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Some Constructions in the Theory of Locally Finite Simple Lie Algebras

Yuri Bahturin and Georgia Benkart *

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Abstract. Some locally finite simple Lie algebras are graded by finite (possibly nonreduced) root systems. Many more algebras are sufficiently close to being root graded that they still can be handled by the techniques from that area. In this paper we single out such Lie algebras, describe them, and suggest some applications of such descriptions.

1. Introduction

In this work we will give an alternative description of the diagonal direct limits of classical simple Lie algebras. These direct limit algebras have appeared in several papers [3, 6, 7, 8] and can be defined as follows. Suppose we have a countable directed family of classical simple Lie algebras, and $\mathcal{L}$ is the limit algebra. We can always restrict to the case where all the algebras are of the same kind (special linear, symplectic, or orthogonal), say $X \in \{\mathfrak{sl}, \mathfrak{sp}, \mathfrak{so}\}$, and view the component algebras as $X(U)$, $X(V)$, etc., for appropriate vector spaces U, V, etc. The distinguishing feature of diagonal direct limits is the following condition. When $\varphi : X(U) \to X(V)$ is a structure homomorphism of the directed family, then V is an $X(U)$-module, and it should have a direct sum decomposition into irreducible $X(U)$-submodules of the form

$$V = U^{\oplus \ell} \oplus (U^*)^{\oplus r} \oplus \mathbb{K}^{\oplus z},$$

where the multiplicities ℓ, r, z are nonnegative integers and $\mathbb{K}$ is the underlying field, which will be assumed to be algebraically closed of characteristic zero throughout the paper. (Note when U is isomorphic to its dual module U^*, we assume that $r = 0$.) It is an easy remark, in fact in [3], that for diagonal direct limits the directed family of algebras can be chosen to form a chain

$$\mathfrak{g}^{(1)} \xrightarrow{\varphi_1} \mathfrak{g}^{(2)} \xrightarrow{\varphi_2} \ \ldots \ \to \mathfrak{g}^{(i)} \xrightarrow{\varphi_i} \mathfrak{g}^{(i+1)} \to \ \ldots\ , \tag{1}$$

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where assuming $\mathfrak{g}^{(i)} = X(V^{(i)})$, we have the decomposition of $V^{(i+1)}$ as a $\mathfrak{g}^{(i)}$-module given by

$$V^{(i+1)} = (V^{(i)})^{\oplus \ell_i} \oplus \big((V^{(i)})^*\big)^{\oplus r_i} \oplus \mathbb{K}^{\oplus z_i}.$$

To obtain the decomposition of $V^{(i+2)}$ over $\mathfrak{g}^{(i)}$, it is necessary to take the product of terms in the structure sequence of triples $\{t_i = (\ell_i, r_i, z_i) \mid i = 1, 2, \dots\}$:

$$\begin{aligned} t_i * t_{i+1} &= (\ell_i, r_i, z_i) * (\ell_{i+1}, r_{i+1}, z_{i+1}) \\ &= (\ell_i\ell_{i+1} + r_ir_{i+1}, \ell_ir_{i+1} + r_i\ell_{i+1}, (\ell_{i+1} + r_{i+1})z_i + z_{i+1}). \end{aligned} \tag{2}$$

It is then easy to see that the sequences $\{\ell_i + r_i\}$, $\{\ell_i - r_i\}$ are multiplicative. The multiplicativity of $\{\ell_i + r_i\}$ was first used in [4] to produce uncountable families of pairwise nonisomorphic simple Lie algebras of special linear type over arbitrary fields. Many diagonal direct limits were classified in [18], and the final classification of the algebras in question was accomplished in [8] in terms of the multiplicative and limit properties of their structure sequences.

As a particular case, the authors of [8] recover the classification of the direct limits of finite-dimensional matrix algebras. As V is the unique irreducible module for End$V$ when V is finite-dimensional, here we can restrict our attention to sequences $\underline{\mathfrak{p}} = \{(p_i, q_i)\}$, where $q_i \geq 0$ and $p_i \geq 1$ for all $i \geq 0$. Thus, to initiate the sequence, there is some vector space $V^{(0)} = V = \mathbb{K}^{\oplus p_0}$. Then $V^{(i+1)} = (V^{(i)})^{\oplus p_i} \oplus \mathbb{K}^{\oplus q_i}$, and

$$(p_i, q_i) * (p_{i+1}, q_{i+1}) = (p_ip_{i+1}, p_{i+1}q_i + q_{i+1}).$$

Each constituent algebra has the form $\mathcal{E}^{(i)} = \text{End } V^{(i)}$. If we represent each $\mathcal{E}^{(i)}$ in matrix form, then the structure embedding $\varphi_i : \mathcal{E}^{(i)} \to \mathcal{E}^{(i+1)}$ is given by

$$A \mapsto \text{diag}(\underbrace{A, \dots, A}_{p_i}, \underbrace{0, \dots, 0}_{q_i}). \tag{3}$$

Let us denote the limit algebra $\mathcal{E} = \varinjlim \mathcal{E}^{(i)}$ by $\mathcal{E}(\underline{\mathfrak{p}})$ when we want to emphasize its dependence on $\underline{\mathfrak{p}}$. We now enumerate some properties of the algebras $\mathcal{E}(\underline{\mathfrak{p}})$.

(1) $\mathcal{E} = \mathcal{E}(\underline{\mathfrak{p}})$ has an involution, namely the limit $a \mapsto a^\tau$ of the standard transpose map. Indeed, if we represent each $\mathcal{E}^{(i)}$ in matrix form as in (3), then $\varphi_i(A^\tau) = \big(\varphi_i(A)\big)^\tau$. Thus the ordinary transpose map is compatible with the direct limit and therefore defines a "transpose" map $\tau : \mathcal{E} \to \mathcal{E}$, with the usual property $(ab)^\tau = b^\tau a^\tau$.

(2) $\mathcal{E}$ has a trace map defined as follows. Suppose $a \in \mathcal{E}$, say $a = A \in \mathcal{E}^{(i)}$. Set

$$\mathfrak{t}(a) = \frac{1}{p_0 \cdots p_{i-1}} \text{tr} A.$$

If also $a = B \in \mathcal{E}^{(j)}$, $j > i$, then we have (after renumbering the diagonal blocks)

$$B = \text{diag}(\underbrace{A, \dots, A}_{p_i \cdots p_{j-1}}, 0, \dots, 0).$$

It follows that

$$\frac{1}{p_0\cdots p_{j-1}}\mathrm{tr}B = \frac{p_i\cdots p_{j-1}}{p_0\cdots p_{j-1}}\mathrm{tr}A = \frac{1}{p_0\cdots p_{i-1}}\mathrm{tr}A,$$

and so the definition of $\mathfrak{t}(a)$ whether given by A or B agrees. Thus $\mathfrak{t}(a)$ is well-defined and satisfies the usual trace properties,

$$\mathfrak{t}(ab) = \mathfrak{t}(ba) \tag{4}$$
$$\mathfrak{t}(\kappa a+\lambda b) = \kappa\mathfrak{t}(a)+\lambda\mathfrak{t}(b), \qquad \kappa,\lambda\in\mathbb{K}. \tag{5}$$

(4) The natural module for $\mathcal{E}$ is defined in the following way. Using the decomposition $V^{(i+1)} = (V^{(i)})^{\oplus p_i}\oplus\mathbb{K}^{\oplus q_i}$, we define the map $\eta_i : V^{(i)}\to V^{(i+1)}$ by

$$\eta_i(v) = (\underbrace{v,\dots,v}_{p_i},\underbrace{0,\dots,0}_{q_i}).$$

Then we can form the limit space $V = \varinjlim V^{(i)}$ using the η_i as the structure maps. Now, for $a\in\mathcal{E}$ and $\beta\in V$, define $a*\beta$ by setting $a*\beta = Av$ where $a\in A\in\mathcal{E}^{(i)}$, $\beta = v\in V^{(i)}$ for an appropriate i. If also $a = B\in\mathcal{E}^{(j)}$, $\beta = w\in V^{(j)}$ for some $j>i$ then $B = \mathrm{diag}(\underbrace{A,\dots,A}_{p_i\cdots p_{j-1}},0,\dots,0)$, $w = (\underbrace{v,\dots,v}_{p_i\cdots p_{j-1}},0,\dots,0)$ after a uniform renumbering of the elements in both tuples. Then we have $Bw = (\underbrace{Av,\dots,Av}_{p_i\cdots p_{j-1}},0,\dots,0)$, which is the image of Av under $\eta_i\cdots\eta_{j-1}$ (with a backward renumbering of the elements of the tuple). This shows that the element $a*\beta$ is well-defined, and makes V into an $\mathcal{E}$-module; that is, we have

$$(ab)*\beta = a*(b*\beta)$$
$$(a+b)*\beta = a*\beta+b*\beta$$
$$a*(\beta+\gamma) = a*\beta+a*\gamma$$
$$(\lambda a)*\beta = a*(\lambda\beta) = \lambda(a*\beta)$$

where $a,b\in\mathcal{E}$, $\beta,\gamma\in V$, $\lambda\in\mathbb{K}$.

(5) A natural nondegenerate symmetric bilinear form can be defined on V as follows. Choose any such form $b^{(0)}$ on $V^{(0)}\times V^{(0)}$ and proceed by induction. If $b^{(i)}$ is such a form on $V^{(i)}\times V^{(i)}$, we define $b^{(i+1)}$ on $V^{(i+1)}\times V^{(i+1)}$ by setting

$$b^{(i+1)}(u,v) = \frac{1}{p_i}b^{(i)}(u,v) \tag{6}$$

if u,v belong to the same copy of $V^{(i)}$ in the decomposition of $V^{(i+1)}$ given by

$$V^{(i+1)} = \underbrace{V^{(i)}\oplus\cdots\oplus V^{(i)}}_{p_i}\oplus\mathbb{K}v_1^{(i)}\oplus\cdots\oplus\mathbb{K}v_{q_i}^{(i)}. \tag{7}$$

Different copies of $V^{(i)}$ are assumed to be pairwise orthogonal, and $b^{(i+1)}(v_j^{(i)}, v_k^{(i)}) = \delta_{j,k}$ (Kronecker delta) for $j, k = 1, \ldots, q_i$. Now if $u, v \in V^{(i)}$, then $\eta_i(u), \eta_i(v) \in V^{(i+1)}$ and

$$b^{(i+1)}(\eta_i(u), \eta_i(v)) = \underbrace{\frac{1}{p_i} b^{(i)}(u,v) + \cdots + \frac{1}{p_i} b^{(i)}(u,v)}_{p_i} = b^{(i)}(u,v). \tag{8}$$

Thus the definition is consistent, and we have a well-defined nondegenerate symmetric bilinear form on V.

(6) If $V = \varinjlim V^{(i)}$ and beginning with some ℓ, there is a nondegenerate skew-symmetric bilinear form $c^{(i)}$ on $V^{(i)}$ for $i \geq \ell$, and the numbers q_i are all even starting with $i = \ell$, then by induction we can define such a form on $V^{(i+1)}$. We do this as in (4), by supposing that the different copies of $V^{(i)}$ in (7) are pairwise orthogonal and are orthogonal to the vectors $\{v_1^{(i)}, \ldots, v_{q_i}^{(i)}\}$. Since q_i is even, we can also split $\mathbb{K}v_1^{(i)} \oplus \cdots \oplus \mathbb{K}v_{q_i}^{(i)}$ into a sum of pairwise orthogonal hyperbolic 2-dimensional subspaces $(\mathbb{K}v_1^{(i)} \oplus \mathbb{K}v_2^{(i)}) \oplus (\mathbb{K}v_3^{(i)} \oplus \mathbb{K}v_4^{(i)}) \oplus \cdots \oplus (\mathbb{K}v_{q_i-1}^{(i)} \oplus \mathbb{K}v_{q_i}^{(i)})$, with $c^{(i+1)}(v_1^{(i)}, v_2^{(i)}) = 1 = -c^{(i+1)}(v_2^{(i)}, v_1^{(i)})$, etc.

Again,

$$c^{(i+1)}(u,v) = \frac{1}{p_i} c^{(i)}(u,v)$$

if u, v belong to the same copy of $V^{(i)}$ in $V^{(i+1)}$. As before, we can check that the collection of forms $\{c^{(i)} \mid i \geq \ell\}$ is compatible with the direct limit and thus defines a nondegenerate skew-symmetric bilinear form on V.

These instruments enable us to introduce four Lie algebras related to the associative algebra $\mathcal{E} = \mathcal{E}(\underline{\mathfrak{p}})$. The first of them, $\mathfrak{gl}(\underline{\mathfrak{p}})$, is simply the set of all elements in $\mathcal{E}$ under the bracket operation $[a,b] = ab - ba$. The second is the subalgebra $\mathfrak{sl}(\underline{\mathfrak{p}})$ of $\mathfrak{gl}(\underline{\mathfrak{p}})$, which is the kernel of the trace function $\mathfrak{t} : \mathcal{E} \longrightarrow \mathbb{K}$ in (2). Given a nondegenerate symmetric form b, as defined in part (4), and the action of $\mathcal{E}$ on V as in (3), we can define $\mathfrak{so}(\underline{\mathfrak{p}}, b)$ as the set of all $a \in \mathfrak{gl}(\underline{\mathfrak{p}})$ such that $b(a * \alpha, \beta) + b(\alpha, a * \beta) = 0$ for all $\alpha, \beta \in V$. Similarly, we can define $\mathfrak{sp}(\underline{\mathfrak{p}}, c)$, where c is a nondegenerate skew-symmetric bilinear form as in (5) above.

We have $[\mathfrak{gl}(\underline{\mathfrak{p}}), \mathfrak{gl}(\underline{\mathfrak{p}})] = \mathfrak{sl}(\underline{\mathfrak{p}})$, as this relation is true on the "finite components" of $\mathfrak{gl}(\underline{\mathfrak{p}})$. Similarly, one can check that $\mathfrak{so}(\underline{\mathfrak{p}}, b)$ and $\mathfrak{sp}(\underline{\mathfrak{p}}, c)$ are Lie subalgebras, and all three families consist of simple Lie algebras (as direct limits of simple Lie algebras). If $a = A \in \mathcal{E}^{(i)}$, $u, v \in V^{(i)}$, and $b^{(i)}(Au, v) + b^{(i)}(u, Av) = 0$, then by (8),

$$\begin{aligned} b^{(i+1)}(\varphi_i(A)\eta_i(u), \eta_i(v)) &+ b^{(i+1)}(\eta_i(u), \varphi_i(A)\eta_i(v)) \\ &= b^{(i+1)}(\eta_i(Au), \eta_i(v)) + b^{(i+1)}(\eta_i(u), \eta_i(Av)) \\ &= b^{(i)}(Au, v) + b^{(i)}(u, Av) = 0. \end{aligned}$$

Since the vectors $v_1^{(i)}, \ldots, v_{q_i}^{(i)}$ are orthogonal to the copies of $V^{(i)}$, and each $v_j^{(i)}$ is annihilated by a, whenever $b^{(i)}$ is a-invariant, then so is $b^{(i+1)}$.

The algebras $\mathfrak{so}(\underline{\mathfrak{p}}, b)$ and $\mathfrak{sp}(\underline{\mathfrak{p}}, c)$ might appear to be dependent on the choice of b and c. Actually, it follows from [8], that the isomorphism classes of such algebras depend only on $\underline{\mathfrak{p}}$ because $\mathbb{K}$ is algebraically closed, and so in the future, we will simply write $\mathfrak{so}(\underline{\mathfrak{p}})$ and $\mathfrak{sp}(\underline{\mathfrak{p}})$ for these Lie algebras.

