplus\mathbb{Z}\delta_2,\quad \text{and}\quad E=[2\Lambda\cup(\mathbb{Z}\delta_1\oplus 2\mathbb{Z}\delta_2)\cup(2\mathbb{Z}\delta_1\oplus\mathbb{Z}\delta_2)]+\delta_1+\delta_2.$$

Then

$$\tilde{R}=(S+S)\cup(\dot{R}_{sh}+S)\cup(\dot{R}_{ex}+E)$$

is an EARS of type BC_1 and nullity 2 in $\mathcal{V}=(\mathrm{span}_{\mathbb{R}}\dot{R})\oplus\mathbb{R}\delta_1\oplus\mathbb{R}\delta_2$. However if we take $G=\mathbb{R}\delta_1$, then $\tilde{R}^{\times}/G$ is not reduced.

(ii) From (i) we conclude that, in the classification of root systems of nullity 2, we should have more EARSs than marked SEARSs (R,G) with reduced quotient R/G. Clearly this difference must occur only in BC-type root systems. According to [1, II. §4], any EARS of type $BC_\ell(\ell\geq 2)$ is isomorphic to one and only one EARS of the form $\tilde{R}=(S+S)\cup(\dot{R}_{sh}+S)\cup(\dot{R}_{lg}+L)\cup(\dot{R}_{ex}+E)$, where the triple (S,L,E) runs through the following table (for notation see [1, Chapter II]):

(S, L, E)
$(S(0,\delta_1,\delta_2),2\Lambda,4\Lambda+2\delta_1+2\delta_2)$
$(\Lambda,\Lambda^{(1)},2\Lambda^{(1)}+\delta_2)$
$(\Lambda,\Lambda^{(1)},2\Lambda+\delta_2)$
$(\Lambda,\Lambda,2\Lambda+\delta_2)$
$(\Lambda,\Lambda,\Lambda^{(1)}+\delta_1)$
$(\Lambda,\Lambda,S(0,\delta_1,\delta_2)+\delta_1+\delta_2)$

Taking $G=\mathbb{R}\delta_1$, one can see that the affine root systems $\tilde{R}^{\times}/G$ corresponding to the first and the last triples (S,L,E) given in the above table are not reduced and the others are reduced. Also any EARS of type BC_1 of nullity 2 is isomorphic to one and only one EARS of the form $\tilde{R}=(S+S)\cup(\dot{R}_{sh}+S)\cup(\dot{R}_{ex}+E)$ where the pair (S,E) runs through the following table:

(S, E)
$(S(0,\delta_1,\delta_2),2\Lambda,4\Lambda+2\delta_1+2\delta_2)$
$(\Lambda,2\Lambda^{(1)}+\delta_2)$
$(\Lambda,2\Lambda+\delta_2)$
$(\Lambda,\Lambda^{(1)}+\delta_1)$
$(\Lambda,S(0,\delta_1,\delta_2)+\delta_1+\delta_2)$

Again, taking $G=\mathbb{R}\delta_1$, one can see that the affine root systems $\tilde{R}^{\times}/G$ corresponding to the first and the last pairs (S,E) given in the above table are not reduced and the others are reduced. Therefore in the classification of EARS of nullity 2 in [1, II.§4], for a given rank, there are four more root systems than in the classification given in [8, §5].

Remark 1.3. (i) Suppose R is a SEARS and G is a subspace of the radical of its form such that $G \cap \langle R \rangle$ is a lattice in G. Then according to [8, §1], R/G is a SEARS. However, in general R/G does not inherit reduceness from R. Also, if $\tilde{R}$ is an EARS and G is a subspace of $\mathcal{V}^0$ such that $G \cap \langle \tilde{R} \rangle$ is a lattice in G, then $\tilde{R}/G$ is not necessarily an EARS. In fact if $\tilde{R}$ has a type different from BC, then $\tilde{R}/G$ is an EARS but for type BC, it is not always the case. (See [4] and part (ii) of this remark.)

