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Invariant Control Sets on Flag Manifolds and Ideal Boundaries of Symmetric Spaces *

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Abstract. Let G be a semisimple real Lie group of non-compact type, K a maximal compact subgroup and $S \subseteq G$ a semigroup with nonempty interior. We consider the ideal boundary $\partial_\infty(G/K)$ of the associated symmetric space and the flag manifolds G/P_Θ. We prove that the asymptotic image $\partial_\infty(Sx_0) \subseteq \partial_\infty(G/K)$, where $x_0 \in G/K$ is any given point, is the maximal invariant control set of S in $\partial_\infty(G/K)$. Moreover there is a surjective projection $\pi\colon \partial_\infty(Sx_0) \to \bigcup_{\Theta \subseteq \Sigma} C_\Theta$, where C_Θ is the maximal invariant control set for the action of S in the flag manifold G/P_Θ, with P_Θ a parabolic subgroup. The points that project over C_Θ are exactly the points of type Θ in $\partial_\infty(Sx_0)$ (in the sense of the type of a cell in a Tits Building).

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1. Introduction

The concept of invariant control sets (i.c.s.) of a semigroup was first introduced by Arnold and Kliemann [1]. We consider the special instance where G is a semisimple real Lie group of non-compact type with finite center and $S \subseteq G$ a semigroup with non-empty interior. If $P \subseteq G$ is a parabolic subgroup, the homogeneous space G/P is a compact manifold (a generalized flag manifolds). The study of invariant control sets for the left action of S on G/P has been systematically used and developed by San Martin [7], [8]. One of the basic results in this context is the existence of a unique i.c.s., whose set of transitivity is given by the points fixed by elements in the interior of S.

From another side, we have the concept of ideal boundary $\partial_\infty(\mathcal{X})$ of a Hadamard manifold $\mathcal{X}$, that was first introduced by Eberlein and Oneil [3] as a way to compactify Hadamard manifolds. Later, the special case of a symmetric space $\mathcal{X} = G/K$ (G is again a semisimple real Lie group of non-compact type with finite center and K a maximal compact subgroup) was exploited by M. Gromov

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[2] in the study of many important results, such as Margulis' Lemma and Mostow's Rigidity Theorem.

In this article we determine the i.c.s. of a semigroup $S \subseteq G$ in the ideal boundary $\partial_\infty(G/K)$: it is just the ideal boundary $\partial_\infty(Sx_0)$, the set of points in the ideal boundary $\partial_\infty(G/K)$ that belong to the closure on any orbit Sx_0, where x_0 is an arbitrary point of the symmetric space G/K (Theorem 4.4). Moreover, we consider a minimal parabolic subgroup $P \subseteq G$ and the set $\{P_\Theta | \Theta \subseteq \Sigma\}$ of parabolic subgroups of G containing P (here Σ is a simple root system determined by P). Then, to each such Θ there is a flag manifold G/P_Θ and a (unique) i.c.s. $C_\Theta \subseteq G/P_\Theta$. All those i.c.s. are incorporated in $\partial_\infty(Sx_0)$ (Theorem 4.3) in the sense that there is a surjective projection $\pi : \partial_\infty(Sx_0) \to \bigcup_{\Theta\subseteq\Sigma} C_\Theta$.

2. Basic constructions

Let $\mathcal{X}$ be a symmetric space of non-compact type. We let $G = \mathrm{Isom}^0(\mathcal{X})$ be the identity component of the isometry group of $\mathcal{X}$ and K the stabilizer (in G) of a point $x_0 \in \mathcal{X}$. Then $\mathcal{X} = G/K$, G is a *real semi-simple Lie group* and K a *maximal compact* subgroup of G. The choice of the base point is immaterial, since their stabilizers are conjugate in G. In this section we introduce the main concepts and notations concerning semisimple Lie algebras and groups and associated symmetric spaces. The standard reference for this section is [5].

2.1. Lie algebra structure.

Since G is semi-simple the *Cartan-Killing form*

$$B(X,Y) = \mathrm{Tr}(\mathrm{adX} \circ \mathrm{adY})$$

is a non-degenerate bilinear form on $\mathfrak{g}\times\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. If we denote by $\mathfrak{k}$ the Lie algebra of K and by $\mathfrak{x}$ its orthogonal complement we get a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{x}$ (direct sum), with $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{x},\mathfrak{x}] \subseteq \mathfrak{k}$ and $[\mathfrak{k},\mathfrak{x}] \subseteq \mathfrak{x}$. We get another Cartan decompositions if we consider another maximal compact subgroup $K' \subset G$. All such subgroups are conjugate and so are their algebras: $\mathfrak{k}' = e^{\mathrm{ad}(X)}(\mathfrak{k})$, for some $X \in \mathfrak{g}$.

A Cartan involution of $\mathfrak{g}$ is an automorphism $\nu : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

$$\nu(X_\mathfrak{k} + X_\mathfrak{x}) = X_\mathfrak{k} - X_\mathfrak{x},$$

where $X_\mathfrak{k}+X_\mathfrak{x}$ is the decomposition of X relative to a given Cartan decomposition of $\mathfrak{g}$. The quadratic form

$$\langle X,Y\rangle = -B(X,\nu(Y))$$

is a positive definite quadratic form on $\mathfrak{g}$ invariant under the action of $\mathrm{Ad}(K)$.

We choose (and fix) a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{x}$. The *real rank* $\mathrm{rank}^{\mathbb{R}}(\mathfrak{g})$ is the dimension of $\mathfrak{a}$. The rank does not depend on the choice neither of $\mathfrak{x}$ nor of $\mathfrak{a}$. The *root space decomposition* of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda\in\Lambda} \mathfrak{g}_\lambda$$

where $\lambda \in \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$,

$$\mathfrak{g}_\lambda = \{Y \in \mathfrak{g} | [H, Y] = \lambda(H) Y, \text{ for all } H \in \mathfrak{a}\}$$

and

$$\Lambda = \{\lambda \in \mathrm{Hom}(\mathfrak{a}, \mathbb{R}) | \mathfrak{g}_\lambda \neq \{0\}\}.$$

The λ's in Λ are called *roots* of $\mathfrak{g}$ and each $\mathfrak{g}_\lambda$ a *root subspace.* Each root $\lambda \in \Lambda$ determines a hyperplane $\mathcal{H}_\lambda = \{H \in \mathfrak{a} | \lambda(H) = \{0\}\}$. Each component of

$$\mathfrak{a} \setminus \bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda$$

is said to be an *open Weyl chamber.* A *Weyl chamber* is the closure of an open Weyl Chamber. A Weyl chamber $\overline{\mathfrak{a}}^+$ determines a set of positive roots

$$\Pi^+ = \left\{\lambda \in \Lambda | \lambda(H) \geq 0 \text{ for every } H \in \overline{\mathfrak{a}}^+\right\}$$

and of negative roots $\Pi^- = -\Pi^+$.