2. Root Gradings

We recall from [13, 11] that a Lie algebra $\mathcal{L}$ over a field $\mathbb{K}$ of characteristic 0 is said to be Δ-graded for a finite reduced root system $\Delta = \mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$ if the following conditions hold:

(Δi) $\mathcal{L}$ contains a split simple subalgebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu\right)$ whose root system is Δ relative to the split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;

(Δii) $\mathcal{L} = \bigoplus_{\mu\in\Delta\cup\{0\}} \mathcal{L}_\mu$ where $\mathcal{L}_\mu = \{x \in \mathcal{L} \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$; and

(Δiii) $\mathcal{L}$ is generated by its root subspaces $\mathcal{L}_\mu$, $\mu \in \Delta$.

It follows from (Δii) that $[\mathcal{L}_\mu, \mathcal{L}_\lambda] \subseteq \mathcal{L}_{\mu+\lambda}$ if $\mu+\lambda \in \Delta\cup\{0\}$ and $[\mathcal{L}_\mu, \mathcal{L}_\lambda] = 0$ otherwise, so that $\mathcal{L}$ has a grading by the elements in $\Delta \cup \{0\}$. We say that $\mathfrak{g}$ is the *grading subalgebra*.

When $\Delta = \mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$, then as a $\mathfrak{g}$-module under the adjoint action, $\mathcal{L}$ decomposes into a direct sum of submodules isomorphic to the adjoint module $\mathfrak{g}$ and the trivial module $\mathbb{K}$. For the doubly-laced root systems $\Delta = \mathrm{B}_n$, C_n, F_4, and G_2, the algebra $\mathcal{L}$ decomposes under $\operatorname{ad}\mathfrak{g}$ into a direct sum of copies of $\mathfrak{g}$, M, and $\mathbb{K}$, where M is the irreducible $\mathfrak{g}$-module whose highest weight is the highest short root. In particular, when $\Delta = \mathrm{B}_n$, we may identify $\mathfrak{g}$ with $\mathfrak{so}(V)$, where V is a $(2n+1)$-dimensional space with a nondegenerate symmetric bilinear form, and $\mathfrak{g}$ is the space of skew transformations relative to the form. In this case, we may identify the $\mathfrak{g}$-module M with V, and $\mathfrak{g}$ with the second exterior power $\Lambda^2(V)$ of V. The second symmetric power $S^2(V)$ is not irreducible, but $S^2(V) = S \oplus \mathbb{K}$, where S is irreducible (and can be identified with the symmetric transformations of trace zero on V).

If $\Delta = \mathrm{C}_n$, we may view $\mathfrak{g}$ as being $\mathfrak{sp}(V)$, where V is a $(2n)$-dimensional space with a nondegenerate skew-symmetric bilinear form, and $\mathfrak{g}$ is the space of skew transformations on V relative to that form. In this case, the second exterior power of V decomposes into irreducible $\mathfrak{g}$-modules as $\Lambda^2(V) = \Lambda\oplus\mathbb{K}$ and $M \cong \Lambda$. Here $\mathfrak{g} \cong S^2(V)$ as $\mathfrak{g}$-modules.

There is a parallel notion of a Lie algebra $\mathcal{L}$ graded by the nonreduced root system $\Delta = \mathrm{BC}_n$ introduced and studied in [2, 10]. Such Lie algebras $\mathcal{L}$ are assumed to contain a split simple subalgebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\mu\in\Delta_\mathfrak{g}} \mathfrak{g}_\mu\right)$ whose root system $\Delta_\mathfrak{g}$ is of type B_n, C_n, or D_n relative to the split Cartan subalgebra $\mathfrak{h}$. Additionally, conditions (Δii) and (Δiii) above must hold in $\mathcal{L}$, where Δ is the root system of type BC_n, (which contains $\Delta_\mathfrak{g}$). As before, the subalgebra $\mathfrak{g}$ is referred to as the grading subalgebra of $\mathcal{L}$. When $n \geq 2$ and $\Delta_\mathfrak{g} \neq \mathrm{D}_2$, a Lie algebra graded by BC_n will decompose as a module for its grading subalgebra $\mathfrak{g}$ into a direct sum of copies of $\mathfrak{g}$, $\mathfrak{s}$, V, and $\mathbb{K}$, where $\mathfrak{s} = S$ for types B_n and D_n, and $\mathfrak{s} = \Lambda$ for type C_n. Conversely, any Lie algebra $\mathcal{L}$ containing such a subalgebra $\mathfrak{g}$, having such a $\mathfrak{g}$-module decomposition, and satisfying (Δiii) automatically will be a BC_n-graded Lie algebra.

Now suppose that we have a diagonal direct limit Lie algebra $\mathcal{L} = X(\underline{\mathfrak{p}})$ for $X \in \{\mathfrak{sl}, \mathfrak{sp}, \mathfrak{so}\}$. If we fix any component $\mathfrak{g} := \mathfrak{g}^{(i)} = X(V)$ of $\mathcal{L}$ in the direct limit (1), then for any $j > i$ we have $V^{(j)} = V^{\oplus \ell_j} \oplus (V^*)^{\oplus r_j} \oplus \mathbb{K}^{\oplus z_j}$. In this case $\mathfrak{gl}(V^{(j)})$ will decompose as a $\mathfrak{g}$-module into a direct sum of submodules isomorphic to $V \otimes V$, $V \otimes V^*$, $V^* \otimes V^*$, V, V^* and $\mathbb{K}$. Since $\mathfrak{g}^{(j)} = X(V^{(j)}) \subset \mathfrak{gl}(V^{(j)})$, we have that $\mathfrak{g}^{(j)}$ is equal to a sum of irreducible $\mathfrak{g}$-modules which are isomorphic to the irreducible summands of those modules. Thus for $X = \mathfrak{sl}$, the only $\mathfrak{g}$-modules that can occur in $\mathfrak{g}^{(j)} = \mathfrak{sl}(V^{(j)})$, are copies of $\mathfrak{g}$, $\mathbb{K}$, V, $S^2(V)$, $\Lambda^2(V)$ and dual modules of the last three.

As $V \cong V^*$ when $\mathcal{L} = X(\underline{\mathfrak{p}})$ for $X = \mathfrak{sp}, \mathfrak{so}$, we need only consider the irreducible summands of $V \otimes V$ along with the modules V and $\mathbb{K}$ in this case. In particular, if $\dim V \geq 5$, then any such direct limit Lie algebra $\mathcal{L}$ is a BC_n-graded Lie algebra for $n = \lfloor (\dim V)/2 \rfloor$ with grading subalgebra $\mathfrak{g} = X(V)$ (condition (Δiii) automatically holds as $\mathcal{L}$ is simple). In summary we have

Lemma 2.1. (i) Assume $\mathcal{L}$ is a diagonal direct limit of the form $\mathcal{L} = X(\underline{\mathfrak{p}})$ for $X = \mathfrak{sp}$ or $\mathfrak{so}$. Fix a component $\mathfrak{g} := \mathfrak{g}^{(i)} = X(V)$ of $\mathcal{L}$ with $\dim V \geq 5$. Then $\mathcal{L}$ is a BC_n-graded Lie algebra for $n = \lfloor (\dim V)/2 \rfloor$ with grading subalgebra $\mathfrak{g}$.

(ii) Assume $\mathcal{L}$ is a diagonal direct limit of the form $\mathcal{L} = \mathfrak{sl}(\underline{\mathfrak{p}})$. Suppose $\mathfrak{g} := \mathfrak{g}^{(i)} = \mathfrak{sl}(V)$ is a component of $\mathcal{L}$. Then as a $\mathfrak{g}$-module under the adjoint action, $\mathcal{L}$ is a direct sum of copies of $\mathfrak{g}$, V, V^*, $\mathbb{K}$, $S^2(V)$, $\Lambda^2(V)$, $S^2(V^*)$ and $\Lambda^2(V^*)$.

We conclude this subsection with an indication of why the diagonal direct limits are of special importance while studying root graded locally finite simple Lie algebras.

The following lemma can be found in [19] (see also [7, Lemma 5.2]).

Lemma 2.2. *Let $\mathfrak{g} \subset \mathfrak{g}' \subset \mathfrak{g}''$ be classical simple Lie algebras. Assume that the rank of $\mathfrak{g}$ is greater than 10 and the embedding $\mathfrak{g} \to \mathfrak{g}''$ is diagonal. Then the embeddings $\mathfrak{g} \to \mathfrak{g}'$ and $\mathfrak{g}' \to \mathfrak{g}''$ are also diagonal.*

Theorem 2.3. *Assume $\mathcal{L}$ is a direct limit of Lie algebras of the form $\mathfrak{sl}(V^{(i)})$, and let $\mathfrak{g}$ be one of the terms of this sequence of rank at least 10. If $\mathcal{L}$ is Δ-graded by the root system Δ of $\mathfrak{g}$, then $\mathcal{L}$ is a diagonal direct limit.*

Proof. According to Lemma 2.2, it is sufficient to prove that the embedding $\mathfrak{g} \to \mathfrak{g}'$ is diagonal, where $\mathfrak{g}'$ is a term of the direct limit whose number is greater than that of $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{sl}(V)$, $\mathfrak{g}' = \mathfrak{sl}(V')$, and we want to establish that if $\mathfrak{g}'$ as a $\mathfrak{g}$-module under the adjoint action has only irreducible submodules $\mathfrak{g}$ and $\mathbb{K}$, then V' as a $\mathfrak{g}$-module has only irreducible submodules V, V^*, and $\mathbb{K}$. Indeed, suppose that

$$V' = \bigoplus_{\omega} V(\omega), \tag{9}$$

where $V(\omega)$ denotes the highest weight $\mathfrak{g}$-module with highest weight ω. Then

$$\mathfrak{sl}(V') \oplus F = V' \otimes (V')^* \tag{10}$$

We choose a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of simple roots for the root system A_l of $\mathfrak{g}$, and let $\{\omega_1, \ldots, \omega_l\}$ denote the corresponding fundamental dominant weights. As $\mathfrak{sl}(V')$ decomposes into copies of $\mathfrak{g}$ and $\mathbb{K}$ which have highest weights $\omega_1 + \omega_l$ and 0 respectively, the same must be true for the summands on the right side of (10). If $\omega = m_1\omega_1 + \ldots + m_l\omega_l \neq 0$ (all $m_i \geq 0$) occurs as a highest weight in the decomposition (9), then $\omega^T := m_l\omega_1 + \ldots + m_1\omega_l \neq 0$ occurs in the decomposition of $(V')^*$. In this case, $V(\omega) \otimes V(\omega^T)$ is a direct summand of the right-hand side of (10), so that $\omega + \omega^T$ is among nonzero dominant weights of the left-hand side. It follows that

$$(m_1 + m_l)\omega_1 + (m_2 + m_{l-1})\omega_1 + \ldots + (m_l + m_1)\omega_l = \omega_1 + \omega_l. \tag{11}$$

Now it is immediate that we have only two options for ω – namely, $\omega = \omega_1$ and then $V(\omega) \cong V$, or $\omega = \omega_l$ and then $V(\omega) \cong V^*$. This proves the diagonality of embedding $\mathfrak{g} \to \mathfrak{g}'$. ∎

The proof of this theorem actually demonstrates a stronger result. It must be in the decomposition of (9) that only copies of $V(\omega_1)$ or only copies of $V(\omega_l)$ occur. In other words, the structure triples must have the form $(\ell_i, 0, 0)$ or $(0, r_i, 0)$. The second case immediately reduces to the first as indicated in [8, Sec. 4]. So we may assume that all triples are of the form $(\ell_i, 0, 0)$. Structure sequences with this property give what have been called *pure limits*, as discussed next.

3. Locally Finite Lie Algebras Graded by the Root Systems of Type A

In this section, we examine certain direct limits of type A that fall into the pattern of root gradings. Here we will suppose that the structure sequence has the form $\underline{\mathfrak{l}} = \{(\ell_i, 0, 0) \mid i = 1, 2, \ldots\}$. In [8] these limits are called *pure*. Then we may assume there is a sequence $\underline{\mathfrak{n}} = \{n_0, n_1, \ldots, n_{t-1}, n_t, \ldots\}$ of natural numbers with $n_0 \geq 2$ and $n_t = n_{t-1}\ell_t$ for all $t = 1, 2, \ldots$. We set $n = n_0$ and $m_t = n_t/n$, and let $\underline{\mathfrak{m}} = \{m_1, m_2, \ldots\}$. Then we can define $\mathfrak{sl}_{\underline{\mathfrak{n}}}$ and $\mathfrak{M}_{\underline{\mathfrak{m}}}$ as the direct limits of the sequences of special linear Lie algebras $\mathfrak{sl}_{n_t}$ and the associative matrix algebras $\mathfrak{M}_{m_t}$, respectively. This is equivalent to a diagonal construction as in previous sections. Indeed, the present procedure can be thought of as starting with a vector space $V^{(0)} = \mathbb{K}^{\oplus n_0}$ of dimension $n_0 \geq 2$ over $\mathbb{K}$ and defining $\mathbb{K}$-spaces $V^{(t)} = (V^{(t-1)})^{\oplus \ell_t}$. The Lie algebras of interest are $\mathfrak{g}^{(t)} = \mathfrak{sl}(V^{(t)})$, which we can identify with the special linear Lie algebra $\mathfrak{sl}_{n_t}$ with entries in $\mathbb{K}$ upon choosing a basis. The corresponding associative algebra is $\mathcal{E}^{(t)} = \operatorname{End} V^{(t)}$, which can be identified with the matrix algebra $\mathfrak{M}_{n_t}$.

A matrix algebra $\mathfrak{M}_n(A)$ with entries in a unital associative algebra A can be viewed as a Lie algebra under the commutator product

$$[aE_{i,j}, bE_{k,l}] = \delta_{j,k} abE_{i,l} - \delta_{l,i} baE_{k,j},$$

and $\mathfrak{sl}_n(A)$ is the Lie subalgebra of $\mathfrak{M}_n(A)$ generated by the matrices $aE_{i,j}$, $i \neq j$, $a \in A$.

If $\varphi : A \longrightarrow B$ is a homomorphism of associative algebras then $\Phi = \mathfrak{sl}(\varphi) : \mathfrak{sl}_n(A) \longrightarrow \mathfrak{sl}_n(B)$ with φ applied entrywise is a homomorphism of Lie algebras (an easy check). Now let us consider two natural sequences of homomorphisms of Lie algebras giving $\mathfrak{sl}_{\underline{n}} = \varinjlim \mathfrak{sl}_{n_t}$ and $\mathfrak{sl}_n(\mathfrak{M}_{\underline{\mathfrak{m}}})$ and establish an isomorphism of these sequences. Thus, we have the following:

$$\mathfrak{sl}_n \xrightarrow{\Phi_1} \mathfrak{sl}_n(\mathfrak{M}_{m_1}) \xrightarrow{\Phi_2} \mathfrak{sl}_n(\mathfrak{M}_{m_2}) \longrightarrow \dots \longrightarrow \mathfrak{sl}_n(\mathfrak{M}_{m_{t-1}}) \xrightarrow{\Phi_t} \mathfrak{sl}_n(\mathfrak{M}_{m_t}) \longrightarrow \quad (12)$$

arising from

$$\mathfrak{M}_{m_0} \xrightarrow{\varphi_1} \mathfrak{M}_{m_1} \xrightarrow{\varphi_2} \mathfrak{M}_{m_2} \longrightarrow \dots \longrightarrow \mathfrak{M}_{m_{t-1}} \xrightarrow{\varphi_t} \mathfrak{M}_{m_t} \longrightarrow \dots , \quad (13)$$

where $\varphi_t(a) = \operatorname{diag}(\underbrace{a, \dots, a}_{\ell_t})$ for $a \in \mathfrak{M}_{m_{t-1}}$. There is an analogous sequence of Lie homomorphisms,

$$\mathfrak{sl}_n \xrightarrow{\vartheta_1} \mathfrak{sl}_{n_1} \xrightarrow{\vartheta_2} \mathfrak{sl}_{n_2} \longrightarrow \dots \longrightarrow \mathfrak{sl}_{n_{t-1}} \xrightarrow{\vartheta_t} \mathfrak{sl}_{n_t} \longrightarrow \dots, \quad (14)$$

which can be defined in the same way. If we can find isomorphisms $\sigma_t : \mathfrak{sl}_n(\mathfrak{M}_{m_t}) \longrightarrow \mathfrak{sl}_{n_t}$ so that $\sigma_t\Phi_t = \vartheta_t\sigma_{t-1}$, then we will establish the following result.