(ii) In [8], K. Saito classifies the (marked) SEARs of nullity 2, using the notion of marking. In fact he introduces a notion of marking for SEARS of arbitrary nullity. A *marking* for a SEARS is a flag $G_0 = \{0\} \subseteq G_1 \subseteq \cdots \subseteq G_r = \mathcal{V}^0$ of subspaces of $\mathcal{V}^0$ such that for each i, $G_i \cap \langle R \rangle$ is a lattice in G_i. Saito gets the concept of marking from a study of primitive forms for the period mapping of simple elliptic singularities. He states that "sometimes it is convenient to fix such a flag for the study of R."

(iii) In Remark 1.2, we compared the classification of marked SEARSs with reduced quotient and EARSs for $\nu = 2$, and saw that the notion of marking with reduced quotient decreases the number of SEARSs appearing in the classification. Now we would like to see what happens to the classification for nullity 3, if we apply such a notion of marking.

In [5], EARSs of type BC and nullity 3 are classified. Accordingly, any EARS of type BC_ℓ ($\ell \geq 3$) is isomorphic to one and only one EARS $\tilde{R} = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E)$ where the triple (S, L, E) runs through the following table (for notation see [5]):

	(S, L, E)
1	$(\Lambda, \Lambda, \delta_1 + \delta_2 + S(0, \delta_1, \delta_2, \delta_3))$
2	$(\Lambda, \Lambda, \delta_1 + \delta_3 + S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2))$
3	$(\Lambda, \Lambda, \delta_2 + \delta_3 + S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3))$
4	$(\Lambda, \Lambda, \delta_1 + \delta_2 + \delta_3 + S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3, \delta_2 + \delta_3))$
5	$(\Lambda, \Lambda^{(1)}, \delta_2 + \delta_3 + 4\Lambda + S(0, \delta_2, \delta_3))$
6	$(\Lambda, \Lambda^{(1)}, \delta_2 + \Lambda^{(1,1,2)})$
7	$(\Lambda, \Lambda, \delta_2 + \Lambda^{(2)})$
8	$(\Lambda, \Lambda, \delta_1 + 2\Lambda)$
9	$(\Lambda, S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \delta_1 + 2\Lambda)$
10	$(\Lambda, \Lambda, \delta_1 + \Lambda^{(1)})$
11	$(\Lambda, \Lambda, \delta_1 + 2\Lambda + S(0, \delta_2, \delta_3))$
12	$(\Lambda, \Lambda, \delta_2 + \delta_3 + 2\Lambda + S(0, \delta_2, \delta_3))$
13	$(\Lambda, \Lambda^{(1)}, \delta_3 + 2\Lambda)$
14	$(\Lambda, \Lambda^{(1)}, \delta_2 + \Lambda^{(2)})$
15	$(\Lambda, \Lambda^{(1)}, \delta_2 + 2\mathbb{Z}\delta_2 + S(0, 2\delta_1, \delta_3))$
16	$(\Lambda, \Lambda^{(2)}, \delta_3 + 2\Lambda)$
17	$(\Lambda, \Lambda^{(2)}, \delta_3 + S(0, 2\delta_1, 2\delta_2) + 2\mathbb{Z}\delta_3)$
18	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, \delta_3 + 2\Lambda)$