It also determines a set of *simple roots*, that is, a linearly independent set $\Sigma = \{\lambda_1, \lambda_2, ..., \lambda_r\}$ of positive roots such that every root may be written as a linear combination $\lambda = \sum_{i=1}^r m_i \lambda_i$ with all the m_i having the same sign: $m_i \geq 0$ if $\lambda \in \Pi^+$ and $m_i \leq 0$ if $\lambda \in \Pi^-$. Geometrically, a root λ is simple (relatively to a given chamber $\overline{\mathfrak{a}}^+$) if and only if $\mathcal{H}_\lambda \cap \overline{\mathfrak{a}}^+$ has dimension $\mathrm{r}(\mathfrak{g}) - 1$.

Given a subset $\Theta \subset \Sigma$, the subspace

$$\mathcal{H}_\Theta = \{H \in \mathfrak{a} | \lambda(H) = 0 \text{ for all } \lambda \in \Theta\}$$

is an abelian subalgebra of $\mathfrak{g}$ and every abelian subalgebra of maximal rank is conjugate to one of those subspaces. The intersection of $\mathcal{H}_\Theta$ with a closed Weyl chamber $\overline{\mathfrak{a}}^+$ is said to be the Θ-*wall* $\overline{\mathfrak{a}}_\Theta$ of $\overline{\mathfrak{a}}^+$. In fact, this wall is determined by the simple roots (relatively to $\mathfrak{a}^+$) contained in Θ, that is, $\overline{\mathfrak{a}}_\Theta = \overline{\mathfrak{a}}_{\Theta \cap \Sigma}$, where Σ is the set of simple roots of the chamber $\mathfrak{a}^+$. An *open wall* $\mathfrak{a}_\Theta$ is the interior of $\overline{\mathfrak{a}}_\Theta$ in $\mathfrak{a}$. We denote by $\overline{\mathfrak{a}}_\Theta^+$ the intersection $\mathcal{H}_\Theta \cap \overline{\mathfrak{a}}^+$ and call the *closed* Θ-*Weyl wall of* $\overline{\mathfrak{a}}^+$. The *open* Θ-*Weyl wall of* $\overline{\mathfrak{a}}^+$ is the interior $\mathfrak{a}_\Theta^+$ of $\overline{\mathfrak{a}}_\Theta^+$ in $\mathcal{H}_\Theta$.

A Weyl chamber $\mathfrak{a}^+$ (alternatively, a set of positive roots Π^+ or a set of associated simple roots Σ) determines maximal nilpotent subalgebras

$$\mathfrak{n}^\pm = \sum_{\lambda \in \Pi^\pm} \mathfrak{g}_\lambda.$$

We denote by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A *minimal parabolic subalgebra* is any algebra conjugate in $\mathfrak{g}$ to

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

The subalgebras $\mathfrak{a}$, $\mathfrak{n}^+$ and $\mathfrak{m}$ are determined by the choice of a Weyl chamber $\mathfrak{a}^+$. Since all such chambers are conjugate, so are the minimal parabolic subalgebras.

More generally, for a subset $\Theta \subseteq \Sigma$ we denote by $\mathfrak{p}_\Theta$ the *parabolic subalgebra*

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}.$$

Here $\mathfrak{n}^-(\Theta)$ stands for the subalgebra spanned by the root spaces $\mathfrak{g}_{-\lambda}$, for $\lambda \in \langle\Theta\rangle$, where $\langle\Theta\rangle$ is the set of (positive) roots generated by Θ. Particularly, $\mathfrak{p}_\emptyset = \mathfrak{p}$ and $\mathfrak{p}_\Sigma = \mathfrak{g}$.

An *Iwasawa decomposition* of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. As in the case of parabolic algebras, all Iwasawa decompositions are conjugate one to the other.

2.2. The structure of the Lie group.

For all objects introduced so far in the Lie algebra $\mathfrak{g}$ we find corresponding objects in the Lie group G and the symmetric space $\mathcal{X} = G/K$. We denote by x_0 the base point $\widetilde{\pi}(\mathrm{Id})$. The subspace $\mathfrak{x} \subset \mathfrak{g}$ given by the Cartan decomposition is identified with the tangent space of $\mathcal{X}$ at x_0 by the map $d\widetilde{\pi}|_{\mathfrak{x}} : \mathfrak{x} \to T_{x_0}\mathcal{X}$. Moreover, geodesics in $\mathcal{X}$ with initial point x_0 are defined as $\eta(t) = \exp(tY)\,x_0$, for some vector $Y \in \mathfrak{x}$, with $\|Y\| = 1$.

By defining $A = \exp\mathfrak{a}$, $K = \exp\mathfrak{k}$ and $N^+ = \exp\mathfrak{n}^+$, we get an *Iwasawa decomposition* $G = KAN^+$.

Here A is a maximal abelian subgroup and N^+ a maximal nilpotent subgroup. A *flat* in $\mathcal{X}$ is an isometrically embedded Euclidean space. It can be easily proved that flats in $\mathcal{X}$ containing the point x_0 correspond (by the exponential map) to subalgebras of $\mathfrak{a}$ and their conjugate. So, $F = Ax_0$ is a maximal flat in $\mathcal{X}$ and every maximal flat in $\mathcal{X}$ is of the form $F' = gF = gAx_0$, with $g \in G$. The *rank* of a symmetric space is the dimension of a maximal flat and, by the preceding argument, it equals the dimension of A.

The structure of Weyl chambers in $\mathfrak{a}$ carries over to the subgroup $A = \exp\mathfrak{a}$ and to the flat $F = Ax_0 \subset \mathcal{X}$: if we denote by $\mathfrak{a}^+$ a Weyl chamber of $\mathfrak{a}$ and by $A^+ = \exp\ \mathfrak{a}^+$ its image in G, we shall call gA^+x_0 a *Weyl chamber*, to any $g \in G$. The point $gx_0 \in gA^+x_0$ is called the *base point* of the chamber. A subalgebra $\mathcal{H}_\Theta$ gives rise to Θ-*flats* $gF_\Theta := g\exp(\mathcal{H}_\Theta)\,x_0$. In a similar way, we say that $g\overline{A}^+_\Theta x_0 := g\exp\left(\overline{\mathfrak{a}}^+_\Theta\right)x_0$ is the Θ-*wall* of the chamber gA^+x_0.

A minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ determines a *minimal parabolic subgroup* $P = MAN^+$, where M is the normalizer of A in K. The subgroup P is the normalizer of the algebra $\mathfrak{p}$ via the adjoint action of G:

$$P = \{g \in G | \mathrm{Ad}(g)\,\mathfrak{p} = \mathfrak{p}\}.$$

Similarly, a parabolic subalgebra $\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}$ determines a *parabolic subgroup*

$$P_\Theta = \{g \in G | \mathrm{Ad}(g)\,\mathfrak{p}_\Theta = \mathfrak{p}_\Theta\}.$$

The Weyl group of G is the quotient $W = M/M'$, where $M' := Z_K(A)$ is the centralizer of A in K. It is a finite group that acts simply transitively on the set of Weyl chambers of $\mathfrak{a}$, respecting the incidence relation of walls.

