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# Homogeneity Rank of Real Representations of Compact Lie Groups

**Claudio Gorodski and Fabio Podestà**

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**Abstract.** The main result of this paper is the classification of the real irreducible representations of compact Lie groups with vanishing homogeneity rank.

## 1. Introduction

Let a compact Lie group $G$ act smoothly on a smooth manifold $M$. The codimension of the principal orbits in $M$ is called the cohomogeneity $\mathrm{cohom}(G,M)$ of the action. Püttmann, starting from an inequality for the dimension of the fixed point set of a maximal torus in $G$ due to Bredon ([1], p. 194), introduced in [14] the *homogeneity rank* of $(G,M)$ as the integer

$$\begin{aligned}\mathrm{homrk}(G,M) &= \mathrm{rk}\,G - \mathrm{rk}\,G_{\mathrm{princ}} - \mathrm{cohom}(G,M)\\ &= \mathrm{rk}\,G - \mathrm{rk}\,G_{\mathrm{princ}} + (\dim G - \dim G_{\mathrm{princ}}) - \dim M,\end{aligned}$$

where $G_{\mathrm{princ}}$ is a principal isotropy subgroup of the action and, for a compact Lie group $K$, $\mathrm{rk}\,K$ denotes its rank, namely the dimension of a maximal torus. We will see in the next section that orbit-equivalent actions have the same homogeneity rank.

This invariant, although not with this name, had already been considered by Huckleberry and Wurzbacher who proved that a Hamiltonian action of a compact Lie group on a symplectic manifold has vanishing homogeneity rank if and only if the principal orbits are coisotropic with respect to the invariant symplectic form (see [7], Theorem 3, p. 267 for this result and other characterizations of this property). If $\rho : G \to \mathbf{U}(V)$ is a complex representation where $V$ is a complex vector space endowed with an invariant symplectic structure, then the $G$-action is automatically Hamiltonian, and it has vanishing homogeneity rank if and only if every principal orbit is coisotropic; this condition can be proved to be equivalent to the fact that a Borel subgroup of the complexified group $G^c$ has an open orbit in $V$, and also to the fact that the naturally induced representation of $G$ on the

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ring of regular functions $\mathbf{C}[V]$ splits into the sum of mutually inequivalent irreducible representations (see e.g. [11], p. 199). Complex representations with these equivalent properties are called *coisotropic* or *multiplicity-free*; Kac [8] classified the irreducible multiplicity-free representations and, later, Benson and Ratcliff [2] and, independently, Leahy [12] classified the reducible ones.

In this paper, we consider the case of an irreducible representation $\rho : G \to \mathbf{O}(V)$ of a compact Lie group $G$ on a real vector space $V$ with vanishing homogeneity rank. Since representations admitting an invariant complex structure have null homogeneity rank if and only if they are multiplicity-free, we will deal only with irreducible representations of real type, also called absolutely irreducible, namely those which admit no invariant complex structure. Our main result is the following theorem.

**Theorem 1.1.** *An absolutely irreducible representation $\rho$ of a compact connected Lie group $G$ has vanishing homogeneity rank if and only if it is either orbit-equivalent to the isotropy representation of an irreducible non-Hermitian symmetric space of inner type or it is one of the following representations:*

| $G$ | $\rho$ | $d$ | $c$ |
|---|---|---|---|
| $\mathbf{Sp}(1) \times \mathbf{Sp}(n),\ n \geq 2$ | $S^3(\mathbf{C}^2) \otimes_{\mathbf{H}} \mathbf{C}^{2n}$ | $8n$ | 3 |
| $\mathbf{SO}(4) \times \mathbf{Spin}(7)$ | $\mathbf{R}^4 \otimes \mathbf{R}^8$ | 32 | 5 |
| $\mathbf{Sp}(1) \times \mathbf{Spin}(11)$ | $\mathbf{C}^2 \otimes_{\mathbf{H}} \mathbf{C}^{32}$ | 64 | 6 |