19	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, \delta_3 + S(0, 2\delta_1, 2\delta_2) + 2\mathbb{Z}\delta_3)$
20	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, 2\delta_1 + 2\delta_2 + \delta_3 + 4\mathbb{Z}\delta_1 + S(0, 2\delta_2, \delta_3))$
21	$(\Lambda, \Lambda^{(1)}, \delta_2 + \delta_3 + S(0, 2\delta_1, \delta_2, \delta_3))$
22	$(\Lambda, \Lambda^{(1)}, \delta_2 + \delta_3 + S(0, 2\delta_1, \delta_2, \delta_3, 2\delta_1 + \delta_2))$
23	$(\Lambda, \Lambda^{(1)}, \delta_2 + \delta_3 + S(0, 2\delta_1, \delta_2, \delta_3, 2\delta_1 + \delta_2, 2\delta_1 + \delta_3))$
24	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, 2\delta_1 + 2\delta_2 + \delta_3 + S(0, 2\delta_1, 2\delta_2, \delta_3))$
25	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, 2\delta_1 + 2\delta_2 + \delta_3 + S(0, 2\delta_1, 2\delta_2, \delta_3, 2\delta_1 + 2\delta_2))$
26	$(S(0, \delta_1, \delta_2, \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 2\delta_3 + 2S(0, \delta_1, \delta_2, \delta_3))$
27	$(S(0, \delta_1, \delta_2, \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 2\delta_3 + 4\mathbb{Z}\delta_1 + S(0, 2\delta_2, 2\delta_3))$
28	$(S(0, \delta_1, \delta_2, \delta_3, \delta_2 + \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 2\delta_3 + 4\mathbb{Z}\delta_1 + S(0, 2\delta_2, 2\delta_3))$
29	$(S(0, \delta_1, \delta_2, \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 2\Lambda^{(2)})$
30	$(S(0, \delta_1, \delta_2, \delta_3, \delta_2 + \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 2\Lambda^{(2)})$
31	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, 2\Lambda, 2\delta_1 + 2\delta_2 + 2\Lambda^{(2)})$
32	$(\Lambda, \Lambda^{(2)}, \delta_3 + \Lambda^{(1,2,3)})$
33	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, \delta_3 + 2\Lambda^{(1,2,3)})$
34	$(\Lambda, \Lambda^{(2)}, \delta_3 + \Lambda^{(2,2,3)})$
35	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, \delta_3 + \Lambda^{(2,2,3)})$
36	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, 2\delta_1 + 2\delta_2 + \Lambda^{(2,2,3)})$
37	$(\Lambda, \Lambda^{(1)}, \delta_2 + \Lambda^{(1,1,3)})$
38	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \Lambda^{(2)}, 2\delta_1 + 2\delta_2 + \Lambda^{(2,2,2)})$
39	$(S(0, \delta_1, \delta_2, \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 4\Lambda)$
40	$(S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2), 2\Lambda, 2\delta_1 + 2\delta_3 + 4\Lambda)$
41	$(S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3), 2\Lambda, 2\delta_2 + 2\delta_3 + 4\Lambda)$
42	$(S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3, \delta_2 + \delta_3), 2\Lambda, 2\delta_1 + 2\delta_2 + 2\delta_3 + 4\Lambda)$