Theorem 3.1. *Assume $\underline{\mathfrak{l}} = (\ell_1, \ell_2, \dots)$ and $\underline{\mathfrak{n}} = (n_0 = n, n_1, n_2, \dots)$, where $n \geq 2$, the l_i are positive integers, and $n_t = n_{t-1}\ell_t$. Set $\underline{\mathfrak{m}} = (m_1, m_2, \dots)$ where $m_t = n_t/n$ for all $t = 1, 2, \dots$. Then $\mathfrak{sl}_n(\mathfrak{M}_{\underline{\mathfrak{m}}}) \cong \mathfrak{sl}_{\underline{\mathfrak{n}}} = \varinjlim \mathfrak{sl}_{n_t}$.*

Proof. First we note that there is an isomorphism $\sigma_t : \mathfrak{M}_n(\mathfrak{M}_{m_t}) \longrightarrow \mathfrak{M}_{n_t}$ of associative algebras. This is a standard argument: there is a basis $\varepsilon_{p,q}E_{i,j}$ $(1 \leq p, q \leq m_t,\ 1 \leq i, j \leq n)$ of $\mathfrak{M}_n(\mathfrak{M}_{m_t})$, where $E_{i,j}$ is a matrix unit in $\mathfrak{M}_n$, and $\varepsilon_{p,q}$ is a matrix unit of $\mathfrak{M}_{m_t}$ considered as the coefficient. But we can also view $E_{i,j}$ as the coefficient and $\varepsilon_{p,q}$ as a matrix unit. Having this in mind, we define σ_t as the linear transformation whose image of $\varepsilon_{p,q}E_{i,j}$ is an $(n_t \times n_t)$-matrix split into square blocks of size n. Thus, there are $m_t = n_t/n$ blocks in each row and column. The (p, q) entry of $\sigma_t(\varepsilon_{p,q}E_{i,j})$ is $E_{i,j}$, and all other entries are 0. Observe that

$$\varepsilon_{p,q}E_{i,j}\varepsilon_{p',q'}E_{i',j'} = \delta_{j,i'}\delta_{q,p'}\varepsilon_{p,q'}E_{i,j'},$$

so the image of the product under σ_t is the matrix whose (p, q') entry is $\delta_{j,i'}\delta_{q,p'}E_{i,j'}$ and all other entries are 0.

The product $\sigma_t(\varepsilon_{p,q}E_{i,j})\,\sigma_t(\varepsilon_{p',q'}E_{i',j'})$ involves the (p, q) block times the (p', q') block. So it gives 0 unless $q = p'$, and in that case, the result is $E_{i,j}E_{i',j'} = \delta_{j,i'}E_{i,j'}$ in the (p, q') place. This is the same as the image of the product above. Thus, σ_t is a homomorphism. But since both algebras are simple and of the same dimension over $\mathbb{K}$, σ_t is an isomorphism.

The map σ_t also is an isomorphism of the corresponding Lie algebras, and thus restricts to an isomorphism $\sigma_t : \mathfrak{sl}_n(\mathfrak{M}_{m_t}) \longrightarrow \mathfrak{sl}_{n_t}$ of their commutators.

Now we want to check that the relation $\vartheta_t\sigma_{t-1} = \sigma_t\Phi_t$ holds. Let us start with $\varepsilon_{p,q}E_{i,j} \in \mathfrak{sl}_n(\mathfrak{M}_{m_{t-1}})$. Then

$$\Phi_t(\varepsilon_{p,q}E_{i,j}) = \big(\varepsilon_{p,q} + \varepsilon_{p+m_{t-1},q+m_{t-1}} + \dots + \varepsilon_{p+(\ell_t-1)m_{t-1},q+(\ell_t-1)m_{t-1}}\big)E_{i,j} \quad (15)$$

(we recall that $m_t = \ell_t m_{t-1}$ and $n_t = m_t n$). The image of this element under σ_t will be the matrix which is the sum of ℓ_t identical blocks $E_{i,j}$ in positions (p, q),

$(p+m_{t-1}, q+m_{t-1}), \ldots, (p+(\ell_t-1)m_{t-1}, q+(\ell_t-1)m_{t-1})$. If instead we first apply σ_{t-1} to $\varepsilon_{p,q}E_{i,j}$, we obtain an $(n_{t-1}\times n_{t-1})$-matrix with block $E_{i,j}$ at the (p,q) location. Applying ϑ_t means placing the matrix just obtained ℓ_t times down the diagonal of an $(n_t\times n_t)$-matrix. This means that the matrix $E_{i,j}$ now appears at the positions (p,q), $(p+m_{t-1}, q+m_{t-1}), \ldots, (p+(\ell_t-1)m_{t-1}, q+(\ell_t-1)m_{t-1})$, which gives the same matrix as above. So the proof of Theorem 3.1 is complete. ■

It is well-known [17, 13, 1] that any Lie algebra L of the form $\mathfrak{sl}_n(A)$, where A is an associative algebra, is A_{n-1}-graded, and as such, has a realization as

$$L = (\mathfrak{sl}_n \otimes A) \oplus D_{A,A}.$$

The multiplication is given by

$$\begin{aligned}
[x\otimes a, y\otimes a'] &= [x,y]\otimes \frac{1}{2}(a\circ a') + (x\circ y)\otimes \frac{1}{2}[a,a'] + (x|y)D_{a,a'} \\
[D_{a,a'}, x\otimes b] &= x\otimes [[a,a'],b] \\
[D_{a,a'}, D_{b,b'}] &= D_{[[a,a'],b],b'} + D_{b,[[a,a'],b']},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
a\circ a' &= aa' + a'a \\
x\circ y &= xy + yx - \frac{2}{n}\mathrm{tr}(xy) \\
(x|y) &= \frac{1}{n}\mathrm{tr}(xy) \qquad \text{and} \\
D_{a,a'}(b) &= [[a,a'],b].
\end{aligned} \tag{17}$$

As any Δ-graded Lie algebra L is perfect ($L=[L,L]$), it has a universal covering algebra $\widehat{L}$ (often called the universal central extension) which is also perfect and is unique up to isomorphism. Any perfect Lie algebra which is a central extension of L is a homomorphic image of $\widehat{L}$. The algebra $\widehat{L}$ is the vector space $\widehat{L} = (\mathfrak{sl}_n\otimes A)\oplus \{A,A\}$ with $\{A,A\} = (A\otimes A)/J$, where J is the subspace of $A\otimes A$ generated by the elements $a\otimes b + b\otimes a$, $ab\otimes c + bc\otimes a + ca\otimes b$ for all $a,b,c\in A$. We know from [9] that any derivation $\delta\in \mathrm{Der}A$ has an action on $\{A,A\}$ by setting $\delta\{a,b\} = \{\delta a, b\} + \{a,\delta b\}$, and $\{A,A\}$ can be made into a Lie algebra by specifying

$$[\{a,b\},\{c,d\}] = D_{a,b}\{c,d\} = \{[[a,b],c],d\} + \{c,[[a,b],d]\},$$

(see [12, Lemma 1.46]). The mapping $\{a,b\}\mapsto D_{a,b}$ is a surjective homomorphism [1, Lemma 4.10]. Now if we endow $\widehat{L} = (\mathfrak{sl}_n\otimes A)\oplus\{A,A\}$ with the multiplication given by

$$\begin{aligned}
[x\otimes a, y\otimes a'] &= [x,y]\otimes \frac{1}{2}(a\circ a') + (x\circ y)\otimes \frac{1}{2}[a,a'] + (x|y)\{a,a'\}, \\
[\{a,a'\}, x\otimes b] &= x\otimes [[a,a'],b], \\
[\{a,b\},\{c,d\}] &= \{[[a,b],c],d\} + \{c,[[a,b],d]\},
\end{aligned} \tag{18}$$

then $(\widehat{L},\widehat{\pi})$ with $\widehat{\pi}:\widehat{L}\longrightarrow L$ given by $\widehat{\pi}: x\otimes a\mapsto x\otimes a$, $\widehat{\pi}:\{a,a'\}\mapsto D_{a,a'}$ is the universal covering algebra of L. The center of $\widehat{L}$ is the so-called full skew-dihedral homology

$$\mathrm{HF}(A)=\left\{\sum_i\{a_i,b_i\}\in\{A,A\}\;\middle|\;\sum_i D_{a_i,b_i}=0\right\},$$

which is often identified with the first (Connes) cyclic homology group $\mathrm{HC}_1(A)$ of A.

In our case $A=\mathfrak{M}_{\underline{\mathfrak{m}}}$, and we can determine $\{A,A\}$ precisely. Let us suppose that $A=\mathfrak{M}_{\underline{\mathfrak{m}}}$ and $A_t=\mathfrak{M}_{m_t}$. It should be noted first that if $\varphi: A\longrightarrow B$ is a homomorphism of associative algebras, then φ has a natural extension not only to $\Phi=\mathfrak{sl}_n(\varphi):\mathfrak{sl}_n(A)\longrightarrow\mathfrak{sl}_n(B)$ but also to $\widehat{\Phi}=\widehat{\mathfrak{sl}_n(\varphi)}:\widehat{\mathfrak{sl}_n(A)}\longrightarrow\widehat{\mathfrak{sl}_n(B)}$, defined by $\widehat{\Phi}(x\otimes a)=x\otimes\varphi(a)$ and $\widehat{\Phi}(\{a,a'\})=\{\varphi(a),\varphi(a')\}$. Clearly then

$$\begin{aligned}
\widehat{\Phi}([x\otimes a,y\otimes a']) &= [x,y]\otimes\varphi\left(\frac{1}{2}(a\circ a')\right)+(x\circ y)\otimes\varphi\left(\frac{1}{2}[a,a']\right)\\
&\qquad +(x|y)\{\varphi(a),\varphi(a')\}\\
&= [x,y]\otimes\frac{1}{2}\big(\varphi(a)\circ\varphi(a')\big)+(x\circ y)\otimes\frac{1}{2}[\varphi(a),\varphi(a')]\\
&\qquad +(x|y)\{\varphi(a),\varphi(a')\}\\
&= [x\otimes\varphi(a),y\otimes\varphi(a')]=[\widehat{\Phi}(x\otimes a),\widehat{\Phi}(y\otimes a')].
\end{aligned}$$

Also,

$$\begin{aligned}
\widehat{\Phi}([\{a,b\},x\otimes c]\} &= x\otimes\varphi([[a,b],c])=x\otimes[[\varphi(a),\varphi(b)],\varphi(c)]\\
&= [\{\varphi(a),\varphi(b)\},x\otimes\varphi(c)]=[\widehat{\Phi}(\{a,b\}),\widehat{\Phi}(x\otimes c)].
\end{aligned}$$

Finally,

$$\begin{aligned}
\widehat{\Phi}([\{a,b\},\{c,d\}]) &= \{\varphi([[a,b],c]),\varphi(d)\}+\{\varphi(c),\varphi([[a,b],d])\}\\
&= \{[[\varphi(a),\varphi(b)],\varphi(c)],\varphi(d)\}+\{\varphi(c),[[\varphi(a),\varphi(b)],\varphi(d)]\}\\
&= [\{\varphi(a),\varphi(b)\},\{\varphi(c),\varphi(d)\}]=[\widehat{\Phi}(\{a,b\}),\widehat{\Phi}(\{c,d\})].
\end{aligned}$$

Now we are going to apply two functors, $\mathfrak{sl}_n(.)$ and $\widehat{\mathfrak{sl}_n(.)}$, to the sequence in (13). This will produce the following diagram which will be shown to be commutative, and each $\widehat{\pi}_t$ will be shown to be injective:

$$\begin{array}{ccccccccc}
\widehat{\mathfrak{sl}_n(\mathfrak{M}_{m_1})} & \xrightarrow{\widehat{\Phi_1}} & \dots & \longrightarrow & \widehat{\mathfrak{sl}_n(\mathfrak{M}_{m_{t-1}})} & \xrightarrow{\widehat{\Phi_{t-1}}} & \widehat{\mathfrak{sl}_n(\mathfrak{M}_{m_t})} & \xrightarrow{\widehat{\Phi_t}} & \dots\\
\downarrow\widehat{\pi_1} & & & & \downarrow\widehat{\pi_{t-1}} & & \downarrow\widehat{\pi_t} & & \\
\mathfrak{sl}_n(\mathfrak{M}_{m_1}) & \xrightarrow{\Phi_1} & \dots & \longrightarrow & \mathfrak{sl}_n(\mathfrak{M}_{t-1}) & \xrightarrow{\Phi_{t-1}} & \mathfrak{sl}_n(\mathfrak{M}_{m_t}) & \xrightarrow{\Phi_t} & \dots
\end{array}$$

If we verify the commutativity of the diagram and the injectivity of the maps $\widehat{\pi}_t$, then we will obtain that $\widehat{\pi}:\widehat{\mathfrak{sl}_n(\mathfrak{M}_{\underline{\mathfrak{m}}})}\longrightarrow\mathfrak{sl}_n(\mathfrak{M}_{\underline{\mathfrak{m}}})$ is injective. As $\widehat{\pi}$ is surjective, it is an isomorphism of Lie algebras. Now, since we know that the Lie algebra $\mathfrak{sl}_n(\mathfrak{M}_{m_t})$ is isomorphic to $\mathfrak{sl}_{n_t}$ for each $t=1,2,\dots$, all central extensions of this algebra are split. But each Δ-graded algebra, including $\widehat{\mathfrak{sl}_n(\mathfrak{M}_{m_t})} = (\mathfrak{sl}_n \otimes \mathfrak{M}_{m_t}) \oplus \{\mathfrak{M}_{m_t}, \mathfrak{M}_{m_t}\}$, is generated by its gradation subspaces corresponding to nonzero roots; hence if $\widehat{\mathfrak{sl}_n(\mathfrak{M}_{m_t})}$ is a split central extension of $\mathfrak{sl}_n(\mathfrak{M}_{m_t})$, they must coincide. Thus, all column maps in the diagram are isomorphisms. It remains to check the commutativity of the diagram. For this we have

$$\begin{aligned}
\widehat{\pi_{t+1}}\widehat{\Phi_t}(x \otimes a) &= \widehat{\pi_{t+1}}(x \otimes \varphi_t(a)) = x \otimes \varphi_t(a), \\
\Phi_t \widehat{\pi}_t(x \otimes a) &= \Phi_t(x \otimes a) = x \otimes \varphi_t(a), \\
\widehat{\pi_{t+1}}\widehat{\Phi_t}(\{a,b\}) &= \widehat{\pi_{t+1}}(\{\varphi_t(a), \varphi_t(b)\}) = D_{\varphi_t(a),\varphi_t(b)} = \Phi_t(D_{a,b}), \\
\Phi_t \widehat{\pi}_t(\{a,b\}) &= \Phi_t(D_{a,b}),
\end{aligned}$$

as required.