*where $\mathbf{Spin}(7)$ acts on $\mathbf{R}^8$ via the real spin representation, $\mathbf{Spin}(11)$ acts on $\mathbf{C}^{32}$ via the complex spin representation, $d$ denotes the dimension of the representation space and $c$ denotes its cohomogeneity.*

## 2. Preliminaries

Let $(G, V)$ be an absolutely irreducible representation of a compact Lie group $G$ on a real vector space $V$. It is shown in Corollary 1.2 in [14] that the homogeneity rank of a linear representation is non positive. In this regard, the representations with vanishing homogeneity rank are precisely those with maximal homogeneity rank. The following monotonicity property that is stated on p. 375 in [14] and is valid for smooth actions on smooth manifolds will be the basis of the method of our classification. Since there is no proof in [14], we include one for the sake of completeness.

**Proposition 2.1.** *Let $(G, M)$ be a smooth action. If $G'$ be a closed subgroup of $G$, then* $\mathrm{homrk}(G', M) \leq \mathrm{homrk}(G, M)$.

**Proof.** We first prove the statement in the case in which $M$ is $G$-homogeneous, i. e. we prove that given a homogeneous space $M = G/H$, where $G$ is a compact Lie group and $H$ is a closed subgroup, for every closed subgroup $G'$ of $G$ we have

$$\mathrm{homrk}(G', G/H) \leq \mathrm{rk}\ G - \mathrm{rk}\ H.$$

We prove this by induction on the dimension of the manifold, the initial case $\dim M = 1$ being clear. Fix the point $o = [H] \in G/H$, a maximal torus $T_H$ of $H$, and a maximal torus $T$ of $G$ containing $T_H$. Since conjugation of $G'$ by elements of $G$ does not affect the homogeneity rank, we can assume that a maximal torus $T'$ of $G'$ sits inside $T$. Then we have

$$\text{rk } G' - \text{rk } G'_o \leq \dim T' - \dim(T' \cap G'_o) = \dim T' \cdot o \leq \dim T \cdot o,$$

where $G'_o$ denotes the isotropy subgroup of $G'$ at $o$. Therefore

$$\text{rk } G' - \text{rk } G'_o \leq \text{rk } G - \text{rk } H.$$

We now consider the slice representation of $G'_o$ on the normal space $W$ to the orbit $G' \cdot o$; we can assume that the dimension $k$ of $W$ is at least $2$, since otherwise $G'_o$ contains a principal isotropy subgroup of $(G, G/H)$ as a subgroup of finite index and the claim follows immediately. Denote by $S$ the unit sphere in $W$ with respect to a $G'_o$-invariant inner product in $W$ and apply the induction hypothesis. Since $G'_o$ is a closed subgroup of $\mathbf{SO}(k)$, we have

$$\begin{aligned} \text{homrk}(G'_o, S) &= \text{rk } G'_o - \text{rk } G'_{\text{princ}} - \text{cohom}(G', M) + 1 \\ &\leq \text{homrk}(\mathbf{SO}(k), S) = \frac{1 + (-1)^k}{2}, \end{aligned}$$

where $G'_{\text{princ}}$ denotes a principal isotropy subgroup of $G'$ on $M$. It then follows that

$$\text{homrk}(G', M) \leq \text{rk } G' - \text{rk } G'_o + \frac{-1 + (-1)^k}{2} \leq \text{rk } G - \text{rk } H,$$

and our claim is proved.