Each parabolic subgroup determines a (compact) *flag manifold* $\mathbb{B}_\Theta = G/P_\Theta$ that is realize as the set $\{\mathrm{Ad}(g)\,\mathfrak{p}_\Theta | g \in G\}$. Since Cartan subalgebras and subgroups are all conjugate, the same is true for the root systems determined by the groups A and gAg^{-1}. So, if λ is a root determined by $\mathfrak{a}$, then $g\lambda$ is the root of $\mathrm{Ad}(g)\,\mathfrak{a}$ defined by the formula

$$g\lambda(H) = \mathrm{Ad}(g) \circ \lambda \circ \mathrm{Ad}\left(g^{-1}\right)(H), \text{ for all } H \in \mathrm{Ad}(g)\,\mathfrak{a}.$$

A parabolic subgroup is said to be of *type* Θ if it is determined by a set of roots of the form $g(\Theta)$. By doing so, the flag manifold $\mathbb{B}_\Theta$ may be viewed as the set of all type Θ parabolic subgroups. In particular, $\mathbb{B} := \mathbb{B}_\emptyset = G/P$ is the set of all minimal parabolic subgroups.

Parabolic subgroups are partially ordered by inclusion, with $P_{\Theta_1} \subset P_{\Theta_2}$ if $\Theta_1 \subset \Theta_2$. Hence there is a natural fibration

$$\widehat{\pi}^{\Theta_1}_{\Theta_2} \;:\; \mathbb{B}_{\Theta_1} \to \mathbb{B}_{\Theta_2}, \quad gP_{\Theta_1} \mapsto gP_{\Theta_2}$$

The set of all Weyl chambers and walls is also partially ordered by inclusion, but with respect to the reverse order: $\overline{\mathfrak{a}}_{\Theta_2} \subset \overline{\mathfrak{a}}_{\Theta_1}$ if $\Theta_1 \subset \Theta_2$.

3. Ideal boundaries and symmetric spaces

The concept of ideal boundary was first introduced by P. Eberlein and B. O'Neill [3]. The approach adopted here is the one found in [2]. Although this concept may be defined for every metric space, this general definition is highly non geometric and give us not much intuition to work with. So, we will restrict ourselves to a geometric definition that holds for a sufficiently wide family of spaces, the so called Hadamard spaces.

3.1. Ideal boundaries of Hadamard spaces.

We start this section with the definition of *Comparision Inequalities of Alexandrov and Topogonov*, shortly denoted by CAT inequalities:

Let $\mathcal{X}$ be a geodesic metric space and $\mathcal{X}(\epsilon)$ a surface of constant curvature ϵ, that is, a sphere of radius $\frac{1}{\epsilon}$ when $\epsilon > 0$, a Euclidean plane when $\epsilon = 0$ or an hyperbolic plane with curvature ϵ when $\epsilon < 0$. Given a geodesic triangle $\triangle(x, y, z)$, with vertices x, y and z in $\mathcal{X}$, we can construct a *comparison triangle* $\widetilde{\triangle}(\widetilde{x}, \widetilde{y}, \widetilde{z})$ in $\mathcal{X}(\epsilon)$ having sides of the same length as $\triangle$ (just taking the care that, when $\epsilon > 0$, $\triangle$ has sides no longer than $\dfrac{\pi}{2\epsilon}$). We say that $\mathcal{X}$ satisfies inequality $\mathrm{CAT}(\epsilon)$ if the following condition holds:

For every triangle $\triangle(x, y, z) \subset \mathcal{X}$ and every point a in the segment $\overline{xy}$ we have that $d(z, a) \leq d_\epsilon(\widetilde{z}, \widetilde{a})$, where $\widetilde{a}$ is the corresponding point in the segment $\overline{\widetilde{x}\widetilde{y}}$ of the comparison triangle $\widetilde{\triangle}$ and d_ϵ is the metric in $\mathcal{X}(\epsilon)$. We note that Riemannian manifolds with curvature bounded from above by ϵ satisfy $\mathrm{CAT}(\epsilon)$.

A *Hadamard space* (manifold) is a simply connected geodesic metric space (manifold) satisfying $\mathrm{CAT}(0)$.

We will now define the ideal boundary of a Hadamard space $(\mathcal{X}, d(\cdot,\cdot))$.

Two geodesic rays $\gamma, \beta : \mathbb{R}^+ \longrightarrow \mathcal{X}$ are said to be *asymptotic* if there is a constant $a \geq 0$ such that $d(\gamma(t), \beta(t)) \leq a$, for every $t \geq 0$. This defines an equivalence relation on the set of all geodesic rays in $\mathcal{X}$. We call the set of equivalence classes of asymptotic geodesic rays the *ideal boundary* of $\mathcal{X}$. We denote this space by $\partial_\infty \mathcal{X}$ and the equivalence class determined by γ we denote by $\gamma(\infty)$.

Ther is a natural metric on the ideal boundary:

1. Given $\eta, \xi \in \partial_\infty \mathcal{X}$ and $x \in \mathcal{X}$, one can prove that there are representative geodesic rays γ, β with $\gamma(0) = \beta(0) = x$ and $\gamma(\infty) = \eta, \beta(\infty) = \xi$. The

function $\frac{1}{t} d(\gamma(t), \beta(t))$ is a bounded convex function, hence its limit exists and we define $d_l(\eta, \xi) = \lim_{t \to \infty} \frac{1}{t} d(\gamma(t), \beta(t))$

If $\mathcal{X}$ is a Hadamard differentiable manifold, we can define two more metrics:

2. Given $\eta, \xi \in \partial_\infty \mathcal{X}$ and $x \in \mathcal{X}$, we choose as before representatives γ, β with $\gamma(0) = \beta(0) = x$ and $\gamma(\infty) = \eta, \beta(\infty) = \xi$ and define$\angle_x(\eta, \xi)$ to be the angle between the geodesics γ and β at x. Then we define the *Tits metric* by

$$d_T(\eta, \xi) = \sup_{x \in \mathcal{X}} \angle_x(\eta, \xi)$$

3. Since $\mathcal{X}$ is simply connected, given any class $\gamma(\infty)$ and any point $x_0 \in \mathcal{X}$ there is one and only one geodesic ray $\beta : \mathbb{R}^+ \longrightarrow \mathcal{X}$ with $\beta(0) = x_0$ and $\beta(\infty) = \gamma(\infty)$, so that we can identify $\partial_\infty \mathcal{X}$ with the unit tangent sphere and give it the usual metric of a unit sphere. We shall denote this metric by $d_S(\cdot, \cdot)$.