As a consequence of these considerations and Theorem 3.1, we have established the following:

Theorem 3.2. (i) *Any A_{n-1}-graded Lie algebra, $n \geq 3$, with coordinate algebra equal to the matrix algebra $\mathfrak{M}_{\underline{m}}$ is isomorphic to $\mathfrak{sl}_n(\mathfrak{M}_{\underline{m}})$, hence to $\mathfrak{sl}_{\underline{n}}$, and has no non-split central extensions.*

(ii) $\mathrm{HC}_1(\mathfrak{M}_{\underline{m}}) = 0$.

4. Lie Superalgebras Having a Prescribed Decomposition Relative to $\mathfrak{sl}_n$ $(n \geq 4)$

Before we proceed to investigate locally finite simple Lie algebras which are of a more general form than those discussed in the previous sections, we need some results on certain Lie algebras which generalize A_{n-1}-graded Lie algebras. As these results hold in the wider context of Lie superalgebras, we will phrase them in that language with an eye towards further applications in the future.

Our object of study here will be Lie superalgebras $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over a field $\mathbb{K}$ of characteristic zero satisfying the following requirements:

(a) $L_{\bar{0}}$ contains a subalgebra $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}_n$ for $n \geq 4$;

(b) As a $\mathfrak{g}$-module, L is a direct sum of copies of $\mathfrak{g}$, $V = V(\omega_1)$ (the natural n-dimensional module of $\mathfrak{g}$ with highest weight ω_1), its dual module $V^* = V(\omega_{n-1})$, and trivial modules;

(c) Relative to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of diagonal matrices, L decomposes into weight spaces, and L is generated by the weight spaces corresponding to the nonzero weights.

Thus, there are $\mathbb{Z}_2$-graded vector spaces A, B, C, D such that

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus D,$$

where D is the sum of the trivial $\mathfrak{g}$-modules (it is the centralizer of $\mathfrak{g}$ in L, hence a subalgebra). We identify the subalgebra $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes A$. Thus,

$$L_{\bar{0}} = (\mathfrak{g} \otimes A_{\bar{0}}) \oplus (V \otimes B_{\bar{0}}) \oplus (V^* \otimes C_{\bar{0}}) \otimes D_{\bar{0}}$$
$$L_{\bar{1}} = (\mathfrak{g} \otimes A_{\bar{1}}) \oplus (V \otimes B_{\bar{1}}) \oplus (V^* \otimes C_{\bar{1}}) \otimes D_{\bar{1}}$$

Because $\mathfrak{g} \cong V(\omega_1 + \omega_{n-1})$ and

$$\begin{aligned}
V(\omega_1+\omega_{n-1}) \otimes V(\omega_1) &= V(2\omega_1+\omega_{n-1}) \oplus V(\omega_2+\omega_{n-1}) \oplus V(\omega_1) \\
V(\omega_1) \otimes V(\omega_{n-1}) &= V(\omega_1+\omega_{n-1}) \oplus V(0) \\
V(\omega_1) \otimes V(\omega_1) &= V(2\omega_1) \oplus V(\omega_2) \\
V(\omega_1+\omega_{n-1}) \otimes V(\omega_{n-1}) &= V(\omega_1+2\omega_{n-1}) \oplus V(\omega_1+\omega_{n-2}) \oplus V(\omega_{n-1}) \\
V(\omega_{n-1}) \otimes V(\omega_{n-1}) &= V(2\omega_{n-1}) \oplus V(\omega_{n-2}),
\end{aligned}$$

there exists a supercommutative product

$$a \times a' \to a \circ a' \in A$$

superanticommutative products

$$\begin{aligned}
a \times a' &\to [a, a'] \in A \\
a \times a' &\to \langle a, a' \rangle \in D
\end{aligned}$$

and products

$$\begin{aligned}
a \times b &\to ab \in B \\
a \times c &\to ca \in C \\
b \times c &\to (b, c) \in A \\
b \times c &\to \langle b, c \rangle \in D \\
d \times a &\to da \in A \\
d \times b &\to db \in B \\
d \times c &\to dc \in C
\end{aligned}$$

for $a, a' \in A$, $b \in B$, $c \in C$, and $d \in D$ so that the product in L is given by

$$\begin{aligned}
[x \otimes a, y \otimes a'] &= [x, y] \otimes \frac{1}{2} a \circ a' + x \circ y \otimes \frac{1}{2}[a, a'] + (x|y)\langle a, a' \rangle \\
[x \otimes a, u \otimes b] &= xu \otimes ab = -(-1)^{\bar{a}\bar{b}}[u \otimes b, x \otimes a] \\
[v^* \otimes c, x \otimes a] &= v^* x \otimes ca = -(-1)^{\bar{a}\bar{c}}[x \otimes a, v^* \otimes c] \\
[u \otimes b, v^* \otimes c] &= (uv^* - \frac{1}{n}\mathrm{tr}(uv^*)I) \otimes (b, c) + \frac{1}{n}\mathrm{tr}(uv^*)\langle b, c \rangle \\
&= (uv^* - \frac{1}{n} v^* u I) \otimes (b, c) + \frac{1}{n} v^* u \langle b, c \rangle \qquad (19) \\
&= -(-1)^{\bar{b}\bar{c}}[v^* \otimes c, u \otimes b] \\
[d, x \otimes a] &= x \otimes da = -(-1)^{\bar{d}\bar{a}}[x \otimes a, d] \\
[d, u \otimes b] &= u \otimes db = -(-1)^{\bar{d}\bar{b}}[u \otimes b, d] \\
[d, v^* \otimes c] &= v^* \otimes dc = -(-1)^{\bar{d}\bar{c}}[v^* \otimes c, d] \\
[d, d'] &\in D
\end{aligned}$$

for $x, y \in \mathfrak{g}$, $u \in V$, $v^* \in V^*$. All other products are zero. When we write such expressions, we assume that the elements in A, B, C, and D are homogeneous, and $\bar{a} = \bar{\imath}$ if $a \in A_{\bar{\imath}}$, etc. The action $x, u \to xu$ of $\mathfrak{g}$ on V is just matrix multiplication, as we may identify V with $\mathbb{K}^n$, that is, with $n \times 1$ matrices over $\mathbb{K}$. We identify V^* with $1 \times n$ matrices over $\mathbb{K}$, and $v^* x$ above is just the matrix product. Similarly,

uv^* is the product of the two matrices u, v^* as is v^*u; and $(x|y)$ and $x \circ y$ are as in (17).

The prototype of such a Lie superalgebra is the special linear Lie superalgebra $L = \mathfrak{sl}_{n,m}$ viewed as a $\mathfrak{sl}_n$-module. In this case, $L \cong \mathfrak{sl}_n \oplus (V \otimes W^*) \oplus (V^* \otimes W) \oplus \big(\mathfrak{sl}_m \oplus \mathbb{K}d\big)$, where W is the natural m-dimensional module for $\mathfrak{sl}_m$, W^* is its dual, and d is the $(n+m) \times (n+m)$ matrix which is $mI_n - nI_m$. The spaces W and W^* are odd ($W = W_{\bar{1}}$, $W^* = (W^*)_{\bar{1}}$). Here $A = \mathbb{K}$ and $\mathfrak{g} \otimes A \cong \mathfrak{sl}_n$ so we have not bothered to write A. As another example, we can consider the Lie algebra $L = \mathfrak{sl}_{n+1}$ regarded as a module for $\mathfrak{g} = \mathfrak{sl}_n$, which we identify with the $(n \times n)$-matrices of trace 0 in the northwest corner of L. Then $L \cong \mathfrak{sl}_n \oplus V \oplus V^* \oplus \mathbb{K}d$ where d is the diagonal $(n+1) \times (n+1)$-matrix with n 1's and $-n$ down its main diagonal.

We wish to derive properties of the products in (19). For this we define

$$aa' := \frac{1}{2}a \circ a' + \frac{1}{2}[a, a'].$$

Therefore,

$$\begin{aligned} a \circ a' &= aa' + (-1)^{\bar{a}\bar{a}'}a'a \\ [a, a'] &= aa' - (-1)^{\bar{a}\bar{a}'}a'a. \end{aligned} \tag{20}$$

From

$$[x \otimes a_1, [y \otimes a_2, z \otimes a_3]] = [[x \otimes a_1, y \otimes a_2], z \otimes a_3] + (-1)^{\bar{a_1}\bar{a_2}}[y \otimes a_2, [x \otimes a_1, z \otimes a_2]]$$

we obtain that

$$\begin{aligned}
&[x, [y, z]] \otimes \frac{1}{4}a_1 \circ (a_2 \circ a_3) + x \circ [y, z] \otimes \frac{1}{4}[a_1, a_2 \circ a_3] + \frac{1}{2}(x|[y, z])\langle a_1, a_2 \circ a_3\rangle \\
&+[x, y \circ z] \otimes \frac{1}{4}a_1 \circ [a_2, a_3] + x \circ (y \circ z) \otimes \frac{1}{4}[a_1, [a_2, a_3]] \\
&+\frac{1}{2}(x|(y \circ z))\langle a_1, [a_2, a_3]\rangle - (-1)^{(\bar{a_2}+\bar{a_3})\bar{a_1}}(y|z)x \otimes \langle a_2, a_3\rangle a_1 \;= \qquad (21)\\
&[[x, y], z] \otimes \frac{1}{4}(a_1 \circ a_2) \circ a_3 + [x, y] \circ z \otimes \frac{1}{4}[a_1 \circ a_2, a_3] \\
&+\frac{1}{2}([x, y]|z)\langle a_1 \circ a_2, a_3\rangle + [x \circ y, z] \otimes \frac{1}{4}[a_1, a_2] \circ a_3 + \frac{1}{2}(x \circ y|z)\langle [a_1, a_2], a_3\rangle \\
&+(x|y)z \otimes \langle a_1, a_2\rangle a_3 + (x \circ y) \circ z \otimes \frac{1}{4}[[a_1, a_2], a_3] \\
&+(-1)^{\bar{a_1}\bar{a_2}}[y, [x, z]] \otimes \frac{1}{4}a_2 \circ (a_1 \circ a_3) + (-1)^{\bar{a_1}\bar{a_2}}y \circ [x, z] \otimes \frac{1}{4}[a_2, a_1 \circ a_3] \\
&+(-1)^{\bar{a_1}\bar{a_2}}\frac{1}{2}(y|[x, z])\langle a_2, a_1 \circ a_3\rangle + (-1)^{\bar{a_1}\bar{a_2}}[y, x \circ z] \otimes \frac{1}{4}a_2 \circ [a_1, a_3] \\
&+(-1)^{\bar{a_1}\bar{a_2}}y \circ (x \circ z) \otimes \frac{1}{4}[a_2, [a_1, a_3]] + (-1)^{\bar{a_1}\bar{a_2}}\frac{1}{2}(y|x \circ z)\langle a_2, [a_1, a_3]\rangle \\
&-(-1)^{\bar{a_2}\bar{a_3}}(x|z)y \otimes \langle a_1, a_3\rangle a_2.
\end{aligned}$$

Now suppose that $x = E_{1,2}, y = E_{2,3}$ and $z = E_{3,1}$ in (21). Then we see that

$$\begin{aligned}
&(E_{1,1} - E_{2,2}) \otimes \frac{1}{4} a_1 \circ (a_2 \circ a_3) + (E_{1,1} + E_{2,2} - \frac{2}{n} I) \otimes \frac{1}{4}[a_1, a_2 \circ a_3] \qquad (22)\\
&+\frac{1}{2n}\langle a_1, a_2 \circ a_3\rangle + (E_{1,1} - E_{2,2}) \otimes \frac{1}{4} a_1 \circ [a_2, a_3]\\
&+(E_{1,1} + E_{2,2} - \frac{2}{n} I) \otimes \frac{1}{4}[a_1, [a_2, a_3]] + \frac{1}{2n}\langle a_1, [a_2, a_3]\rangle \ =\\
&(E_{1,1} - E_{3,3}) \otimes \frac{1}{4}(a_1 \circ a_2) \circ a_3 + (E_{1,1} + E_{3,3} - \frac{2}{n} I) \otimes \frac{1}{4}[a_1 \circ a_2, a_3]\\
&+\frac{1}{2n}\langle a_1 \circ a_2, a_3\rangle + (E_{1,1} - E_{3,3}) \otimes \frac{1}{4}[a_1, a_2] \circ a_3\\
&+(E_{1,1} + E_{3,3} - \frac{2}{n} I) \otimes \frac{1}{4}[[a_1, a_2], a_3] + \frac{1}{2n}\langle [a_1, a_2], a_3\rangle\\
&+(-1)^{\bar{a_1}\bar{a_2}}(E_{3,3} - E_{2,2}) \otimes \frac{1}{4} a_2 \circ (a_1 \circ a_3)\\
&-(-1)^{\bar{a_1}\bar{a_2}}(E_{2,2} + E_{3,3} - \frac{2}{n} I) \otimes \frac{1}{4}[a_2, a_1 \circ a_3] - (-1)^{\bar{a_1}\bar{a_2}}\frac{1}{2n}\langle a_2, a_1 \circ a_3\rangle\\
&+(-1)^{\bar{a_1}\bar{a_2}}(E_{2,2} - E_{3,3}) \otimes \frac{1}{4} a_2 \circ [a_1, a_3]\\
&+(-1)^{\bar{a_1}\bar{a_2}}(E_{2,2} + E_{3,3} - \frac{2}{n} I) \otimes \frac{1}{4}[a_2, [a_1, a_3]] + (-1)^{\bar{a_1}\bar{a_2}}\frac{1}{2n}\langle a_2, [a_1, a_3]\rangle.
\end{aligned}$$

Now as $n \geq 4$, the elements $E_{1,1} - E_{2,2}$, $E_{1,1} - E_{3,3}$, and $E_{1,1} + E_{2,2} - \frac{2}{n} I$ are linearly independent. Moreover,

$$\begin{aligned}
&E_{3,3} - E_{2,2} = (E_{1,1} - E_{2,2}) - (E_{1,1} - E_{3,3})\\
&E_{2,2} + E_{3,3} - \frac{2}{n} I = (E_{1,1} + E_{2,2} - \frac{2}{n} I) - (E_{1,1} - E_{3,3})\\
&E_{1,1} + E_{3,3} - \frac{2}{n} I = (E_{1,1} - E_{2,2}) - (E_{1,1} - E_{3,3}) + (E_{1,1} + E_{2,2} - \frac{2}{n} I).
\end{aligned}$$

Thus, the coefficient of $E_{1,1} - E_{2,2}$ in (22) says that

$$\begin{aligned}
&\frac{1}{4} a_1 \circ (a_2 \circ a_3) + \frac{1}{4} a_1 \circ [a_2, a_3]\\
&\qquad = \frac{1}{4}[a_1 \circ a_2, a_3] + \frac{1}{4}[[a_1, a_2], a_3]\\
&\qquad\qquad +(-1)^{\bar{a_1}\bar{a_2}}\frac{1}{4} a_2 \circ (a_1 \circ a_3) - (-1)^{\bar{a_1}\bar{a_2}}\frac{1}{4} a_2 \circ [a_1, a_3],
\end{aligned}$$

or

$$a_1 \circ (a_2 a_3) = [a_1 a_2, a_3] + (-1)^{\bar{a_1}(\bar{a_2}+\bar{a_3})} a_2 \circ (a_3 a_1).$$