In the general case, we fix a $G$-regular point $p \in M$ and observe that a point $q \in G{\cdot}p$ is principal for the $G'$-action on $G{\cdot}p$ if and only if it is principal for the $G'$-action in $M$; this means that $\text{cohom}(G', M) = \text{cohom}(G, M) + \text{cohom}(G', G \cdot p)$. We know from the previous case that

$$\text{homrk}(G', G \cdot p) \leq \text{homrk}(G, G \cdot p),$$

and now our claim follows by subtracting $\text{cohom}(G, M)$ from both members of the above inequality. ∎

**Corollary 2.2.** *Let $(G, V)$ be a representation of a compact Lie group $G$ on a real vector space $V$. If $(G, V)$ is not of vanishing homogeneity rank, then the action of a closed subgroup of $G$ on $V$ is never of vanishing homogeneity rank.*

The preceding corollary indicates a strategy to classify representations with vanishing homogeneity rank. First we observe that the standard representation of $\mathbf{SO}(n)$ on $\mathbf{R}^n$ is of vanishing homogeneity rank if and only if $n$ is even. Then we need to decide which of the maximal subgroups of $\mathbf{SO}(n)$, where $n$ is even, act absolutely irreducibly on $\mathbf{R}^n$ with vanishing homogeneity rank. For each example that we encounter, we examine which of its maximal subgroups still act absolutely irreducibly on $\mathbf{R}^n$ with vanishing homogeneity rank, and so on. The process will eventually yield all the closed subgroups of $\mathbf{SO}(n)$ that act absolutely irreducibly on $\mathbf{R}^n$ with vanishing homogeneity rank. The effectiveness of this strategy is elucidated by the following well known result of Dynkin [3].

**Theorem 2.3.** (Dynkin) *1. Let $G$ be a maximal connected subgroup of $\mathbf{SO}(n)$. Then $G$ is conjugate in $\mathbf{O}(n)$ to one of the following:*

*(a)* $\mathbf{SO}(k) \times \mathbf{SO}(n-k)$, *where* $1 \leq k \leq n-1$;

*(b)* $\rho(\mathbf{SO}(p) \times \mathbf{SO}(q))$, *where* $pq = n$ *and* $3 \leq p \leq q$, *and* $\rho$ *is the real tensor product of the vector representations;*

*(c)* $\mathbf{U}(k)$, *where* $2k = n$;

*(d)* $\rho(\mathbf{Sp}(p) \times \mathbf{Sp}(q))$, *where* $4pq = n \neq 4$, *and* $\rho$ *is the quaternionic tensor product of the vector representations;*

*(e)* $\rho(G_1)$, *where* $G_1$ *is simple and* $\rho$ *is a real form of a complex irreducible representation of degree* $n$ *of real type.*

*2. Let $G$ be a maximal connected subgroup of $\mathbf{SU}(n)$. Then $G$ is conjugate to one of the following:*

*(a)* $\mathbf{SO}(n)$;

*(b)* $\mathbf{Sp}(k)$, *where* $2k = n$;

*(c)* $\mathbf{S}(\mathbf{U}(k) \times \mathbf{U}(n-k))$, *where* $1 \leq k \leq n-1$;

*(d)* $\rho(\mathbf{SU}(p) \times \mathbf{SU}(q))$, *where* $pq = n$ *and* $p \geq 3$ *and* $q \geq 2$, *and* $\rho$ *is the complex tensor product of the vector representations;*

*(e)* $\rho(G_1)$, *where* $G_1$ *is simple and* $\rho$ *is a complex irreducible representation of degree* $n$ *of complex type.*

*3. Let $G$ be a maximal connected subgroup of $\mathbf{Sp}(n)$. Then $G$ is conjugate to one of the following:*

*(a)* $\mathbf{U}(n)$;

*(b)* $\mathbf{Sp}(k) \times \mathbf{Sp}(n-k)$, *where* $1 \leq k \leq n-1$;

*(c)* $\rho(\mathbf{SO}(p) \times \mathbf{Sp}(q))$, *where* $pq = n$ *and* $p \geq 3$ *and* $q \geq 1$, *and* $\rho$ *is the real tensor product of the vector representations;*

*(d)* $\rho(G_1)$, *where* $G_1$ *is simple and* $\rho$ *is a complex irreducible representation of degree* $2n$ *of quaternionic type.*

Recall that a symmetric space of compact type $X = L/G$ is said to be of *inner type* if $\mathrm{rk}\ L = \mathrm{rk}\ G$; otherwise, $X$ is said to be of *outer type* (compare Theorem 8.6.7 on p. 255 in [15]). Moreover, the isotropy representation of $X$ is absolutely irreducible if and only if $X$ is irreducible and non-Hermitian. The following lemma implies that the isotropy representations of symmetric spaces of semisimple type that have vanishing homogeneity rank are precisely those coming from symmetric spaces of inner type.