The metrics d_l and d_T do not only determines the same topology, but are surprisingly related [2] by the same relation we find between the extrinsic and the intrinsic metric of a unit sphere embedded in $\mathbb{R}^n$

$$d_l(\eta, \xi) = 2 \sin\left(\frac{d_T(\eta, \xi)}{2}\right)$$

while the metric d_S usually defines a different topology. These topologies coincide only when $\mathcal{X}$ is a Euclidean space. On the other hand, $(\partial_\infty \mathcal{X}, d_l)$ is a discrete metric space when the curvature of $\mathcal{X}$ is bounded from above by a constant $C < 0$, whereas $(\partial_\infty \mathcal{X}, d_S)$ is never discrete. The discreetness of $(\partial_\infty \mathcal{X}, d_l)$ reflects the fact that we can find a geodesic asymptotic to any given pair of geodesic rays.

Let $\mathcal{X}$ be a Hadamard manifold. We fix a point $x_0 \in \mathcal{X}$ and for a given $\eta \in \partial_\infty \mathcal{X}$ we choose the unique geodesic ray $\eta(s)$ such that $\eta(0) = x_0$ and $\eta(\infty) = \eta$. Given a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $\mathcal{X}$, consider the sequence of geodesic rays $(\eta_n(s))_{n=1}^\infty$ such that $\eta_n(0) = x_0$ and $\eta_n(d(x_0, x_n)) = x_n$. We say that x_n converges to η if $\lim_{n \to \infty} d(x_0, x_n) = \infty$ and $\lim_{n \to \infty} \eta_n'(0) = \eta'(0)$, or equivalently, if

$$\lim_{n \to \infty} \eta_n(\infty) = \eta$$

in $(\partial_\infty \mathcal{X}, d_S)$. This defines a topology on $\overline{\mathcal{X}} := \mathcal{X} \cup \partial_\infty \mathcal{X}$ that coincides with the metric topology on $\mathcal{X}$ and the sphere metric d_S in $\partial_\infty \mathcal{X}$, and such that $\partial_\infty \mathcal{X}$ is closed. The convergence from points of $\mathcal{X}$ to a point in $\partial_\infty \mathcal{X}$ is defined as above. This turns $\mathcal{X}$ into a compact topological space.

The rest of this section will be devoted to the study of the ideal boundary of a symmetric space $\mathcal{X} = G/K$ of non-compact type and real rank at least 2.

For a subset $\mathcal{C} \subset \mathcal{X}$, we define its ideal boundary $\partial_\infty \mathcal{C} := \partial \mathcal{C} \cap \partial_\infty \mathcal{X}$, where $\partial \mathcal{C}$ stands for the usual boundary in $\overline{\mathcal{X}}$. If $\mathcal{C}$ is convex, then

$$\partial_\infty \mathcal{C} = \{\eta(\infty) \,|\, \eta(s) \text{ is a geodesic ray contained in } \mathcal{C}\}.$$

Since every Weyl chamber $B_g = g\overline{A}^+ x_0$ is convex, we have that

$$\begin{aligned} g\overline{A}^+(\infty) &= \partial_\infty \left(g\overline{A}^+ x_0\right) \\ &= \left\{\eta(\infty) \,|\eta(s) = g(\exp sX)\, x_0,\ X \in \mathfrak{a}^+\right\}. \end{aligned}$$

This is called a *Weyl chamber at infinity.* The Weyl chambers at infinity are either equal or disjoint.

Similar definitions hold also for *walls at infinity* and *flats at infinity*, denoted by $gA_\Theta^+(\infty)$ and $gA(\infty)$ respectively. We will consider only *closed* chambers $g\overline{A}^+(\infty)$ and walls at infinity:

$$g\overline{A}_\Theta^+(\infty) = \partial_\infty \left(g\overline{A}_\Theta^+ x_0\right) = \left\{\eta(\infty) \,|\eta(s) = g(\exp sX)\, x_0,\ X \in \mathfrak{a}_\Theta^+\right\}.$$

3.2. The structure of ideal chambers.

In order to prove our results, we must characterize the ideal points of a symmetric space according to their parabolic type. To put it explicitly, we define a map

$$\pi : \partial_\infty \mathcal{X} \to \bigcup_{\Theta \subset \Sigma} G/P_\Theta,$$

as follows: each $\eta \in \partial_\infty \mathcal{X}$ is of the form $\eta(\infty)$ with $\eta(s) = (g \exp sX)\, x_0$ with $|X| = 1$ and $X \in \mathring{\bigcup}_{\Theta\subseteq\Sigma} \mathfrak{a}_\Theta^+$, where we are considering the open chambers and open walls, so that the union is disjoint. So, we associate to η the parabolic subgroup $\pi(\eta) := gP_\Theta g^{-1}$ (where $X \in \mathfrak{a}_\Theta^+$). This association is independent of the choice of g. Alternatively, we could associate to η the (open) Weyl chamber or wall at infinity $gA_{\Theta_0}(\infty)$ that contains it.

We denote by $\partial_\infty^\Theta \mathcal{X}$ the inverse image $\pi^{-1}(G/P_\Theta)$, the set of all Θ-*singular* geodesic rays. We note that $\pi^{-1}(G/P_\emptyset) = \partial_\infty^\emptyset \mathcal{X}$ is an open and dense subset of $\partial_\infty \mathcal{X}$, whenever we consider either the Tits metric or the spherical metric. Also, $\mathring{\bigcup}_{\lambda\in\Sigma} \partial_\infty^{\{\lambda\}} \mathcal{X}$ is open and dense in $\partial_\infty \mathcal{X} \backslash \partial_\infty^\emptyset \mathcal{X}$. In the same way, we find that $\mathring{\bigcup}_{\substack{\Theta\subseteq\Sigma \\ |\Theta|=k}} \partial_\infty^\Theta \mathcal{X}$ is open and dense in $\partial_\infty \mathcal{X} \backslash \left(\mathring{\bigcup}_{\substack{\Phi\subseteq\Sigma \\ |\Phi|<k}} \partial_\infty^\Phi \mathcal{X}\right)$, where $|\Theta|$ is just the cardinality of Θ and $k \leq \mathrm{r}(\mathfrak{g})$. Again, this fact is independent of the metric topology we work with. The projection $\pi : \partial_\infty \mathcal{X} \to \bigcup_{\Theta\subseteq\Sigma} G/P_\Theta$ splits as a set of projections

$$\pi^\Theta : \partial_\infty^\Theta \mathcal{X} \to G/P_\Theta, \quad \Theta \subseteq \Sigma.$$

Remark 3.1. This structure is actually a geometric realization of a spherical Tits building. Shortly, the set of apartments is

$$\mathcal{A} = \{gA(\infty) \,|g \in G\} = \{\text{All flats at infinity}\},$$

and the chambers and walls are given by

$$\Delta = \left\{g\overline{A}_\Theta^+(\infty) \,|g \in G, \Theta \subseteq \Sigma\right\} = \{\text{All chambers and walls at infinity}\}.$$

Since for every pair of chambers or walls at infinity there is a flat at infinity that contains both of them, the adjacency relation is just the usual one defined in the flats at infinity. More details about this structure may be found in [2, Appendix 5].