Simplifying, we obtain

$$\begin{aligned}
a_1(a_2 a_3) - (a_1 a_2) a_3 \ &= \ (-1)^{\bar{a_1}(\bar{a_2}+\bar{a_3})}\Big(a_2(a_3 a_1) - (a_2 a_3) a_1\Big)\\
&\qquad -(-1)^{\bar{a_3}(\bar{a_1}+\bar{a_2})}\Big(a_3(a_1 a_2) - (a_3 a_1) a_2\Big).
\end{aligned}$$

Letting $(a_1, a_2, a_3) = a_1(a_2 a_3) - (a_1 a_2) a_3$, the associator, and multiplying this equation by $(-1)^{\bar{a_1}\bar{a_3}}$ shows that

$$(-1)^{\bar{a_1}\bar{a_3}}(a_1, a_2, a_3) - (-1)^{\bar{a_1}\bar{a_2}}(a_2, a_3, a_1) + (-1)^{\bar{a_2}\bar{a_3}}(a_3, a_1, a_2) = 0.$$

Cyclically permuting gives

$$(-1)^{\bar{a}_1\bar{a}_2}(a_2, a_3, a_1) - (-1)^{\bar{a}_2\bar{a}_3}(a_3, a_1, a_2) + (-1)^{\bar{a}_1\bar{a}_3}(a_1, a_2, a_3) = 0,$$

and adding these two relations shows that

$$(a_1, a_2, a_3) = 0.$$

As a consequence we deduce that

Proposition 4.1. *A with the product $a \times a' \to aa'$ is an associative superalgebra (i.e. a $\mathbb{Z}_2$-graded associative algebra).*

Let us return to (21) but this time substitute $x = E_{1,2}$, $y = E_{2,1}$, and $z = E_{2,3}$. As $[y, z] = 0 = y \circ z$, equation (21) in this instance reduces to

$$\begin{aligned} 0 \;=\; & -E_{2,3} \otimes \frac{1}{4}(a_1 \circ a_2) \circ a_3 - E_{2,3} \otimes \frac{1}{4}[a_1 \circ a_2, a_3] + E_{2,3} \otimes \frac{1}{4}[a_1, a_2] \circ a_3 \\ & + \frac{n-4}{n} E_{2,3} \otimes \frac{1}{4}[[a_1, a_2], a_3] + E_{2,3} \otimes \frac{1}{n}\langle a_1, a_2\rangle a_3 \\ & + (-1)^{\bar{a}_1\bar{a}_2} E_{2,3} \otimes \frac{1}{4} a_2 \circ (a_1 \circ a_3) + (-1)^{\bar{a}_1\bar{a}_2} E_{2,3} \otimes \frac{1}{4}[a_2, a_1 \circ a_3] \\ & + (-1)^{\bar{a}_1\bar{a}_2} E_{2,3} \otimes \frac{1}{4} a_2 \circ [a_1, a_3] + (-1)^{\bar{a}_1\bar{a}_2} E_{2,3} \otimes \frac{1}{4}[a_2, [a_1, a_3]] \end{aligned}$$

so that

$$\begin{aligned} 0 \;=\; & -\frac{1}{2}(a_1 \circ a_2) a_3 + \frac{1}{4}[a_1, a_2] \circ a_3 + \frac{n-4}{4n}[[a_1, a_2], a_3] + \frac{1}{n}\langle a_1, a_2\rangle a_3 \\ & + (-1)^{\bar{a}_1\bar{a}_2}\frac{1}{2} a_2 \circ (a_1 a_3) + (-1)^{\bar{a}_1\bar{a}_2}\frac{1}{2}[a_2, a_1 a_3], \end{aligned}$$

which implies that

$$\langle a_1, a_2\rangle a_3 = [[a_1, a_2], a_3]. \tag{23}$$

Remark 4.2. We note that the results in Proposition 4.1 and in (23) alternately could be derived from known results for A_{n-1}-graded Lie algebras using Grassmann envelopes, as $(\mathfrak{g} \otimes A) \otimes D$ is a subalgebra of L. See also [14].

We turn our attention next to discovering properties of the spaces B, C and D. First consider

$$\begin{aligned} [x \otimes a_1, [y \otimes a_2, u \otimes b]] \;=\; & [[x \otimes a_1, y \otimes a_2], u \otimes b] \\ & + (-1)^{\bar{a}_1\bar{a}_2}[y \otimes a_2, [x \otimes a_1, u \otimes b]] \end{aligned} \tag{24}$$

with $x = E_{1,2}$, $y = E_{2,3}$ and $u = e_3$ (a standard basis element of V). This gives

$$e_1 \otimes a_1(a_2 b) = e_1 \otimes \frac{1}{2}(a_1 \circ a_2) b + e_1 \otimes \frac{1}{2}[a_1, a_2] b$$

from which we see that

Proposition 4.3. *B is an A-module: $a_1(a_2b) = (a_1a_2)b$.*

Now (24) with $x = E_{1,2}$, $y = E_{2,1}$, $u = e_1$, says that

$$e_1 \otimes a_1(a_2b) = e_1 \otimes \frac{1}{2}(a_1 \circ a_2)b + (E_{1,1} + E_{2,2} - \frac{2}{n}I)e_1 \otimes \frac{1}{2}[a_1, a_2]b + e_1 \otimes \frac{1}{n}\langle a_1, a_2\rangle b.$$

This gives the result

$$\langle a_1, a_2\rangle b = [a_1, a_2]b \tag{25}$$

for all $a_1, a_2 \in A$, $b \in B$. Likewise

$$\begin{aligned}[v^* \otimes c, [x \otimes a_1, y \otimes a_2]] \quad = \quad & [[v^* \otimes c, x \otimes a_1], y \otimes a_2] \\ & + (-1)^{a_1c}[x \otimes a_1, [v^* \otimes c, y \otimes a_2]]\end{aligned} \tag{26}$$

with $x = E_{1,2}$, $y = E_{2,3}$, and $v^* = e_1^*$ produces the relation

$$e_3^* \otimes \frac{1}{2}c\Big(a_1 \circ a_2 + [a_1, a_2]\Big) = e_3^* \otimes (ca_1)a_2$$

which says

Proposition 4.4. *C is a right A-module: $c(a_1a_2) = (ca_1)a_2$.*

Choosing instead $x = E_{1,2}$, $y = E_{2,1}$ and $v^* = e_1^*$ in (26) shows that

$$e_1^* \otimes \Big(\frac{1}{2}c(a_1 \circ a_2) + \frac{n-2}{n}e_1^* \otimes \frac{1}{2}c[a_1, a_2] - (-1)^{(\bar{a_1}+\bar{a_2})\bar{c}}\frac{1}{n}\langle a_1, a_2\rangle c\Big) = e_1^* \otimes (ca_1)a_2.$$

Consequently,

$$\langle a_1, a_2\rangle c = (-1)^{(\bar{a_1}+\bar{a_2})\bar{c}}c[a_1, a_2]. \tag{27}$$

Now let us tackle substitutions of the identity

$$[x \otimes a, [u \otimes b, v^* \otimes c]] = [xu \otimes ab, v^* \otimes c] - (-1)^{\bar{a}(\bar{b}+\bar{c})}[u \otimes b, v^*x \otimes ca], \tag{28}$$

which rephrased gives

$$\begin{aligned}&[x, uv^*] \otimes \frac{1}{2}a \circ (b, c) + x \circ \Big(uv^* - \frac{1}{n}\mathrm{tr}(uv^*)I\Big) \otimes \frac{1}{2}[a, (b, c)] \\ &\quad +(x|uv^*)\langle a, (b, c)\rangle - (-1)^{\bar{a}(\bar{b}+\bar{c})}\frac{1}{n}\mathrm{tr}(uv^*)x \otimes \langle b, c\rangle a \\ &\quad = \Big(xuv^* - \frac{1}{n}\mathrm{tr}(xuv^*)I\Big) \otimes (ab, c) + \frac{1}{n}\mathrm{tr}(xuv^*)\langle ab, c\rangle \\ &\qquad -(-1)^{\bar{a}(\bar{b}+\bar{c})}\left(\Big(uv^*x - \frac{1}{n}\mathrm{tr}(uv^*x)I\Big) \otimes (b, ca) + \frac{1}{n}\mathrm{tr}(uv^*x)\langle b, ca\rangle\right)\end{aligned} \tag{29}$$

Starting with $x = E_{1,2}$, $u = e_2$, $v^* = e_3^*$, we see that

$$a(b, c) = (ab, c). \tag{30}$$

Proceeding with $x = E_{1,2}$, $u = e_3$, and $v^* = e_1^*$, we determine that

$$-E_{3,2} \otimes \frac{1}{2}a \circ (b, c) + E_{3,2} \otimes \frac{1}{2}[a, (b, c)] = -(-1)^{\bar{a}(\bar{b}+\bar{c})}E_{3,2} \otimes (b, ca),$$

which implies

$$(b,c)a=(b,ca). \tag{31}$$

It is apparent from (29), that

$$\langle a,(b,c)\rangle=\langle ab,c\rangle-(-1)^{\bar a(\bar b+\bar c)}\langle b,ca\rangle. \tag{32}$$

Also, we see by setting $u=e_1, v^*=e_1^*$, and $x=E_{2,3}$ in (29) that

$$\langle b,c\rangle a=[(b,c),a]. \tag{33}$$

In fact, relations (30-33) imply that (29) holds.

Next we compute the Jacobi identity

$$[[u_1\otimes b_1,v^*\otimes c],u_2\otimes b_2]=-(-1)^{\bar{b_2}\bar c}[u_1\otimes b_1,[u_2\otimes b_2,v^*\otimes c]], \tag{34}$$

which says

$$\begin{aligned}\left(u_1v^*-\frac{1}{n}\mathrm{tr}(u_1v^*)I\right)u_2\otimes(b_1,c)b_2+\frac{1}{n}\mathrm{tr}(u_1v^*)u_2\otimes\langle b_1,c\rangle b_2\\ =(-1)^{\bar{b_1}\bar{b_2}+\bar{b_1}\bar c+\bar{b_2}\bar c}\Bigg(\left(u_2v^*-\frac{1}{n}\mathrm{tr}(u_2v^*)I\right)u_1\otimes(b_2,c)b_1\\ +\frac{1}{n}\mathrm{tr}(u_2v^*)u_1\otimes\langle b_2,c\rangle b_1\Bigg)\end{aligned}$$

Therefore, as $v^*u=\mathrm{tr}(uv^*)$ for all $u\in V,\ v^*\in V^*$,

$$\begin{aligned}v^*u_2u_1\otimes\left((b_1,c)b_2-(-1)^{\bar{b_1}\bar{b_2}+\bar{b_1}\bar c+\bar{b_2}\bar c}\frac{1}{n}\langle b_2,c\rangle b_1+(-1)^{\bar{b_1}\bar{b_2}+\bar{b_1}\bar c+\bar{b_2}\bar c}\frac{1}{n}(b_2,c)b_1\right)\\ =v^*u_1u_2\otimes\left((-1)^{\bar{b_1}\bar{b_2}+\bar{b_1}\bar c+\bar{b_2}\bar c}(b_2,c)b_1-\frac{1}{n}\langle b_1,c\rangle b_2+\frac{1}{n}(b_1,c)b_2\right)\end{aligned}$$

giving

$$\langle b_1,c\rangle b_2=(b_1,c)b_2+(-1)^{\bar{b_1}\bar{b_2}+\bar{b_1}\bar c+\bar{b_2}\bar c}n(b_2,c)b_1 \tag{35}$$

There is an analogous relation for C:

$$\langle b,c_1\rangle c_2=-(-1)^{\bar b\bar{c_1}}nc_1(b,c_2)-(-1)^{(\bar b+\bar{c_1})\bar{c_2}}c_2(b,c_1). \tag{36}$$

Now observe that the Jacobi identity with d and various other substitutions show the following hold:

$$\begin{array}{rcl}d(aa') &=& d(a)a'+(-1)^{\bar a\bar d}ad(a')\\ d(ab) &=& d(a)b+(-1)^{\bar a\bar d}ad(b)\\ d(ca) &=& d(c)a+(-1)^{\bar c\bar d}cd(a)\\ d(b,c) &=& (d(b),c)+(-1)^{\bar b\bar d}(b,d(c))\\ [d,\langle b,c\rangle] &=& \langle d(b),c\rangle+(-1)^{\bar b\bar d}\langle b,d(c)\rangle.\end{array} \tag{37}$$

To summarize these results we have the following:

Theorem 4.5. *Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra over a field $\mathbb{K}$ of characteristic zero satisfying the assumptions of (4.). Then there exist an associative superalgebra A with unit element, a ($\mathbb{Z}_2$-graded) left A-module B, a ($\mathbb{Z}_2$-graded) right A-module C, and a Lie superalgebra D over $\mathbb{K}$, such that*

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus D, \tag{38}$$

where D is the sum of the trivial $\mathfrak{g}$-modules. The multiplication is as in (19), where the maps $b \times c \to (b, c) \in A$ and $b \times c \to \langle b, c\rangle \in D$ satisfy (30-37). Conversely, a $\mathbb{Z}_2$-graded algebra L as in (38), where A is an associative superalgebra with unit element, B is a ($\mathbb{Z}_2$-graded) left A-module, C is a ($\mathbb{Z}_2$-graded) right A-module, and D is a Lie superalgebra with a representation on the superspace $A \oplus B \oplus C$, is a Lie superalgebra if the multiplication is as in (19), and relations (23), (25), (27), (30-33), and (35-37) hold.

Proof. We have shown one direction already. For the converse, observe that the associativity of A along with (23) implies the Jacobi (super)identity (21) for three elements from $\mathfrak{g} \otimes A$. The fact that B is a left A-module and (25) imply that (24) holds. Similarly, (26) follows from the fact that C is a right A-module and (27). Then (30-33) give (28); while (35) is equivalent to (34) holding. The Jacobi (super)identity for two elements from $V^* \otimes C$ and one from $V \otimes B$, is equivalent to (36). Commutators of three elements from $V \otimes B$ or three elements from $V^* \otimes C$ are 0. As D is a Lie superalgebra and elements from D are assumed to satisfy (37), then L is a Lie superalgebra. ∎

5. Locally Finite Lie Algebras That Are Close to Root Graded Lie Algebras

Some locally finite simple Lie algebras are not root graded for any root system but closely resemble root graded Lie algebras. The one-sided limits of special linear Lie algebras, which are a particular kind of diagonal limit, provide a natural class of such algebras. For one-sided limit algebras, if $\mathfrak{g} = \mathfrak{sl}_n$ for $n \geq 4$ is a certain fixed term of the sequence (1), then each $V^{(i)}$ has a decomposition $V^{(i)} = V^{\oplus \ell_i} \oplus \mathbb{K}^{\oplus z_i}$ into copies of the natural n-dimensional $\mathfrak{g}$-module V and of the trivial $\mathfrak{g}$-module $\mathbb{K}$ for nonnegative ℓ_i, z_i. Then the Lie algebra $\mathfrak{g}^{(i)}$ decomposes as a $\mathfrak{g}$-module in the following way:

$$\mathfrak{g}^{(i)} = \mathfrak{g}^{\oplus a_i} \oplus V^{\oplus b_i} \oplus (V^*)^{\oplus c_i} \oplus \mathbb{K}^{\oplus d_i}. \tag{39}$$

Consequently, the limit Lie algebra $\mathcal{L} = \varinjlim \mathfrak{g}^{(i)}$ admits a decomposition

$$\mathcal{L} = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus D \tag{40}$$

for $\mathfrak{g} = \mathfrak{sl}_n$ as in the previous section. The general properties of the spaces A, B, C, D have been described in Section 4. (But here the $\mathbb{Z}_2$-gradings are trivial.) However, we can determine more precise information about these algebras and modules in this case.