**Lemma 2.4.** *Let $(G, V)$ be the isotropy representation of a symmetric space of compact type $X = L/G$. Then* $\mathrm{homrk}(G, V) = 0$ *if and only if* $\mathrm{rk}\ G = \mathrm{rk}\ L$.

**Proof.** Let $\mathfrak{l} = \mathfrak{g} + V$ be the Cartan decomposition of $X$ with respect to the involution, where $\mathfrak{l}$ and $\mathfrak{g}$ respectively denote the Lie algebras of $L$ and $G$. Let $\mathfrak{a} \subset V$ be a maximal Abelian subspace. By the structural theory of symmetric spaces, it is known that the dimension of $\mathfrak{a}$ is equal to the cohomogeneity of $(G,V)$, and that the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{g}$ is the Lie algebra of a principal isotropy subgroup of $(G,V)$. It follows that $\mathrm{homrk}(G,V) = \mathrm{rk}\ \mathfrak{g} - \mathrm{rk}\ \mathfrak{m} - \dim \mathfrak{a}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$. Then it is easily seen that $\mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{l}$. Now $\mathrm{rk}\ \mathfrak{m} = \dim \mathfrak{t}$, $\mathrm{rk}\ \mathfrak{l} = \dim \mathfrak{t} + \dim \mathfrak{a}$, and hence $\mathrm{homrk}(G,V) = \mathrm{rk}\ \mathfrak{g} - \mathrm{rk}\ \mathfrak{l} = \mathrm{rk}\ G - \mathrm{rk}\ L$ which proves our thesis. ∎

We will also use Theorem 1.3 of [14] which, for convenience of the reader, we restate here.

**Theorem 2.5.** (Püttmann) *Let $(G,M)$ be an isometric action of the compact Lie group $G$ on a Riemannian manifold $M$. Then, for any $x \in M$, we have*

$$\dim \nu_x(G\cdot x)^{G_x} \leq \mathrm{cohom}(G,M) - (\mathrm{rk}\ G_x - \mathrm{rk}\ G_{\mathrm{princ}}),$$

*where $\nu_x(G\cdot x)$ denotes the normal space to the orbit $G\cdot x$ at $x$, $\nu_x(G\cdot x)^{G_x}$ denotes the fixed point subspace of $G_x$ in $\nu_x(G\cdot x)$, and $G_{\mathrm{princ}}$ is a principal isotropy subgroup of $(G,M)$.*

The following proposition implies that orbit-equivalent actions have the same homogemeity rank.

**Proposition 2.6.** *Let $(G,M)$ be a smooth action. If $G'$ is a closed subgroup of $G$, and $G$ and $G'$ have the same orbits in $M$, then* $\mathrm{homrk}(G',M) = \mathrm{homrk}(G,M)$.

**Proof.** It is clearly enough to prove that if $G'$ and $G$ act transitively on the same manifold $M$, then $\mathrm{homrk}(G',M) = \mathrm{homrk}(G,M)$. If we represent $M = G/H = G'/H'$ for suitable closed subgroups $H \subset G$ and $H' \subset G'$, then we claim that

$$\mathrm{rk}\ G - \mathrm{rk}\ H = \mathrm{rk}\ G' - \mathrm{rk}\ H'.$$

This follows from the fact that, given a homogeneous space $M = G/H$ with $G$ compact, the number $\chi_\pi(M) := \mathrm{rk}\ H - \mathrm{rk}\ G$ is a homotopy invariant of $M$ (see [13], p. 207) ∎

Finally, we state the following direct consequences of the definition of homogeneity rank, which we shall repeatedly use in our arguments.