4. Semigroups and invariant control sets

Let $X = G/L$ be a homogeneous manifold. We denote respectively by cl$D$ and int$D$ the closure and the interior of a subset D (of X or G, to be clearly understood from the context). A set S of diffeomorphisms of M is a *semigroup* if the composition of elements of S (with possible restrictions of domains) is still in S. An *invariant control set* for S (an S-i.c.s.) is a subset $\mathrm{C} \subseteq X$ with $\mathrm{int}(C) \neq \emptyset$ satisfying the conditions:

(i) For all $x \in \mathrm{C}$, $\mathrm{cl}\, Sx = \mathrm{cl}\, \mathrm{C}$,

(ii) C is maximal with property *(i)*.

For the simplicity of the presentation, we assume that G is a semisimple Lie group of non-compact type and S a subsemigroup of G, even if some of the results do not depend on the semisimplicity of G. Regarding the control sets in a compact homogeneous space $X = G/L$ we have the following:

Proposition 4.1. 7, Proposition 2.1 *Let $X = G/L$ be a compact homogeneous space and S a subsemigroup of G with* $\mathrm{int}S \neq \emptyset$. *Let $\mathrm{C} \subset X$ be an S-i.c.s and let $\mathrm{C}_0 = (\mathrm{int}\, S)\, \mathrm{C}$. Then:*

(i) $\mathrm{C}_0 = \mathrm{int}\ (Sx)$ *for all* $x \in \mathrm{C}_0$.

(ii) $S\mathrm{C}_0 \subset \mathrm{C}_0 = Sy = (\mathrm{int}\ S)\, y$ *for all* $y \in \mathrm{C}_0$.

(iii) $\mathrm{C}_0 = \{x \in C | \exists g \in \mathrm{int}\ S \textit{ with } gx = x\}$.

(iv) $\mathrm{C}_0 = \{x \in C | \exists g \in \mathrm{int}\ S \textit{ with } g^{-1}x \in \mathrm{C}\}$.

(v) $\mathrm{cl}\ \mathrm{C}_0 = \mathrm{C}$.

Because of property *(ii)* in the proposition above, C_0 is called the *set of transitivity* of C.

The product MA is a closed subgroup of G. The homogeneous space G/MA may be seen as the set of Weyl chambers in either $\mathfrak{g}$ or Weyl chambers in G with base point at the identity. Alternatively, it may be seen as the choice of a Weyl chamber decomposition in each of the flats gAx_0 of $\mathcal{X}$. Each Weyl chamber $b = gMA$ is conjugate to the base chamber A^+: $b = gA^+g^{-1}$.

We assume throughout the rest of the paper that S has non-empty interior. Then, it has a unique S invariant control set C [7, Theorem 3.1]. If we put

$$\Delta = \left\{ b = gA^+g^{-1} \in G/MA | b \cap \mathrm{int}\ S \neq \emptyset \right\},$$

we have the following:

Theorem 4.2. *[8, Theorem 3.1] Let C be the unique S-i.c.s. in G/P and C_0 be its set of transitivity. Let*

$$p : G/MA \to G/MAN^+$$

be the canonical projection. Then

$$\mathrm{C}_0 = p\,(\Delta)\,.$$

4.1. Ideal Boundaries and invariant control sets.

In this section we consider subsemigroups of a semisimple Lie group of non-compact type G. Without loss of generality, we assume that G has finite center. We also assume that S has non-empty interior and show how to construct invariant control sets in the ideal boundary $\partial_\infty(\mathcal{X})$ of the associated symmetric space $\mathcal{X} = G/K$. This is done simply by considering the ideal boundary of an orbit of S in $\mathcal{X}$, as stated in our main Theorem:

Theorem 4.3. *Let S be a sub-semigroup of a semisimple Lie group G, with non-empty interior. Consider the boundary of an orbit Sx_0 in G/K and let D be the ideal boundary $\partial_\infty(Sx_0)$. Then D is the invariant control set of S. Moreover, let C_Θ denote the unique S-i.c.s. in G/P_Θ and $\mathrm{D}^\Theta = \mathrm{D} \cap \partial_\infty^\Theta(\mathcal{X})$. Then,*

$$\pi^\Theta\left(\mathrm{D}^\Theta\right) = \mathrm{C}_\Theta.$$

We start proving a particular instance of this theorem, the case $\Theta = \emptyset$.

Theorem 4.4. *Let S be a sub-semigroup of a semisimple Lie group G, with non-empty interior. Let C be the unique S-i.c.s. in $G/P = G/P_\emptyset$. Consider the boundary of an orbit Sx_0 in G/K and let $\mathrm{D} = \partial_\infty(Sx_0)$ and $\mathrm{D}^\emptyset = \mathrm{D} \cap \partial_\infty^\emptyset(\mathcal{X})$. Then,*

$$\pi^\emptyset\left(\mathrm{D}^\emptyset\right) = \mathrm{C}.$$

Actually, we could consider the orbit of any point $gx_0 \in \mathcal{X}$ instead of the orbit Sx_0:

Proposition 4.5. *For any two points $gx_0, hx_0 \in \mathcal{X}$, the ideal boundaries of their S-orbits coincide, that is, $\partial_\infty(Sgx_0) = \partial_\infty(Shx_0)$.*

Proof. Given $\eta \in \partial_\infty(Sgx_0)$, there is a sequence $(s_i)_{i=1}^\infty, s_i \in S$ such that $\lim_{i\to\infty} s_i g x_0 = \eta$, the limit considered in $\overline{\mathcal{X}} = \mathcal{X} \cup \partial_\infty x$. But

$$d(s_i g x_0, s_i h x_0) = d(gx_0, hx_0)$$

is bounded, so that $\lim_{i\to\infty} s_i h x_0 = \eta$. ■

We note that $\mathrm{D} := \partial_\infty(Sgx_0) = \mathrm{cl}_{\overline{\mathcal{X}}}(Sx_0) \setminus \mathcal{X}$.

The proof of Theorem 4.4 is obtained from the next two lemmas.