Let $V = \mathbb{K}^{\oplus p_0}$ be a vector space, and set $E = \operatorname{End} V$. We consider E as the initial term of the sequence of algebras $E^{(i)} = \operatorname{End} V^{(i)}$ where $V^{(0)} = V$ and $V^{(i)} = \left(V^{(i-1)} \otimes P^{(i)}\right) \oplus Q^{(i)}$ for some spaces $P^{(i)}$, $Q^{(i)}$ of dimensions p_i, q_i, respectively, $i = 1, 2, \ldots$. We also define embeddings $\varphi^{(i)} : E^{(i-1)} \longrightarrow E^{(i)}$ by setting, for $x \in E^{(i-1)}$,

$$\varphi^{(i)}(x)\left(v^{(i-1)} \otimes a^{(i)} + b^{(i)}\right) = x(v^{(i-1)}) \otimes a^{(i)}. \tag{41}$$

Here $v^{(i-1)} \in V^{(i-1)}$, $a^{(i)} \in P^{(i)}$, $b^{(i)} \in Q^{(i)}$, and $\varphi^{(i)} : E^{(i-1)} \longrightarrow E^{(i)}$ is a homomorphism of algebras. In appropriate bases for $V^{(i-1)}$ and $V^{(i)}$, if $\mathcal{X}$ is the matrix of x, then $(\underbrace{\mathcal{X}, \ldots, \mathcal{X}}_{p_i}, \underbrace{0, \ldots, 0}_{q_i})$ is a matrix for $\varphi^{(i)}(x)$, and the direct limit of the family $\{E^{(i)}, \varphi^{(i)}\}$ is the direct limit $\mathcal{E} = \mathcal{E}(\underline{\mathfrak{p}})$ of associative matrix algebras corresponding to the sequence $\underline{\mathfrak{p}} = \{(p_i, q_i)\}$ as in Section 1. If we restrict ourselves to the Lie subalgebras $L^{(i)} = \mathfrak{sl}(V^{(i)}) \subset E^{(i)}$ under the bracket operation, then $\varphi^{(i)}$ induces a homomorphism of Lie algebras $\varphi^{(i)} : L^{(i-1)} \longrightarrow L^{(i)}$, and in this way we obtain a *one-sided* diagonal direct limit $\mathcal{L}$ of the family $\{L^{(i)}, \varphi^{(i)}\}$ of special linear Lie algebras (see [8]).

To obtain the structure of $\mathcal{L}$ we need to determine the structure of $\mathcal{E}$. First of all, we give each $E^{(i)}$ a grading by the semigroup Γ of matrix units of 2×2 matrices. For any algebra A, this amounts to writing A as the direct sum of subspaces $A = A_{1,1} \oplus A_{1,2} \oplus A_{2,1} \oplus A_{2,2}$ such that $A_{i,j}A_{i',j'} = 0$ if $j \neq i'$ and $A_{i,j}A_{j,j'} \subset A_{i,j'}$. Simple algebras with this decomposition property often have been referred to as generalized matrix algebras. Such gradings arise naturally on an algebra A with identity element 1 and an idempotent e, if one sets $A_{1,1} = eAe$, $A_{1,2} = eA(1-e)$, $A_{2,1} = (1-e)Ae$, $A_{2,2} = (1-e)A(1-e)$. This is just the *Peirce decomposition* of A with respect to e. In particular, if $A = \operatorname{End} X$ and we fix a vector space decomposition $X = Y \oplus Z$, then taking the idempotent $e \in A$ equal to the projection of X onto Y along Z produces a Γ-grading of A.

Lemma 5.1. *Suppose A, B are associative $\mathbb{K}$-algebras with unit elements and Γ-gradings by the idempotents e, f, respectively. If $\varphi : A \longrightarrow B$ is a homomorphism (a $\mathbb{K}$-linear transformation satisfying $\varphi(aa') = \varphi(a)\varphi(a')$), then φ is a Γ-graded homomorphism provided that $\varphi(1)f = f\varphi(1) = \varphi(e)$.*

Proof. Indeed, if $x = eae \in eAe$ for some $a \in A$, then

$$\begin{aligned} f\varphi(x)f = f\varphi(e)\varphi(a)\varphi(e)f &= f\varphi(1)\varphi(e)\varphi(a)\varphi(e)\varphi(1)f \\ &= \varphi(e)^2\varphi(a)\varphi(e)^2 = \varphi(e^2ae^2) = \varphi(x). \end{aligned}$$

Therefore, $\varphi(eAe) \subseteq fBf$. If $x = ea(1-e)$ for some $a \in A$, then

$$\begin{aligned} f\varphi(x)(1-f) &= f\varphi(1)\varphi(e)\varphi(a)\varphi(1-e)\varphi(1)(1-f) \\ &= \varphi(e)\varphi(e)\varphi(a)\varphi(1-e)\big(\varphi(1) - \varphi(e)\big) = \varphi(ea(1-e)) = \varphi(x), \end{aligned}$$

so that $\varphi(eA(1-e)) \subseteq fB(1-f)$. The remaining two cases can be handled in a similar manner. ∎

Suppose now that $V^{(0)} = V = \mathbb{K}^{\oplus p_0}$– is a vector space as before, and $V^{(i)} = (V^{(i-1)} \otimes P^{(i)}) \oplus Q^{(i)}$ for $i = 1, 2, \ldots$. To obtain a Γ-grading on $E^{(i)} = \operatorname{End} V^{(i)}$, we decompose space $V^{(i)}$ as $V^{(i)} = U^{(i)} \oplus W^{(i)}$ according to the following procedure. We set $U^{(0)} = V^{(0)} = V$, $W^{(0)} = 0$, and for $i \geq 1$, we suppose that $U^{(i)} = \left(U^{(i-1)} \otimes P^{(i)}\right) \subseteq \left(V^{(i-1)} \otimes P^{(i)}\right) \subseteq V^{(i)}$ and $W^{(i)} = \left(W^{(i-1)} \otimes P^{(i)}\right) \oplus Q^{(i)} \subseteq \left(V^{(i-1)} \otimes P^{(i)}\right) \oplus Q^{(i)} \subseteq V^{(i)}$. Applying induction we have the following

$$\begin{aligned} V^{(i)} = \left(V^{(i-1)} \otimes P^{(i)}\right) \oplus Q^{(i)} &= \left(\left(U^{(i-1)} \oplus W^{(i-1)}\right) \otimes P^{(i)}\right) \oplus Q^{(i)} \\ &= \left(U^{(i-1)} \otimes P^{(i)}\right) \oplus \left(\left(W^{(i-1)} \otimes P^{(i)}\right) \oplus Q^{(i)}\right) = U^{(i)} \oplus W^{(i)}. \end{aligned}$$

In particular,

$$\begin{aligned} U^{(i)} &= V \otimes P^{(1)} \otimes \cdots \otimes P^{(i)} \\ W^{(i)} &= \sum_{j=1}^{i} Q^{(j)} \otimes P^{(j+1)} \otimes \cdots \otimes P^{(i)}. \end{aligned}$$

Each $E^{(i)}$ becomes Γ-graded by the idempotent $e^{(i)}$ which is the projection onto $U^{(i)}$ along $W^{(i)}$. Now, by our definition, $\varphi^{(i)}(1)$ is exactly the projection of $V^{(i)}$ onto $V^{(i-1)} \otimes P^{(i)}$ along $Q^{(i)}$ because

$$\varphi^{(i)}(1)\left(v^{(i-1)} \otimes a^{(i)} + b^{(i)}\right) = 1(v^{(i-1)}) \otimes a^{(i)} = v^{(i-1)} \otimes a^{(i)}.$$

We have

$$V^{(i-1)} \otimes P^{(i)} = \left(U^{(i-1)} \otimes P^{(i)}\right) \oplus \left(W^{(i-1)} \otimes P^{(i)}\right)$$

and $e^{(i)}$ is the projection of $V^{(i)}$ onto $U^{(i)}$ along $W^{(i)} = \left(W^{(i-1)} \otimes P^{(i)}\right) \oplus Q^{(i)}$. As $e^{(i-1)}$ is the projection of $V^{(i-1)}$ onto $U^{(i-1)}$ along $W^{(i-1)}$,

$$\varphi^{(i)}(e^{(i-1)})\left(\left(u^{(i-1)} + w^{(i-1)}\right) \otimes a^{(i)} + b^{(i)}\right) = u^{(i-1)} \otimes a^{(i)}.$$

Thus, $\varphi^{(i)}(1)e^{(i)} = \varphi^{(i)}(e^{(i-1)}) = e^{(i)}\varphi^{(i)}(1)$, and by Lemma 5.1, each $\varphi^{(i)}$ is a Γ-graded homomorphism. Consequently, the direct limit $\mathcal{E}$ of $\{E^{(i)}, \varphi^{(i)}\}$ can be endowed with a natural Γ-grading. In particular, $a \in \mathcal{E}_{r,s}$ if and only if there exists such $i > 0$ such that $a \in (E^{(i)})_{r,s}$. So we have that $\mathcal{E}$ is a Γ-graded (simple) associative algebra.

Now, via the same construction giving $\mathcal{E}$, we define another associative algebra $\overline{\mathcal{E}}$, but starting with the initial vector space $V^{(0)} = \mathbb{K}$. (This just amounts to setting $p_0 = 1$.) Thus, the spaces $\overline{V}^{(i)}$ decompose into the direct sum of subspaces $\overline{U}^{(i)} \cong P^{(1)} \otimes \cdots \otimes P^{(i)}$ and $\overline{W}^{(i)}$, where $\overline{W}^{(i)}$ is the same as $W^{(i)}-$. The direct limit $\overline{\mathcal{E}} = \varinjlim\{\overline{E}^{(i)}, \overline{\varphi}^{(i)}\}$ is another Γ-graded algebra.

Finally, we consider a one-dimensional extension $\widetilde{V}$ of the "initial" space V, which we write simply as $\widetilde{V} = V \oplus \mathbb{K}$. We endow $\widetilde{E} = \operatorname{End}\widetilde{V}$ with the Γ-grading defined by the projection of $\widetilde{V}$ on V along the line $\mathbb{K}$. Our first result in this section will express $\mathcal{E}$ in terms of $\widetilde{E}$ and $\overline{\mathcal{E}}$. To do this we need to introduce an auxiliary construction.

Given any semigroup Ξ and two Ξ-graded associative algebras A, B, one can consider the vector space

$$C = \bigoplus_{\xi \in \Xi} (A_\xi \otimes B_\xi).$$

This carries a natural multiplication by setting, for $x \in A_\xi$, $y \in B_\xi$, $x' \in A_{\xi'}$, $y' \in B_{\xi'}$: $(x \otimes y)(x' \otimes y') = (xx') \otimes (yy')$. Let us denote C by $A_\Xi B$.

Lemma 5.2. *$C = A_\Xi B$ is a Ξ-graded associative algebra with $C_\xi = A_\xi \otimes B_\xi$.*

Proof. Obvious. ■

Now we return to our endomorphism algebras and establish the desired connection between the algebras $\mathcal{E}$, $\overline{\mathcal{E}}$ and $\widetilde{E}$. We set $\mathcal{S} = \widetilde{E}_\Gamma \overline{\mathcal{E}}$, using the Γ-grading as above. If $S^{(i)} = \widetilde{E}_\Gamma \overline{E}^{(i)}$, then the following is true.

Theorem 5.3. *For each $i \geq 1$, there is a natural Γ-graded isomorphism $\Psi^{(i)} : S^{(i)} \longrightarrow E^{(i)}$, and it extends to a Γ-graded isomorphism*

$$\Psi : \mathcal{S} = \widetilde{E}_\Gamma \overline{\mathcal{E}} \longrightarrow \mathcal{E}.$$

Proof. For a vector space $X = Y \oplus Z$, every $\alpha \in \operatorname{End} X$ can be written in the form $\alpha = \alpha_{1,1} + \alpha_{1,2} + \alpha_{2,1} + \alpha_{2,2}$ with respect to the Γ-grading arising from the projection idempotent onto Y along Z. Here $(\operatorname{End} X)_{1,1}$ can be identified naturally with $\operatorname{End} Y$, $(\operatorname{End} X)_{1,2}$ with $\operatorname{Hom}(Z, Y)$, $(\operatorname{End} X)_{2,1}$ with $\operatorname{Hom}(Y, Z)$, and $(\operatorname{End} X)_{2,2}$ with $\operatorname{End} Z$. The products are natural, as well. For example, $\alpha_{2,1}\alpha_{1,2}$ corresponds to the composition of mappings $\alpha_{1,2} \in \operatorname{Hom}(Z, Y)$ and $\alpha_{2,1} \in \operatorname{Hom}(Y, Z)$ and thus is an element of $\operatorname{End} Z$, which corresponds to $(\operatorname{End} X)_{2,2}$.