**Remark 2.7.** *Let $(G,M)$ be a smooth action of a compact Lie group $G$ on a smooth manifold $M$. Then:*

*(a) If* $\mathrm{homrk}(G,M) = 0$, *then* $\dim M \leq \dim G + \mathrm{rk}\ G$.

*(b) If $G'$ is a connected closed subgroup of $G$ having the same homogeneity rank, then* $\mathrm{rk}\ G' \geq \mathrm{rk}\ G - \mathrm{rk}\ G_{\mathrm{princ}}$. *Moreover if $G_{\mathrm{princ}}$ is finite, then $G = G'$ (indeed, $G$ and $G'$ are orbit-equivalent both with finite principal isotropy, hence $G$ and $G'$ have the same Lie algebra).*

## 3. The classification

In this section, we apply the strategy discussed in the previous section to classify absolutely irreducible representations with vanishing homogeneity rank. It is enough to consider orthogonal representations of even degree $2n$. According to Theorem 2.3, the maximal connected subgroups of $\mathbf{SO}(2n)$ acting absolutely irreducibly on $V=\mathbf{R}^{2n}$ are: $\rho(\mathbf{SO}(p)\times\mathbf{SO}(q))$, where $pq=2n$ and $3\leq p\leq q$, and $\rho$ is the real tensor product of the vector representations; $\rho(\mathbf{Sp}(p)\times\mathbf{Sp}(q))$, where $4pq=2n\neq 4$, and $\rho$ is the quaternionic tensor product of the vector representations; and $\rho(G_1)$, where $G_1$ is simple and $\rho$ is a real form of a complex irreducible representation of degree $2n$ of real type.

### 3.1. The case of $\rho(\mathbf{SO}(p)\times\mathbf{SO}(q))$ and its maximal subgroups.

Here $pq=2n$ and $3\leq p\leq q$. We have that $\rho$ is the isotropy representation of the symmetric space $\mathbf{SO}(p+q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, and $\mathrm{rk}\ \mathbf{SO}(p+q)=\mathrm{rk}\ \mathbf{SO}(p)+\mathrm{rk}\ \mathbf{SO}(q)$ because not both of $p$, $q$ are odd. It follows from Lemma 2.4 that this is an example.

Next we must investigate maximal connected subgroups $G$ of $\rho(\mathbf{SO}(p)\times\mathbf{SO}(q))$. We shall consider separately three cases which cover all the possibilities.

#### 3.1.1. $G=\rho(G_1)$, where $G_1=K\times\mathbf{SO}(q)$, and $K\subset\mathbf{SO}(p)$ is a maximal connected subgroup.

Set $\hat{G}_1=\mathbf{SO}(p)\times\mathbf{SO}(q)$, $\hat{G}=\rho(\hat{G}_1)$. There is a $\hat{G}_1$-regular point $x\in V$ whose connected principal isotropy subgroup is given by $\hat{G}_{1x}=\mathbf{SO}(q-p)\subset\mathbf{SO}(q)$. The isotropy subgroup of $G_1$ at $x$ is the intersection $\hat{G}_{1x}\cap G_1$, and its connected component is $\mathbf{SO}(q-p)$. If $G$ has vanishing homogeneity rank on $V$, then Theorem 2.5 applied to $x$ gives

$$\begin{aligned}\dim\nu_x(Gx)^{G_x} &\leq \mathrm{cohom}(G,V)-(\mathrm{rk}\ G_x-\mathrm{rk}\ G_{\mathrm{princ}})=\mathrm{rk}\ G-\mathrm{rk}\ G_x\\ &= \mathrm{rk}\ K+\mathrm{rk}\,\mathbf{SO}(q)-\mathrm{rk}\,\mathbf{SO}(q-p)=\mathrm{rk}\ K+\mathrm{rk}\,\mathbf{SO}(p),\end{aligned}$$