Taking a marking $(\tilde{R}^\times, G_1 \subseteq G_2)$ where $G_1 = \mathbb{R}\delta_1$ and $G_2 = \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2$, one can see that the affine root systems $\tilde{R}^\times/G_2$ corresponding to the triples from lines 1, 2, 4, 8, 9, 11, 20, 24–31, 36, 38, 39, 40 and 42 are not reduced. Also the SEARS $\tilde{R}^\times/G_1$ corresponding to the triples (S, L, E) from lines 1–4, 6–12, 14, 15, 20–31 and 36–42 are not reduced. Note also that any EARS of type BC_1 of nullity 3 is isomorphic to one and only one EARS of the form $\tilde{R} = (S+S) \cup \dot{R}_{sh}+S) \cup (\dot{R}_{ex}+E)$ where the pair (S, E) runs through the following table:

	(S, E)
1	$(\Lambda, \delta_1 + \delta_2 + S(0, \delta_1, \delta_2, \delta_3))$
2	$(\Lambda, \delta_1 + \delta_3 + S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2))$
3	$(\Lambda, \delta_2 + \delta_3 + S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3))$
4	$(\Lambda, \delta_1 + \delta_2 + \delta_3 + S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3, \delta_2 + \delta_3))$
5	$(\Lambda, \delta_2 + \delta_3 + 4\Lambda + S(0, \delta_2, \delta_3))$
6	$(\Lambda, \delta_2 + \Lambda^{(1,1,2)})$
7	$(\Lambda, \delta_1 + \Lambda^{(1)})$
8	$(\Lambda, \delta_1 + 2\Lambda + S(0, \delta_2, \delta_3))$
9	$(\Lambda, \delta_2 + \Lambda^{(2)})$
10	$(\Lambda, \delta_2 + 2\mathbb{Z}\delta_2 + S(0, 2\delta_1, \delta_3))$
11	$(\Lambda, \delta_3 + 2\Lambda)$
12	$(\Lambda, \delta_3 + S(0, 2\delta_1, 2\delta_2) + 2\mathbb{Z}\delta_3)$
13	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \delta_3 + 2\Lambda)$
14	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \delta_3 + S(0, 2\delta_1, 2\delta_2) + 2\mathbb{Z}\delta_3)$
15	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, 2\delta_1 + 2\delta_2 + \delta_3 + 4\mathbb{Z}\delta_1 + S(0, 2\delta_2, \delta_3))$
16	$(\Lambda, \delta_2 + \delta_3 + S(0, 2\delta_1, \delta_2, \delta_3))$
17	$(\Lambda, \delta_2 + \delta_3 + S(0, 2\delta_1, \delta_2, \delta_3, 2\delta_1 + \delta_2))$
18	$(\Lambda, \delta_2 + \delta_3 + S(0, 2\delta_1, \delta_2, \delta_3, 2\delta_1 + \delta_2, 2\delta_1 + \delta_3))$
19	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, 2\delta_1 + 2\delta_2 + \delta_3 + S(0, 2\delta_1, 2\delta_2, \delta_3))$
20	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, 2\delta_1 + 2\delta_2 + \delta_3 + S(0, 2\delta_1, 2\delta_2, \delta_3, 2\delta_1 + 2\delta_2))$
21	$(S(0, \delta_1, \delta_2, \delta_3), 2\delta_1 + 2\delta_2 + 2\delta_3 + 2S(0, \delta_1, \delta_2, \delta_3))$
22	$(S(0, \delta_1, \delta_2, \delta_3), 2\delta_1 + 2\delta_2 + 2\delta_3 + 4\mathbb{Z}\delta_1 + S(0, 2\delta_2, 2\delta_3))$
23	$(S(0, \delta_1, \delta_2, \delta_3, \delta_2 + \delta_3), 2\delta_1 + 2\delta_2 + 2\delta_3 + 4\mathbb{Z}\delta_1 + S(0, 2\delta_2, 2\delta_3))$
24	$(S(0, \delta_1, \delta_2, \delta_3), 2\delta_1 + 2\delta_2 + 2\Lambda^{(2)})$
25	$(S(0, \delta_1, \delta_2, \delta_3, \delta_2 + \delta_3), 2\delta_1 + 2\delta_2 + 2\Lambda^{(2)})$
26	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, 2\delta_1 + 2\delta_2 + 2\Lambda^{(2)})$
27	$(\Lambda, \delta_3 + \Lambda^{(1,2,3)})$
28	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \delta_3 + 2\Lambda^{(1,2,3)})$
29	$(\Lambda, \delta_3 + \Lambda^{(2,2,3)})$
30	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, \delta_3 + \Lambda^{(2,2,3)})$
31	$(S(0, \delta_1, \delta_2) + \mathbb{Z}\delta_3, 2\delta_1 + 2\delta_2 + \Lambda^{(2,2,2)})$
32	$(S(0, \delta_1, \delta_2, \delta_3), 2\delta_1 + 2\delta_2 + 4\Lambda)$
33	$(S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2), 2\delta_1 + 2\delta_3 + 4\Lambda)$
34	$(S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3), 2\delta_2 + 2\delta_3 + 4\Lambda)$
35	$(S(0, \delta_1, \delta_2, \delta_3, \delta_1 + \delta_2, \delta_1 + \delta_3, \delta_2 + \delta_3), 2\delta_1 + 2\delta_2 + 2\delta_3 + 4\Lambda)$

Again taking $G_1 = \mathbb{R}\delta_1$ and $G_2 = \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2$, one can see that the affine root systems $\tilde{R}^\times/G_2$ corresponding to the pairs from lines 1, 2, 4, 8, 15, 19–26, 31–33 and 35 are not reduced. Also the SEARS $\tilde{R}^\times/G_1$ corresponding to the pairs (S, E) from lines 1–10, 15–26 and 31–35 are not reduced. The above discussion shows that applying such a notion of marking with reduced quotients decreases significantly the number of root systems. It is clear that as the nullity increases, the number of roots omitted by the notion of marking with reduced quotients also increases.

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