Applying this to the particular case of $\widetilde{V} = V \oplus \mathbb{K}$, we have that every $\Phi \in \widetilde{E} = \operatorname{End} \widetilde{V}$ can be expressed as $\Phi = \phi + u + u^* + \lambda$, where $\phi \in \operatorname{End} V$, $u \in \operatorname{Hom}(\mathbb{K}, V) \cong V$, $u^* \in \operatorname{Hom}(V, \mathbb{K}) \cong V^*$, $\lambda \in \operatorname{End} \mathbb{K} \cong \mathbb{K}$. The action of Φ on $v + \kappa \in \widetilde{V} = V \oplus \mathbb{K}$, $v \in V$, $\kappa \in \mathbb{K}$, is given by

$$\Phi(v + \kappa) = \underbrace{[\phi(v) + \kappa u]}_{\in V} + \underbrace{[u^*(v) + \lambda\kappa]}_{\in \mathbb{K}}.$$

Now in the case of $\overline{E}^{(i)} = \operatorname{End} \overline{V}^{(i)}$ for $\overline{V}^{(i)} = \overline{U}^{(i)} \oplus \overline{W}^{(i)}$, we have $\overline{\alpha}^{(i)} = \overline{\alpha}^{(i)}_{1,1} + \overline{\alpha}^{(i)}_{1,2} + \overline{\alpha}^{(i)}_{2,1} + \overline{\alpha}^{(i)}_{2,2}$ where $\overline{\alpha}^{(i)}_{1,1} \in \operatorname{End} \overline{U}^{(i)}$, $\overline{\alpha}^{(i)}_{1,2} \in \operatorname{Hom}(\overline{W}^{(i)}, \overline{U}^{(i)})$, $\overline{\alpha}^{(i)}_{2,1} \in \operatorname{Hom}(\overline{U}^{(i)}, \overline{W}^{(i)})$, $\overline{\alpha}^{(i)}_{2,2} \in \operatorname{End} \overline{W}^{(i)}$, where for $\overline{u}^{(i)} \in \overline{U}^{(i)}$, $\overline{w}^{(i)} \in \overline{W}^{(i)}$,

$$\overline{\alpha}^{(i)}(\overline{u}^{(i)} + \overline{w}^{(i)}) = \underbrace{[\overline{\alpha}^{(i)}_{1,1}(\overline{u}^{(i)}) + \overline{\alpha}^{(i)}_{1,2}(\overline{w}^{(i)})]}_{\in \overline{U}^{(i)}} + \underbrace{[\overline{\alpha}^{(i)}_{2,1}(\overline{u}^{(i)}) + \overline{\alpha}^{(i)}_{2,2}(\overline{w}^{(i)})]}_{\in \overline{W}^{(i)}}.$$

Finally, for $E^{(i)} = \operatorname{End} V^{(i)}$ we want to use

$$\begin{aligned}
\operatorname{End} U^{(i)} &= \operatorname{End}(V \otimes \overline{U}^{(i)}) \cong \operatorname{End} V \otimes \operatorname{End} \overline{U}^{(i)} && (42)\\
\operatorname{Hom}(W^{(i)}, U^{(i)}) &= \operatorname{Hom}(W^{(i)}, V \otimes \overline{U}^{(i)}) && (43)\\
&\cong \operatorname{Hom}(\mathbb{K}, V) \otimes \operatorname{Hom}(\overline{W}^{(i)}, \overline{U}^{(i)})\\
&\cong V \otimes \operatorname{Hom}(\overline{W}^{(i)}, \overline{U}^{(i)})\\
\operatorname{Hom}(U^{(i)}, W^{(i)}) &= \operatorname{Hom}(V \otimes \overline{U}^{(i)}, W^{(i)}) && (44)\\
&\cong \operatorname{Hom}(V, \mathbb{K}) \otimes \operatorname{Hom}(\overline{U}^{(i)}, \overline{W}^{(i)})\\
&\cong V^* \otimes \operatorname{Hom}(\overline{U}^{(i)}, \overline{W}^{(i)})\\
\operatorname{End} W^{(i)} &= \operatorname{End} \overline{W}^{(i)}. && (45)
\end{aligned}$$

The isomorphisms in formulas (42)-(45) define the mappings $\Psi_{r,s}^{(i)}$, $r,s=1,2$ of the Γ-graded components $S_{r,s}^{(i)}=\widetilde{E}_{r,s}\otimes\overline{E}_{r,s}^{(i)}$ of $S^{(i)}$ to the respective graded components $E_{r,s}^{(i)}$ of $E^{(i)}$.

The explicit expressions for the action of Ψ are displayed below. In these formulas we give the action only on the elements of $V^{(i)}$ of the form $v\otimes\overline{u}^{(i)}+w^{(i)}$, $v\in V$, $\overline{u}^{(i)}\in\overline{U}^{(i)}$, $w^{(i)}\in W^{(i)}$; for the other elements, the linearity and bilinearity of the operations involved can be used to determine the images.

$$\begin{aligned}
\Psi_{1,1}^{(i)}(\psi\otimes\overline{\alpha}_{1,1}^{(i)})(v\otimes\overline{u}^{(i)}+w^{(i)}) &= \psi(v)\otimes\overline{\alpha}_{1,1}^{(i)}(\overline{u}^{(i)})\in V\otimes\overline{U}^{(i)} \\
&= U^{(i)} \qquad (46)\\
\Psi_{1,2}^{(i)}(u\otimes\overline{\alpha}_{1,2}^{(i)})(v\otimes\overline{u}^{(i)}+w^{(i)}) &= u\otimes\overline{\alpha}_{1,2}^{(i)}(w^{(i)})\in U^{(i)}, \qquad (47)\\
\Psi_{2,1}^{(i)}(u^*\otimes\overline{\alpha}_{2,1}^{(i)})(v\otimes\overline{u}^{(i)}+w^{(i)}) &= u^*(v)\overline{\alpha}_{2,1}^{(i)}(\overline{u}^{(i)})\in W^{(i)}, \qquad (48)\\
\Psi_{2,2}^{(i)}(\kappa\otimes\overline{\alpha}_{2,2}^{(i)})(v\otimes\overline{u}^{(i)}+w^{(i)}) &= \kappa\overline{\alpha}_{2,2}^{(i)}(w^{(i)})\in W^{(i)}. \qquad (49)
\end{aligned}$$

Again, we see that

$$\Psi_{r,s}^{(i)}(S_{r,s}^{(i)})\subseteq E_{r,s}^{(i)} \qquad (50)$$

for all $r,s=1,2$. Now it is a routine computation to verify that $\Psi^{(i)}=\Psi_{1,1}^{(i)}\oplus\Psi_{1,2}^{(i)}\oplus\Psi_{2,1}^{(i)}\oplus\Psi_{2,2}^{(i)}$ is in fact a homomorphism of $S^{(i)}$ into $E^{(i)}$. Thanks to (50) there are "only" eight cases to deal with. They are especially simple if both factors come from $S_{1,1}^{(i)}$ or $S_{2,2}^{(i)}$. To demonstrate one of the "harder" cases, we check what happens for $S_{1,2}^{(i)}$ and $S_{2,1}^{(i)}$. If we first multiply inside $S^{(i)}$, then we obtain

$$(u\otimes\overline{\alpha}_{1,2}^{(i)})(u^*\otimes\overline{\alpha}_{2,1}^{(i)})=uu^*\otimes\overline{\alpha}_{1,2}^{(i)}\overline{\alpha}_{2,1}^{(i)}$$

where $uu^*\in\operatorname{End}V$ is the linear transformation acting on $v\in V$ as $uu^*(v)=u^*(v)\,u$. (Note $u^*(v)$ is just the matrix product u^*v.) Thus the action of the image of the product on a sample element $v\otimes\overline{u}^{(i)}+w^{(i)}$ of $V^{(i)}$ produces $u^*(v)u\otimes(\overline{\alpha}_{1,2}^{(i)}\overline{\alpha}_{2,1}^{(i)})(\overline{u}^{(i)})$. The product of the images of the factors yields

$$\begin{aligned}
\Psi^{(i)}(u\otimes\overline{\alpha}_{2,1}^{(i)})\left(u^*(v)\overline{\alpha}_{2,1}^{(i)}(\overline{u}^{(i)})\right) &= u\otimes\overline{\alpha}_{2,1}^{(i)}\left(u^*(v)\overline{\alpha}_{1,2}^{(i)}(\overline{u}^{(i)})\right)\\
&= u^*(v)u\otimes\left(\overline{\alpha}_{1,2}^{(i)}(\overline{\alpha}_{2,1}^{(i)}(\overline{u}^{(i)})\right),
\end{aligned}$$

the same thing. All the remaining cases may be treated in a similar fashion. Moreover, if we look at formulas (42)-(45) we can convince ourselves easily that each $\Psi^{(i)}$ is actually an isomorphism of the respective algebras $S^{(i)}$ and $E^{(i)}$ (recall that each $E^{(i)}$ is a simple algebra). Thus, we have proved the first claim of Theorem 5.3.

To establish the second claim we need to show that the family of isomorphisms $\Psi=\left\{\Psi^{(i)}\right\}$ is compatible with the structure mappings $\left\{\varphi^{(i)}\right\}$ and $\left\{1_\Gamma\overline{\varphi}^{(i)}\right\}$, where 1 stands for the identity mapping of $\widetilde{E}$ and the Γ-product $1_\Gamma\overline{\varphi}^{(i)}$ has quite a natural meaning: this is the same as the tensor product but on the Γ-homogeneous components only. That is, we have to check that for any $i\geq 1$ and every $r,s=1,2$ that

$$\varphi^{(i)}\Psi_{r,s}^{(i-1)}=\Psi_{r,s}^{(i)}(1_\Gamma\overline{\varphi}_{r,s}^{(i)}). \qquad (51)$$

holds on Γ-homogeneous elements. Since all mappings $\varphi^{(i)}$ and $\overline{\varphi}^{(i)}$ are Γ-graded, their restrictions $\varphi^{(i)}_{r,s} : E^{(i-1)}_{r,s} \longrightarrow E^{(i)}_{r,s}$ and $\overline{\varphi}^{(i)}_{r,s} : \overline{E}^{(i-1)}_{r,s} \longrightarrow \overline{E}^{(i)}_{r,s}$ are well-defined.

To prove (51) we have to apply both sides to the elements of $V^{(i-1)}$. By the above, it is sufficient to consider only the elements of the form

$$v^{(i)} = v \otimes \overline{u}^{(i-1)} \otimes a^{(i)} + w^{(i-1)} \otimes b^{(i)} + c^{(i)}, \tag{52}$$

where $v \in V$, $\overline{u}^{(i-1)} \in \overline{U}^{(i-1)}$, $w^{(i-1)} \in W^{(i-1)}$, $a^{(i)}, b^{(i)} \in P^{(i)}$, $c^{(i)} \in Q^{(i)}$. It is important to indicate the homogeneous decomposition of $v^{(i)} = u^{(i)} + w^{(i)}$. We have

$$u^{(i)} = v \otimes \overline{u}^{(i-1)} \otimes a^{(i)} \quad \text{and} \quad w^{(i)} = w^{(i-1)} \otimes b^{(i)} + c^{(i)}. \tag{53}$$

It is necessary to consider four cases, and in each one we apply both sides of (51) to an element of $S^{(i-1)}$ to obtain two linear transformations of $V^{(i)}$. Then we have to apply both these transformations to (52). Since their patterns are similar, we put more emphasis on the $r = 1$, $s = 1$ case, and in the remaining cases, simply write down the final conclusion.

Case 1: $r = 1, s = 1$. The result of applying both sides of (51) to a homogeneous element of $S^{(i-1)}$ of degree $1, 1$, i.e. to an element $\psi \otimes \overline{\alpha}^{(i-1)}_{1,1}$, is a linear transformation from $E^{(i)}_{1,1}$. Hence we only have to consider how it acts on $u^{(i)} = v \otimes \overline{u}^{(i-1)} \otimes a^{(i)}$ in (53). Let us start with the left-hand side. According to (41), we have

$$\begin{aligned}\Big(\big(\varphi^{(i)}\Psi^{(i-1)}_{1,1}\big)\big(\psi \otimes \overline{\alpha}^{(i-1)}_{1,1}\big)\Big)&\big(v \otimes \overline{u}^{(i-1)} \otimes a^{(i)}\big) \\ &= \Big(\psi(v) \otimes \overline{\alpha}^{(i-1)}_{1,1}\big(\overline{u}^{(i-1)}\big)\Big) \otimes a^{(i)}.\end{aligned}$$

Now if we consider the right-hand side, then we have to use (46), with the transformation $(1_\Gamma \varphi^{(i)})(\psi \otimes \overline{\alpha}^{(i-1)}_{1,1}) = \psi \otimes \varphi^{(i)}(\overline{\alpha}^{(i-1)}_{1,1})$. From (41), we obtain the element $\psi(v) \otimes \Big(\overline{\alpha}^{(i-1)}_{1,1}(\overline{u}^{(i-1)}) \otimes a^{(i)}\Big)$. So, in this case, we are done.

Case 2: $r = 1, s = 2$. This case involves applying both sides of (51) on a homogeneous element $u \otimes \overline{\alpha}^{(i-1)}_{1,2}$ of $S^{(i-1)}$ of degree $1, 2$, then computing the action on $w^{(i)} = w^{(i-1)} \otimes b^{(i)} + c^{(i)}$ in (53). The final result on both sides is $u \otimes \overline{\alpha}^{(i-1)}_{1,2}(w^{(i-1)}) \otimes b^{(i)}$.

Case 3: $r = 2, s = 1$. The effect of applying (51) to a homogeneous element $u^* \otimes \overline{\alpha}^{(i-1)}_{2,1}$ of $S^{(i-1)}$ of degree $2, 1$ and computing the action on $u^{(i)} = v \otimes \overline{u}^{(i-1)} \otimes a^{(i)}$ in (53) is $u^*(v) \otimes \overline{\alpha}^{(i-1)}_{2,1}(\overline{u}^{(i-1)}) \otimes a^{(i)}$ for both sides.

Case 4: $r = 2, s = 2$. The result of applying both sides of (51) to an element $\kappa \otimes \overline{\alpha}^{(i-1)}_{2,2} \in S^{(i-1)}$ of degree $2, 2$ and then acting on $w^{(i)} = w^{(i-1)} \otimes b^{(i)} + c^{(i)}$ in (53) gives $\kappa \overline{\alpha}^{(i-1)}_{2,2}(w^{(i-1)}) \otimes b^{(i)}$ on both sides.

We have shown that the sets $\{\Psi^{(i)}\}$ and $\{\varphi^{(i)}\}$ have the desired properties. Finally, we conclude that the family $\{\Psi^{(i)} \mid i = 1, 2, \ldots\}$ is a Γ-graded isomorphism of algebras $\mathcal{S}$ and $\mathcal{E}$, as required. This completes the proof of Theorem 5.3. ■

Now we are in a position to formulate a “Coordinatization Theorem” for one-sided direct limits of special linear Lie algebras. It was mentioned earlier that if $\mathcal{L}$ is such a limit, then $\mathcal{L}$ is a Lie subalgebra of the associative algebra $\mathcal{E}$ studied above regarded as a Lie algebra under the commutator product. Viewed as a module over its subalgebra $\mathfrak{g} = \mathfrak{sl}(V)$, $\mathcal{L}$ is a $\mathfrak{g}$-submodule of $\mathcal{E}$. We can apply Theorem 5.3 and write $\mathcal{E}$ as $\mathcal{E} \cong \mathcal{S} = (\operatorname{End}\widetilde{V})_\Gamma\overline{\mathcal{E}}$. Then expressing $\operatorname{End}\widetilde{V}$ as $\operatorname{End}V \oplus V \oplus V^* \oplus \mathbb{K}$ produces a decomposition for $\mathcal{E}$,

$$\mathcal{E} = \big((\operatorname{End}V)\otimes\overline{\mathcal{E}}_{1,1}\big) \oplus (V\otimes\overline{\mathcal{E}}_{1,2}) \oplus (V^*\otimes\overline{\mathcal{E}}_{2,1}) \oplus (\mathbb{K}\otimes\overline{\mathcal{E}}_{2,2}). \tag{54}$$

It is convenient to express the multiplication in $\mathcal{E}$ in terms of the decomposition in (54). Using the notation from the proof of Theorem 5.3, we list all the nonzero products:

(1) $(\psi\otimes\overline{\alpha}_{1,1})(\psi'\otimes\overline{\beta}_{1,1}) = \psi\psi'\otimes\overline{\alpha}_{1,1}\overline{\beta}_{1,1}$

(2) $(\psi\otimes\overline{\alpha}_{1,1})(u\otimes\overline{\alpha}_{1,2}) = \psi u\otimes\overline{\alpha}_{1,1}\overline{\alpha}_{1,2}$

(3) $(u\otimes\overline{\alpha}_{1,2})(v^*\otimes\overline{\alpha}_{2,1}) = uv^*\otimes\overline{\alpha}_{1,2}\overline{\alpha}_{2,1}$

(4) $(u\otimes\overline{\alpha}_{1,2})(\kappa\otimes\overline{\alpha}_{2,2}) = \kappa u\otimes\overline{\alpha}_{1,2}\overline{\alpha}_{2,2}$

(5) $(v^*\otimes\overline{\alpha}_{2,1})(\psi\otimes\overline{\alpha}_{1,1}) = v^*\psi\otimes\overline{\alpha}_{2,1}\overline{\alpha}_{1,1}$

(6) $(v^*\otimes\overline{\alpha}_{2,1})(u\otimes\overline{\alpha}_{1,2}) = v^*u\otimes\overline{\alpha}_{2,1}\overline{\alpha}_{1,2}$

(7) $(\kappa\otimes\overline{\alpha}_{2,2})(v^*\otimes\overline{\alpha}_{2,1}) = \kappa v^*\otimes\overline{\alpha}_{2,2}\overline{\alpha}_{2,1}$

(8) $(\kappa\otimes\overline{\alpha}_{2,2})(\lambda\otimes\overline{\beta}_{2,2}) = \kappa\lambda\otimes\overline{\alpha}_{2,2}\overline{\beta}_{2,2}$.