where we have used that not both of $p$, $q$ are odd. Note that $\dim\nu_x(\hat{G}x)=p$ and $\dim\nu_x(Gx)=p+\dim\mathbf{SO}(p)-\dim K$. It is clear that

$$\nu_x(Gx)^{G_x}\supset\nu_x(\hat{G}x),$$

since $\nu_x(Gx)\supset\nu_x(\hat{G}x)$, $G_x\subset\hat{G}_x$ and $x$ is $\hat{G}$-regular. It follows that

$$\dim\nu_x(Gx)^{G_x}\geq p.$$

Combining with the above we get that

$$\mathrm{rk}\ K+\mathrm{rk}\ \mathbf{SO}(p)\geq p\geq 2\mathrm{rk}\ \mathbf{SO}(p),$$

and therefore

$$\mathrm{rk}\ K=\mathrm{rk}\ \mathbf{SO}(p).$$

By the classification of maximal subgroups of maximal rank of $\mathbf{SO}(p)$, (see, for example, section 8.10 in [15]), since $K$ is irreducible and of real type on $\mathbf{R}^p$, we must have $K=\mathbf{SO}(p)$.

#### 3.1.2. $G=\rho(G_1)$, where $G_1=\mathbf{SO}(p)\times K$, and $K\subset\mathbf{SO}(q)$ is a maximal connected subgroup.

According to Theorem 2.3, we need to consider three cases.

**(a) $K = \mu(K_1)$, where $K_1$ is simple and $\mu$ is an absolutely irreducible representation of degree $q$.** Of course we need only to consider representations $\mu$ such that $\mu(K_1)$ is a proper subgroup of $\mathbf{SO}(q)$.

We may assume $p < q$. Remark 2.7(a) gives that $p^2 < pq \leq \frac{p(p-1)}{2} + \left[\frac{p}{2}\right] + r$, where $r = \dim K_1 + \mathrm{rk}\ K_1$, and $[x]$ denotes the greatest integer contained not exceeding $x$. This implies that

$$p^2 < 2r, \tag{1}$$

and that

$$p^2 - 2qp + 2r \geq 0. \tag{2}$$

Equation (2) and $p \geq 3$ then imply that

$$q \leq \frac{r}{3} + \frac{3}{2}. \tag{3}$$

Let $s$ be the minimal degree of an absolutely irreducible representation of $K$ and such that its image is not the full $\mathbf{SO}(s)$. Then

$$s > \frac{r}{3} + \frac{3}{2} \tag{4}$$

is a sufficient condition for $(G,V)$ not to be of vanishing homogeneity rank. We next run through the possibilities for $K_1$.

- $K_1 = \mathbf{SU}(m)$, where $m \geq 2$. Here $r = m^2 + m - 2$. If $m = 2$, then $s = 5$, and (4) holds. If $m \geq 3$, then $s = m^2 - 1$ (realized by the adjoint representation), and (4) holds.

- $K_1 = \mathbf{Sp}(m)$, where $m \geq 2$. Here $r = 2m^2 + 2m$. If $m = 2$, then $s = 10$ (realized by the adjoint representation) and (3) holds. If $m \geq 3$ then $s = 2m^2 - m - 1$ (realized by the second fundamental representation) and (3) holds.