Lemma 4.6. *With the notation above defined,*

$$\mathrm{C}_0 \subseteq \pi^\emptyset\left(\mathrm{D}^\emptyset\right).$$

Proof. Let $\widetilde{b} \in \mathrm{C}_0$. By Theorem 4.2 there is $b \in \Delta$ such that $p(b) = \widetilde{b}$ and an element $g \in b \cap \mathrm{int} S$. We consider an open ball $B_r(g) \subset \mathrm{int} S$. For each $h \in b \cap B_r(g)$, we have some $Y_h \in \mathfrak{g}$ with $\|Y_h\| = 1$ such that $hx_0 = \exp(t_h Y_h)x_0$. Then, $h^n = \exp(nt_h Y_h) \in b \cap \mathrm{int} S$ for every $n \geq 1$ so that $h^n x_0 \subset Sx_0$. Moreover, if we put $\eta_h(t) = \exp(tY_h)x_0$, we find that

$$\lim_{n\to\infty} h^n x_0 = \lim_{n\to\infty} \eta_h(nt_h) = \eta_h(\infty).$$

Since b may be considered as an open Weyl chamber, we get that $\eta_h(\infty) \in \mathrm{D}^\emptyset = \mathrm{D} \cap \partial_\infty^\emptyset(\mathcal{X})$ and, from the fact that $h^n \in b$ we deduce that $\pi^\emptyset(\eta_h(\infty)) = p(b) = \widetilde{b}$. ■

Lemma 4.7. *With the notation above defined, let* $\operatorname{int}\left(\mathrm{D}^{\varnothing}\right)$ *stands for the intersection of* $\mathrm{D}^{\varnothing}$ *with the interior of* D *as a subset of* $\partial_\infty \mathcal{X}$. *Then,*

$$\pi^{\varnothing}\left(\operatorname{int}\left(\mathrm{D}^{\varnothing}\right)\right) \subseteq \overline{\mathrm{C}_0}.$$

Proof. First of all we notice that $\mathrm{D} := \partial_\infty (Sx_0) = \partial_\infty ((\operatorname{int} S) x_0)$. Indeed, if h_n is a point in the boundary of S, there is an interior point g_n of S which distance (in G, with any left invariant metric) from h_n at most 1, so that the sequences $h_n x_0$ and $g_n x_0$ have bounded distance in $\mathcal{X}$ and one of them converges to an ideal point if and only if the other one converges to the same ideal point.

Let $\eta = \eta(\infty) \in \operatorname{int}\left(\mathrm{D}^{\varnothing}\right)$, and put $\eta(t) = \exp(tY_0)x_0$, with $\|Y_0\| = 1$, with $Y_0 \in \mathfrak{x}$. Since $\eta \in \mathrm{D}$, there is a sequence $(g_n)_{n=1}^{\infty}, g_n \in S$ such that $\lim_{n\to\infty} g_n x_0 = \eta$. If we put $g_n = \exp t_n Y_n$, with $Y_n \in \mathfrak{x}$ and $\|Y_n\|$, it means that $t_n \to \infty$ and the angle $\theta_n = \sphericalangle(Y_0, Y_n)$ between Y_0 and Y_n goes to 0. Since $\partial_\infty (Sx_0) = \partial_\infty ((\operatorname{int} S) x_0)$, we may assume that $g_n \in \operatorname{int} S$. The same reasoning used in Lemma 4.7 implies that $\eta_n(t) = \exp(tY_n)x_0$ defines an element $\eta_n := \eta_n(\infty)$ such that $\pi^{\varnothing}(\eta_n) \in \mathrm{C}_0$.

On the other hand, since η is an interior point of D, considering the spherical metric in $\partial_\infty \mathcal{X}$, the sets

$$B_{\theta,r}(\eta) = \{\exp(tY)x_0 | t > r; \sphericalangle(Y_0, Y) < \theta\}, r > 0, \theta > 0$$

constitute a base for the neighborhoods of η in $\overline{\mathcal{X}}$. Since $\lim_{n\to\infty} g_n x_0 = \eta$, we find that $\lim_{n\to\infty} Y_n = Y$ so that

$$\lim_{n\to\infty} \eta_n = \eta_0.$$

But the projection $\pi^{\varnothing} : \partial_\infty^{\varnothing}(\mathcal{X}) \to G/P$ is continuous and this implies that $\pi^{\varnothing}(\eta) \in \overline{\mathrm{C}_0}$. ■

Now we can prove Theorem 4.4.

Proof. [Proof (of Theorem 4.4)] Since $\overline{\mathrm{C}_0} = \mathrm{C}$, Lemma 4.6 assures that

$$\overline{\mathrm{C}_0} = \mathrm{C} \subseteq \pi^{\varnothing}\left(\mathrm{D}^{\varnothing}\right).$$

Since $\pi^{\varnothing}$ is continuous, we find that $\pi^{\varnothing}\left(\overline{\mathrm{A}}\right) \subseteq \overline{\pi^{\varnothing}(\mathrm{A})}$ for every subset $\mathrm{A} \subseteq \partial_\infty(\mathcal{X})$, the closures being taken, respectively, in $\partial_\infty(\mathcal{X})$ and $G/P_{\varnothing}$. So, by Lemma 4.7, we find that

$$\pi^{\varnothing}\left(\overline{\operatorname{int}\left(\mathrm{D}^{\varnothing}\right)}\right) = \pi^{\varnothing}\left(\left(\mathrm{D}^{\varnothing}\right)\right) \subseteq \overline{\overline{\mathrm{C}_0}} = \overline{\mathrm{C}} = \mathrm{C}$$

and therefore

$$\pi^{\varnothing}\left(\mathrm{D}^{\varnothing}\right) = \mathrm{C}.$$

■

Now we can prove our main theorem:

Proof. [Proof of Theorem 4.3] We consider now another parabolic subgroup P_Θ, $\Theta \subseteq \Sigma$. Again, the compactness of the manifold G/P_Θ assures the existence of a unique S-invariant control set $\mathrm{C}_\Theta \subseteq G/P_\Theta$. It is known that $\mathrm{C}_\Theta = \rho_\Theta(\mathrm{C})$, where $\rho_\Theta : G/P \to G/P_\Theta$ is the natural projection.

The closed Weyl chamber $\overline{\mathfrak{a}}^+$ is a cone generated by a family $\left\{H_\alpha^\perp\right\}_{\alpha\in\Sigma}$ of unit vectors where $\beta\left(H_\alpha^\perp\right) = 0$ whenever $\beta \neq \alpha$. It means that $H_\alpha^\perp$ is contained in the intersection of all the hyperplanes orthogonal to H_β for every root $\beta \neq \alpha$. Then, the open Weyl chamber may be described as

$$\mathfrak{a}^+ = \left\{\sum_{\alpha\in\Sigma} c_\alpha H_\alpha^\perp | c_\alpha > 0 \text{ for every } \alpha \in \Sigma\right\}.$$

Then, we may define a projection $p_\Theta : \mathfrak{a}^+ \to \mathfrak{a}_\Theta^+$ by

$$p_\Theta\left(\sum_{\alpha\in\Sigma} c_\alpha H_\alpha^\perp\right) = \sum_{\alpha\in\Sigma\setminus\Theta} c_\alpha H_\alpha^\perp.$$

This projection is not orthogonal (with respect to the Cartan-Killing form) but it is clearly surjective. Moreover, the diagram is clearly commutative.

$$\begin{array}{ccc} \partial_\infty^\varnothing(\mathcal{X}) & \xrightarrow{\pi^\varnothing} & G/P_\varnothing \\ \downarrow p_\Theta & & \downarrow \rho_\Theta \\ \partial_\infty^\Theta(\mathcal{X}) & \xrightarrow{\pi^\Theta} & G/P_\Theta \end{array}$$