Recall that the products $\psi\psi'$, ψu, $v^*\psi$ are just matrix multiplication, so for example, the linear function $v^*\psi$ is defined as $(v^*\psi)(u) = v^*\psi u$ for $u\in V$.

The algebra $\overline{\mathcal{E}}_{1,1}$ has an identity e, which is the limit of the idempotents $e^{(i)}$. Moreover, the image of $E = \operatorname{End}V$ in $\mathcal{S}$ under Ψ^{-1} can be identified with $\operatorname{End}V\otimes e$. Indeed, for any $\psi\in\mathcal{E}$ and $v^{(1)} = v\otimes a^{(1)} + b^{(1)}$ with $v\in V$, $a^{(1)}\in P^{(1)}$, $b^{(1)}\in Q^{(1)}$, we have

$$\big(\Psi^{(1)}(\psi\otimes e^{(1)})\big)\,(v\otimes a^{(1)}+b^{(1)}) = \psi(v)\otimes a^{(1)} = \varphi^{(1)}(\psi)(v\otimes a^{(1)}+b^{(1)}).$$

Thus,

$$\Psi^{(1)}(\operatorname{End}V\otimes e^{(1)}) = \varphi^{(1)}(\operatorname{End}V)$$

and

$$\operatorname{End}V\otimes e^{(1)} = ((\Psi)^{(1)})^{-1}\varphi^{(1)}(\operatorname{End}V).$$

Applying $1\otimes\varphi^{(2)}$ to both sides of the latter equation and using (51), we obtain

$$\operatorname{End}V\otimes e^{(2)} = (1\otimes\varphi^{(2)}(\Psi^{(1)})^{-1})\varphi^{(1)}(\operatorname{End}V) = \varphi^{(2)}\varphi^{(1)}(\operatorname{End}V).$$

Continuing in this fashion, we find that $\operatorname{End}V$ can be identified in the limit with $\operatorname{End}V\otimes e$ as claimed above.

For each $\overline{r}_{1,1}\in\overline{\mathcal{E}}_{1,1}$, the subspace $\operatorname{End}V\otimes\overline{r}_{1,1}$ is a natural $\operatorname{End}V$-bimodule. In addition, $V\otimes\overline{r}_{1,2}$ is a natural left $\operatorname{End}V$-module for each $\overline{r}_{1,2}\in\overline{\mathcal{E}}_{1,2}$; $V^*\otimes\overline{r}_{2,1}$

is a natural right $\operatorname{End} V$-module for each $\overline{r}_{2,1} \in \overline{\mathcal{E}}_{2,1}$; and $\overline{\mathcal{E}}_{2,2}$ is annihilated by $\operatorname{End} V$.

These remarks enable us to describe the structure of the direct limit Lie algebra $\mathcal{L}$ as a module for $\mathfrak{g} = \mathfrak{sl}(V)$. First, we decompose $\operatorname{End} V$ as $\mathfrak{g} \oplus \mathbb{K}$ and let I denote the identity element of $\operatorname{End} V$ and 1 the identity element of the ground field $\mathbb{K}$. Thus,

$$\begin{aligned} \mathcal{E} &= \Big((\mathfrak{g} \oplus \mathbb{K}) \otimes \overline{\mathcal{E}}_{1,1}\Big) \oplus (V \otimes \overline{\mathcal{E}}_{1,2}) \oplus (V^* \otimes \overline{\mathcal{E}}_{2,1}) \oplus (1 \otimes \overline{\mathcal{E}}_{2,2}) \qquad (55) \\ &= (\mathfrak{g} \otimes \overline{\mathcal{E}}_{1,1}) \oplus (V \otimes \overline{\mathcal{E}}_{1,2}) \oplus (V^* \otimes \overline{\mathcal{E}}_{2,1}) \oplus \Big((I \otimes \overline{\mathcal{E}}_{1,1}) \oplus (1 \otimes \overline{\mathcal{E}}_{2,2})\Big). \end{aligned}$$

We digress to provide an alternate realization of $\mathcal{L}$ via traces. Recall from (2) of Section 1, that there is a trace function $\mathfrak{t}$ on $\mathcal{E}$ defined as follows. When $a \in \mathcal{E}$, say $a = A \in \mathcal{E}^{(i)}$, then $\mathfrak{t}(a) = (p_0 \cdots p_{i-1})^{-1} \operatorname{tr} A$, where "tr" denotes the ordinary trace in $\operatorname{End} V$. It is immediate that $\mathcal{L}$ is precisely the subset of elements of trace 0 in $\mathcal{E}$. In entirely the same way (in fact, just by setting $p_0 = 1$), there is a trace function $\overline{\mathfrak{t}}$ on $\overline{\mathcal{E}}$, and the following holds.

Proposition 5.4. *If $x = \psi \otimes \overline{\alpha}_{1,1} + u \otimes \overline{\alpha}_{1,2} + u^* \otimes \overline{\alpha}_{2,1} + 1 \otimes \overline{\alpha}_{2,2}$, then*

$$\mathfrak{t}(x) = \operatorname{tr}(\psi)\ \overline{\mathfrak{t}}(\overline{\alpha}_{1,1}) + \overline{\mathfrak{t}}(\overline{\alpha}_{2,2}). \qquad (56)$$

Proof. Since both $\mathfrak{t}$ and $\overline{\mathfrak{t}}$ are stable under the structure homomorphisms of the respective direct limits, it is sufficient to establish (56) on a finite "level" i where all the components belong. Since $u \otimes \overline{\alpha}^{(i)}_{1,2}$ and $u^* \otimes \overline{\alpha}^{(i)}_{2,1}$ contribute nothing to the trace, we only have to compute the trace of $\psi \otimes \overline{\alpha}^{(i)}_{1,1} + 1 \otimes \overline{\alpha}^{(i)}_{2,2}$, which gives the desired expression in (56). ■

Now set

$$D = \left\{ I \otimes \overline{\alpha}_{1,1} + 1 \otimes \overline{\alpha}_{2,2} \in \left(I \otimes \overline{\mathcal{E}}_{1,1}\right) \oplus \left(1 \otimes \overline{\mathcal{E}}_{2,2}\right) \mid n\overline{\mathfrak{t}}(\overline{\alpha}_{1,1}) + \overline{\mathfrak{t}}(\overline{\alpha}_{2,2}) = 0 \right\}. \qquad (57)$$

Then D is the centralizer of $\mathfrak{g}$ in $\mathcal{L}$, and we have the main result of this section.

Theorem 5.5. *Let $\underline{\mathfrak{p}} = \{(p_i, q_i)\}$ where the p_i are positive integers, $q_0 = 0$, and the q_i are nonnegative integers, and let $\underline{\mathfrak{q}} = \{(\overline{p}_i, \overline{q}_i)\}$ where $\overline{q}_i = q_i$ for all $i \geq 0$, $\overline{p}_i = p_i$ for all $i \geq 1$ and $\overline{p}_0 = 1$. Set $V^{(0)} = \mathbb{K}^{\oplus p_0}$, and let $V^{(i+1)} = (V^{(i)})^{\oplus p_i} \oplus \mathbb{K}^{\oplus q_i}$ for all $i \geq 0$. Similarly, assume $\overline{V}^{(0)} = \mathbb{K}^{\oplus \overline{p}_0}$, and let $\overline{V}^{(i+1)} = (\overline{V}^{(i)})^{\oplus \overline{p}_i} \oplus \mathbb{K}^{\oplus \overline{q}_i}$ for all $i \geq 0$. Then the direct limit $\mathcal{L}$ of the family of special linear algebras $\mathfrak{sl}(V^{(i)}), i = 0, 1, \ldots$, with the structure homomorphisms $\varphi^{(i)}$ defined in (41) decomposes as a module for $\mathfrak{g} = \mathfrak{sl}(V)$ ($V = V^{(0)}$) as*

$$\mathcal{L} = (\mathfrak{g} \otimes \overline{\mathcal{E}}_{1,1}) \oplus (V \otimes \overline{\mathcal{E}}_{1,2}) \oplus (V^* \otimes \overline{\mathcal{E}}_{2,1}) \oplus D, \qquad (58)$$

where $\overline{\mathcal{E}} = \overline{\mathcal{E}}(\underline{\mathfrak{q}})$ is the Γ-graded simple associative algebra, which is the direct limit of the algebras $\operatorname{End} \overline{V}^{(i)}$ with similarly defined structure mappings $\overline{\varphi}^{(i)}$, and D is as in (57). The bracket in $\mathcal{L}$ is completely determined by (1) - (8) above.

The explicit form of the bracket in $\mathcal{L}$ can be derived from the formulas in (19) using (1) - (8). We simply have to compute the Lie bracket in $\mathcal{E}$ viewed in the form $\mathcal{E} = (\mathrm{End}\,\widetilde{V})_\Gamma\overline{\mathcal{E}}$. The circle $\circ$ and bracket $[\ ,\ ]$ operations have their usual meaning in the sense of derived operations on the associative algebra $\mathcal{E}$. Thus, using our standard notation in this section, we have

$$\begin{aligned}
[\psi\otimes\overline{\alpha}_{1,1},\psi'\otimes\overline{\beta}_{1,1}] &= [\psi,\psi']\otimes\frac{1}{2}(\overline{\alpha}_{1,1}\circ\overline{\beta}_{1,1}) + (\psi\circ\psi')\otimes\frac{1}{2}[\overline{\alpha}_{1,1},\overline{\beta}_{1,1}]\\
&\quad +(\psi|\psi')I\otimes[\overline{\alpha}_{1,1},\overline{\beta}_{1,1}]\\
[\psi\otimes\overline{\alpha}_{1,1},u\otimes\overline{\alpha}_{1,2}] &= \psi u\otimes\overline{\alpha}_{1,1}\overline{\alpha}_{1,2} = -[u\otimes\overline{\alpha}_{1,2},\psi\otimes\overline{\alpha}_{1,1}]\\
[v^*\otimes\overline{\alpha}_{2,1},\psi\otimes\overline{\alpha}_{1,1}] &= v^*\psi\otimes\overline{\alpha}_{2,1}\overline{\alpha}_{1,1} = -[\psi\otimes\overline{\alpha}_{1,1},v^*\otimes\overline{\alpha}_{2,1}]\\
[u\otimes\overline{\alpha}_{1,2},v^*\otimes\overline{\alpha}_{2,1}] &= \left(uv^* - \frac{1}{n}\mathrm{tr}(uv^*)I\right)\otimes\overline{\alpha}_{1,2}\overline{\alpha}_{2,1}\\
&\quad +\frac{1}{n}\mathrm{tr}(uv^*)I\otimes\overline{\alpha}_{1,2}\overline{\alpha}_{2,1} - \mathrm{tr}(uv^*)1\otimes\overline{\alpha}_{2,1}\overline{\alpha}_{1,2}\\
&= -[v^*\otimes\overline{\alpha}_{2,1},u\otimes\overline{\alpha}_{1,2}].
\end{aligned}$$

Now if $d = I\otimes\overline{\alpha}_{1,1} + 1\otimes\overline{\alpha}_{2,2}$, then

$$\begin{aligned}
[d,\psi\otimes\overline{\beta}_{1,1}] &= \psi\otimes[\overline{\alpha}_{1,1},\overline{\beta}_{1,1}] = -[\psi\otimes\overline{\beta}_{1,1},d]\\
[d,u\otimes\overline{\alpha}_{1,2}] &= u\otimes(\overline{\alpha}_{1,1}\overline{\alpha}_{1,2} - \overline{\alpha}_{1,2}\overline{\alpha}_{2,2}) = -[u\otimes\overline{\alpha}_{1,2},d]\\
[v^*\otimes\overline{\alpha}_{2,1},d] &= v^*\otimes(\overline{\alpha}_{2,1}\overline{\alpha}_{1,1} - \overline{\alpha}_{2,2}\overline{\alpha}_{2,1}) = -[d,v^*\otimes\overline{\alpha}_{2,1}]\\
[d,d']\in D &\qquad \text{for any } d,d'\in D.
\end{aligned}$$

It is easy to see from last four equations that each element $d = I\otimes\overline{\alpha}_{1,1} + 1\otimes\overline{\alpha}_{2,2}\in D$ induces a derivation of the algebras $\overline{\mathcal{E}}_{1,1}$ and $\overline{\mathcal{E}}_{2,2}$ and a derivation of the bimodules $\overline{\mathcal{E}}_{1,2}$ and $\overline{\mathcal{E}}_{2,1}$. All these transformations are induced by the inner derivation of $\overline{\mathcal{E}}$ by the element $\overline{\alpha}_{1,1} + \overline{\alpha}_{2,2}$.

6. Concluding Remarks

In our first paper [3], we studied irreducible highest weight representations of locally finite simple Lie algebras. Our results in this paper allow us to construct an entirely different kind of representation for some of the direct limit Lie algebras considered here. More specifically, for the universal covering algebras $\widehat{L}$ from Section 3., we have the following:

Theorem 6.1. *Let $\widehat{L}$ be a Lie algebra of the form $\widehat{L} = (\mathfrak{g}\otimes A)\oplus\{A,A\}$, with $\mathfrak{g} = \mathfrak{sl}_n$ for $n\geq 3$; A a unital associative algebra; $\{A,A\} = (A\otimes A)/J$, where J the subspace spanned by the elements $a\otimes b + b\otimes a, ab\otimes c + bc\otimes a + ca\otimes b$, $a,b,c\in A$; and with product as in (18). If M is a module for $\mathfrak{M}_n$ and W is a left A-module then $M\otimes W$ becomes an $\widehat{L}$-module under the action*

$$\begin{aligned}
(x\otimes a)(v\otimes w) &= xv\otimes aw\\
\{a,a'\}(v\otimes w) &= v\otimes[a,a']w.
\end{aligned}$$

We leave this theorem without proof, since it is simply a direct verification of the definitions, and also because we believe more general results can be achieved. We postpone this until a subsequent paper.

Recall from Theorem 3.2 that $\widehat{\mathfrak{sl}_n(\mathfrak{M}_{\underline{m}})} \cong \mathfrak{sl}_n(\mathfrak{M}_{\underline{m}}) \cong \mathfrak{sl}_{\underline{n}} = \varinjlim \mathfrak{sl}_{n_t}$. The significance of Theorem 6.1 for the representation theory of $\mathfrak{sl}_{\underline{n}}$ can be seen because of the vast number of irreducible modules for $A = \mathfrak{M}_{\underline{\mathfrak{m}}}$ (this is true of any infinite-dimensional simple associative algebra [15, 16]). A good candidate for an irreducible $\mathfrak{sl}_{\underline{n}}$-module comes from taking the tensor product of irreducible modules $M = \mathbb{K}^n$ and W.

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Yuri Bahturin
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, NF, Canada A1C 5S7
bahturin@mun.ca

Georgia Benkart
Department of Mathematics
University of Wisconsin - Madison
Madison, Wisconsin 53706 USA
benkart@math.wisc.edu

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