- $K_1 = \mathbf{Spin}(m)$, where $m \geq 7$. Here $r = \frac{m(m-1)}{2} + \left[\frac{m}{2}\right]$. All irreducible representations of real type violate (3), except possibly the (half-)spin representations of $\mathbf{B}_{4k-1}$, $\mathbf{B}_{4k}$, $\mathbf{D}_{4k}$. These have respectively $q = 2^{4k-1}$, $2^{4k}$, $2^{4k-1}$. Condition (3) is respectively

$$3 \cdot 2^{4k} \leq 64k^2 - 16k + 9, \quad 3 \cdot 2^{4k+1} \leq 64k^2 + 16k + 9,$$

$$3 \cdot 2^{4k} \leq 64k^2 + 9.$$

  The only cases that survive are $\mathbf{B}_3$ and $\mathbf{D}_4$. In the case of $\mathbf{B}_3$ we have $q = 8$ and $r = 24$. Then (1) implies that $p = 3, 4, 5, 6$. Next we use (2) to get rid of $p = 5, 6$. We end up with $p = 3$ and $p = 4$, and this gives the admissible cases $(\mathbf{SO}(3) \times \mathbf{Spin}(7), \mathbf{R}^3 \otimes \mathbf{R}^8)$ and $(\mathbf{SO}(4) \times \mathbf{Spin}(7), \mathbf{R}^4 \otimes \mathbf{R}^8)$, but note that the first one of these is orbit-equivalent to $(\mathbf{SO}(3) \times \mathbf{SO}(8), \mathbf{R}^3 \otimes \mathbf{R}^8)$. In the case of $\mathbf{D}_4$, we have that $\mu(\mathbf{Spin}(8)) = \mathbf{SO}(8)$, and we rule this out.

- $K_1$ is an exceptional group. Here (4) holds in each case, so there are no examples, see the table below.

| $K_1$ | $r$ | $s$ |
|---|---|---|
| $\mathbf{G}_2$ | 16 | 7 |
| $\mathbf{F}_4$ | 56 | 26 |
| $\mathbf{E}_6$ | 84 | 78 |
| $\mathbf{E}_7$ | 140 | 133 |
| $\mathbf{E}_8$ | 256 | 248 |

**(b)** $K = \mu(\mathbf{SO}(k) \times \mathbf{SO}(l))$**, where** $3 \leq k \leq l$ **and** $q = kl$**, and** $\mu$ **is the real tensor product of the vector representations.** Here $r = \frac{k^2+l^2}{2} + \underbrace{\left[\frac{k}{2}\right] - \frac{k}{2} + \left[\frac{l}{2}\right] - \frac{l}{2}}_{=\theta}$. Note that $-1 \leq \theta \leq 0$. Then (3) is $k^2+l^2-6kl+2\theta+9 \geq 0$. Set $m = l-k \geq 0$. Then $m^2 - 4km - 4k^2 + 9 + 2\theta \geq 0$. This implies that

$$2k + \sqrt{8k^2 - 9 - 2\theta} \leq m. \tag{5}$$

If $pk \leq l$, since the action of $\mathbf{SO}(p) \times \mu(\mathbf{SO}(k) \times \mathbf{SO}(l)) \subset \mathbf{SO}(p) \times \mathbf{SO}(q)$ on $\mathbf{R}^p \otimes \mathbf{R}^q$ is the same thing as the action of $\mu'(\mathbf{SO}(p) \times \mathbf{SO}(k)) \times \mathbf{SO}(l) \subset \mathbf{SO}(pk) \times \mathbf{SO}(l)$ on $\mathbf{R}^{pk} \otimes \mathbf{R}^l$, where $\mu'$ is the real tensor product of the vector representations, this case has already been considered in section 3.1.1. So now we assume that

$$pk > l. \tag{6}$$

Note that $q > \sqrt{2r}$. This implies via (2) that $3 \leq p \leq q - \sqrt{q^2 - 2r}$. Combining this with (6) we have $l < kq - k\sqrt{q^2-2r}$ and then $l^2(1-2k^2) + 2k^2r > 0$. Substituting the value of $r$ we get $k \leq l < k\sqrt{\frac{k^2+2\theta}{k^2-1}}$. We deduce that $\theta = 0$ and

$$0 \leq m < k\left(\frac{k}{\sqrt{k^2-1}} - 1\right). \tag{7}$$

Now (5) and (7) combined imply that $3k + \sqrt{8k^2-9} < \frac{k^2}{\sqrt{k^2-1}}$, which is impossible for $k \geq 3$.