Diagram 1

Since $\partial_\infty^\varnothing(\mathcal{X})$ is open and dense in $\bigcup_{\varnothing\neq\Theta\subseteq\Sigma} \partial_\infty^\Theta(\mathcal{X})$, the same is true for the ideal boundary of an orbit of the semigroup: $\partial_\infty^\varnothing(Sx_0)$ is open and dense in $\partial_\infty(Sx_0)$ and its boundary is contained in $\bigcup_{\varnothing\neq\Theta\subseteq\Sigma} \partial_\infty^\Theta(Sx_0)$, so that

$$\partial_\infty^\Theta(Sx_0) \subseteq p_\Theta\left(\partial_\infty^\varnothing(Sx_0)\right)$$

From this and the fact that the Diagram 1 commutes, we find that

$$\begin{aligned} \pi^\Theta\left(\partial_\infty^\Theta(Sx_0)\right) &\subseteq \pi^\Theta\left(p_\Theta\left(\partial_\infty^\varnothing(Sx_0)\right)\right) \\ &= \rho_\Theta \circ \pi^\varnothing\left(\partial_\infty^\varnothing(Sx_0)\right) \\ &= \rho_\Theta(\mathrm{C}) \\ &= \mathrm{C}_\Theta. \end{aligned}$$

Moreover, given $\widetilde{b} \in (\mathrm{C}_\Theta)_0$ (the set of transitivity of C_Θ), by definition there is an element $g \in \operatorname{int}(S)$ such that $g\widetilde{b} = \widetilde{b}$. As in the proof of Lemma 4.6, we find that $g^n x_0$ converges (in $\overline{\mathcal{X}}$) to a point $\eta = \eta(\infty) \in \partial_\infty^\Theta(Sx_0)$. The theorem follows by taking the appropriates closures, as we did in the proof of Theorem 4.4. ■

Example 4.8. If G is semisimple Lie group of non-compact type and real rank 1, then the Weyl chambers are one-dimensional, so that their asymptotic images in $\partial_\infty \mathcal{X}$ have dimension 0, that is, they are just points. It follows that the projection $\pi : \partial_\infty \mathcal{X} \to G/P$ is actually a bijection so that the ideal boundary and the Furstenberg boundary may be identified.

As a special case, we consider the group $G = SL(2,\mathbb{R})$ and the semigroup $S = SL_+(2,\mathbb{R})$ of matrices with non-negative entries. The symmetric space is just the hyperbolic plane. We consider the Lobatchevsky model, the halfplane $\mathbb{H}^2 = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. Its ideal boundary is just the set $\{z \in \mathbb{C} | \mathrm{Im}(z) = 0\} \cup \{\infty\}$. The group G acts on $\mathbb{H}^2$ by *Möbius transformations*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to A(z) := \frac{az+b}{cz+d}$$

If $A \in SL_+(2,\mathbb{R})$, the image of the point $z = i$

$$\begin{aligned} \frac{ai+b}{ci+d} &= \frac{(ac+bd)+i(ad-bc)}{c^2+d^2} \\ &= \frac{ac+bd+i}{c^2+d^2} \end{aligned}$$

has non-negative real part, so that $\partial^- \infty(Si) \subset \{0 \le t \in \mathbb{R}\} \cup \infty$.
On the other hand, for any $\lambda \ge 0, t > 0$, the matrices

$$A_t = \begin{pmatrix} \frac{1}{t} + \lambda t & \lambda t \\ t & t \end{pmatrix}$$

belong to $SL_+(2,\mathbb{R})$ and $lim_{t\to\infty} A_t(i) = (\lambda + \frac{1}{2t^2}) + \frac{i}{2t^2} = \lambda$, so that $\{0 \le t \in \mathbb{R}\} \subset Si$ The ideal point ∞ is attained as the limit of the orbit of the one-parameter subgroup of hyperbolic isometries

$$B_t = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}.$$

Example 4.9. We consider the group $SL(3,\mathbb{R})$. The associated symmetric space $\mathcal{X} = Sl(3,\mathbb{R})/SO(3,\mathbb{R})$ may be identified with the space of all 3×3 symmetric matrices, such that $vBv^T > 0$ for every $0 \ne v \in \mathbb{R}^3$. This identification is obtained from the action of $Sl(3,\mathbb{R})$ on $\mathcal{X}$ defined by $A \cdot x = AxA^T$. This space is 5-dimensional and its ideal boundary is thus a 4-dimensional sphere. This boundary may be identified with a set of 2-dimensional strips in $\mathbb{R}^3$, as in [2]. Since $\mathcal{X} = Sl(3,\mathbb{R})/SO(3,\mathbb{R})$ is a Hadamard manifold, its ideal boundary is determined by the asymptotic image of geodesic rays starting at the point $x_0 = \mathrm{Id}_{3\times 3}$ (section 3.). The tangent vector $\eta\prime(0)$ of a unit speed geodesic ray $\eta : [0,\infty] \to \mathcal{X}$ is a symmetric matrix A with trace 0 and norm $\|A\|^2 = \mathrm{Tr}(AA^T) = 1$. Since A is symmetric, it has 3 eigenvalues a, b and c. The norm 1 and trace 0 conditions imply that $a^2 + b^2 + c^2 = 1$ and $a + b + c = 0$. We lose no generality by assuming that $a \ge b \ge c$. Since A is symmetric, it has orthogonal eigenvectors v_a, v_b

and v_c. We associate to this ray the strip $s_\eta := (\mathbb{R}v_a + [-r, r]v_b) \subset \mathbb{R}^3$, where $r := (b-c)/(a-b) \in [0, \infty]$ equals 0 if $b = c$ and ∞ if $a = b$. The map $\eta \mapsto s_\eta$ is well defined and a bijection onto the space

$$\mathrm{Str} := \{\mathbb{R}v_1 + [-r, r]v_2 \subset \mathbb{R}^3 |\ \|v_1\| = \|v_2\| = 1, v_1 \text{ orthogonal to } v_2, r \in [0, \infty]\}$$

of all strips in $\mathbb{R}^3$. We note that, $\{v_a, v_b, v_c\}$ being an orthogonal base, at least one of the vectors $\{\pm v_a, \pm v_b, \pm v_c\}$ is contained in the positive orthant

$$\mathbb{R}^3_{(+,+,+)} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\, x_1, x_2, x_3 \geq 0\}$$

We consider now the set

$$\mathfrak{sl}_+(3, \mathbb{R}) := \{(a_{ij})^3_{i,j=1} \in \mathfrak{sl}(3, \mathbb{R}) |\, a_{ij} \geq 0 \text{ for } i \neq j\}$$

of all matrices with nonnegative entries (outside the diagonal) and trace 0. It is clearly a closed convex cone with non empty interior. So, its image under the exponential map generates a semigroup S with non empty interior [6]. We note that S is contained (but not equal) to the semigroup $SL_+(3, \mathbb{R})$ of the matrices with nonnegative entries.

We claim that the invariant control set of S in $\partial_\infty \mathcal{X}$, is contained in the set of strips

$$\mathcal{S}_{(+,+,+)} := \{(\mathbb{R}v_a + [-r, r]v_b) \in \mathrm{Str} |\, : v_a \in \mathbb{R}^3_{(+,+,+)}\}.$$

By definition, given $\eta \in \partial_\infty(Sx_0)$, there is a sequence $A_n \in S$ such that $A_n x_0$ converges to η. We start proving there is an element $B \in S$ such that $\eta = \lim_{n \mapsto \infty} B^n x_0$. First of all we note that, if we repeat terms of the sequence (e.g. considering $A_1, A_1, A_2, A_2, \ldots$ instead of $A_1, A_2, \ldots$) or delete terms of the sequence (i.e., consider subsequences of the given one), we get a sequence A'_n such that $\lim_{n \mapsto \infty} A_n x_0 = \lim_{n \mapsto \infty} A'_n x_0$. However, the first substitution (adding repeated terms) may increase the rate of growth of $d(x_0, A_n x_0)$, while the second substitution may decrease it. So, we lose no generality by assuming that $\lim_{n \mapsto \infty} \frac{1}{n} d(x_0, A_n x_0) = k$, for any given $k \geq 0$. We consider now a vector $X \in \mathfrak{sl}(3, \mathbb{R})$ with $\|X\| = 1$ such that $\lim_{t \mapsto \infty} \exp(tX)x_0 = \eta$. If $\eta \in \mathrm{int}(\partial_\infty(Sx_0))$, we find that $B := \exp(t_0 X) \in \mathrm{int}(S)$ for some $t_0 > 0$. We let k_0 be the distance $d(x_0, Bx_0)$. Since $B^n = \exp(n \cdot (t_0 X))$, we find that $\lim_{n \mapsto \infty} B^n x_0 = \lim_{n \mapsto \infty} A_n x_0 = \eta$ and $B^n \in \mathrm{int}(S)$, for every $n \in \mathbb{N}$. It follows that we may consider only elements $\eta \in \partial_\infty(Sx_0)$ such that $\eta = \lim_{n \mapsto \infty} B^n x_0$, for some element $B \in \mathrm{int}(S)$. Moreover, since the set of matrices in S with different eigenvalues (the split regular elements) is open and dense in S, we loose no generality by restricting ourselves to matrices $B \in S$ with different eigenvalues.

We consider a split regular matrix $B \in S$ and let $\eta := \lim_{n \mapsto \infty} B^n x_0$. Let $a > b > c$ be the eigenvalues of B, with corresponding eigenvectors v_a, v_b and v_c. We want to prove that v_a or its opposite $-v_a$ is in the orthant $\mathbb{R}^3_{(+,+,+)}$.

If we look at the projective space $\mathbb{P}(\mathbb{R}^3)$ over $\mathbb{R}^3$, the sequence $(B^n)_{n \in \mathbb{N}}$ determines a quasi-projective transformation in the sense of Goldsheid and Margulis [4]. It follows that every line $\mathbb{R}v \in (\mathbb{P}(\mathbb{R}^3) - \mathbb{P}(\mathbb{R}v_b + \mathbb{R}v_c))$ is attracted by the line $\mathbb{R}v_a$, since a is the greatest eigenvalue, that is,

$$\lim_{n\to\infty} B^n(\mathbb{R}v) = \mathbb{R}v_a$$

the limit being taken in $\mathbb{P}(\mathbb{R}^3)$ [4, Corollary 2.4]. However, any matrix B with nonnegative entries leaves the positive orthant invariant (actually, S is the compression semigroup of this orthant [7]). It follows that the eigenvector v_a (or its opposite $-v_a$) must be contained in this orthant and so $\partial_\infty(Sx_0) \subseteq \mathcal{S}_{(+,+,+)}$.

We note that the inclusion above is a strict one, that is, not every strip in $\mathcal{S}_{(+,+,+)}$ is contained in $\partial_\infty(Sx_0)$. Indeed, given an orthogonal base $\{v_a, v_b, v_c\}$, and $a, b, c \in \mathbb{R}$, the unique symmetric matrix that has eigenvalues a, b, c with corresponding eigenvectors (relative to a fixed orthogonal base of unit length vectors) $\{v_a, v_b, v_c\}$ is the matrix

$$B := V \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} V^T$$

where V has columns $\{v_a, v_b, v_c\}$. If the coordinates of the eigenvectors are given by

$$v_a = (x_1, y_1, z_1), v_b = (x_2, y_2, z_2), \text{ and } v_c = (x_3, y_3, z_3),$$

the condition that B has nonnegative entries outside the diagonal is equivalent to the inequalities

$$* \begin{cases} x_1y_1a + x_2y_2b + x_3y_3c & \geq & 0 \\ x_1z_1a + x_2z_2b + x_3z_3c & \geq & 0 \\ y_1z_1a + y_2z_2b + y_3z_3c & \geq & 0. \end{cases}$$

Since $r = \frac{b-c}{a-b}$ and $a + b + c = 0$, this system of equations is equivalent to

$$** \begin{cases} x_1y_1(r+2) + x_2y_2(r-1) + x_3y_3(2r+1) & \geq & 0 \\ x_1z_1(r+2) + x_2z_2(r-1) + x_3z_3(2r+1) & \geq & 0 \\ y_1z_1(r+2) + y_2z_2(r-1) + y_3z_3(2r+1) & \geq & 0. \end{cases}$$

If we take for example the pair of orthogonal vectors $v_a := \frac{1}{\sqrt{3}}(1,1,1)$ and $v_b := \frac{1}{\sqrt{14}}(3,-1,-2)$, then, by direct calculations, we can show that those inequalities are satisfied only for $r \leq \frac{1}{10}$, that is, the strip $(\mathbb{R}v_a + [-r, r]v_b) \in \partial_\infty(Sx_0)$ if and only if $r \leq \frac{1}{10}$. So, not every strip in $S_{(+,+,+)}$ is in the ideal boundary of Sx_0. However, since the projections π^Θ are surjective, we find that, for any choice of unit length orthogonal vectors v_a, v_b, with v_a in the first orthant, the system $(**)$ of inequalities has a solution. Actually, this system is the precise description of the strips attained in $\partial_\infty(Sx_0)$.

Remark 4.10. As we saw in remark 3.1, the ideal boundary $\partial_\infty(\mathcal{X})$ may be considered, in a canonical way as the geometric realization of a spherical Tits Building. With this structure, each connected component of $\partial_\infty^\Theta(\mathcal{X})$ is a cell of type Θ in the building structure. This fact encourages the investigation of invariant control sets semigroups of algebraic groups and (B, N) pairs.

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