**(c)** $K = \mu(\mathbf{Sp}(k) \times \mathbf{Sp}(l))$**, where** $q = 4kl \neq 4$**, and** $\mu$ **is the quaternionic tensor product of the vector representations.** We postpone this case to section 3.4.

**3.1.3.** $G = \{(x, \sigma(x)) : x \in \mathbf{SO}(p)\}$**, where** $p = q$ **and** $\sigma$ **is an automorphism of** $\mathbf{SO}(p)$**.**

Here Remark 2.7(a) immediately implies that $(G, V)$ cannot have vanishing homogeneity rank.

**3.2. The case of** $\rho(\mathbf{Sp}(p) \times \mathbf{Sp}(q))$ **and its maximal subgroups.**

Here $4pq = 2n \neq 4$ and $p \leq q$. We have that $\rho$ is the isotropy representation of the symmetric space $\mathbf{Sp}(p+q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, and $\text{rk}\ \mathbf{Sp}(p+q) = \text{rk}\ \mathbf{Sp}(p) + \text{rk}\ \mathbf{Sp}(q)$. It follows from Lemma 2.4 that this is an example.

Next we must investigate maximal connected subgroups $G$ of $\rho(\mathbf{Sp}(p) \times \mathbf{Sp}(q))$. We shall consider three cases separately which cover all the possibilities.

**3.2.1.** $G = \rho(G_1)$**, where** $G_1 = K \times \mathbf{Sp}(q)$**, and** $K \subset \mathbf{Sp}(p)$ **is a maximal connected subgroup.**

Set $\hat{G}_1 = \mathbf{Sp}(p) \times \mathbf{Sp}(q)$. There is a $\hat{G}_1$-regular point $x \in V$ whose principal isotropy subgroup is given by $\hat{G}_{1x} = \mathbf{Sp}(1)^p \times \mathbf{Sp}(q-p)$. Let $\mathcal{O}$ be the orbit of $\hat{G}_1$ through $x$. A point $y \in \mathcal{O}$ is $G_1$-regular in $\mathcal{O}$ if and only if it is $G_1$-regular in $V$; moreover the isotropy subgroup $G_{1y}$ is given by the intersection of $G_1$ with a suitable conjugate of $\hat{G}_{1x}$ in $\hat{G}_1$. This means that a principal isotropy subgroup of $G_1$ contains a subgroup isomorphic to $\mathbf{Sp}(q-p)$. If $(G, V)$ has vanishing homogeneity rank, it then follows that

$$\dim V = 4pq \leq \dim K + \text{rk } K + 2q^2 + 2q - (2(q-p)^2 + 2(q-p)),$$

hence

$$2p^2 - 2p \leq \dim K + \text{rk } K. \tag{8}$$

According to Theorem 2.3, there are two cases to be considered. But if $K$ is of the form $\mu(\mathbf{SO}(k) \times \mathbf{Sp}(l))$, where $\mu$ is the real tensor product of the vector representations, we refer to section 3.4. So we can assume that $K$ is of the form $\mu(K_1)$, where $K_1$ is simple and $\mu$ is a complex irreducible representation of degree $2p$ of quaternionic type. Let $s$ be half the minimal degree of a complex irreducible representation of quaternionic type of $K_1$ and such that its image is not the full $\mathbf{Sp}(s)$. The list of the values of $s$ for each compact simple group $K_1$ is given by the following table (groups not appearing in the table do not admit quaternionic representations):

| $K_1$ | $s$ |
|---|---|
| $\mathbf{SU}(4a+2),\ a \geq 1$ | $\frac{1}{2}\binom{4a+2}{2a+1}$ |
| $\mathbf{Spin}(8a+3),\ a \geq 1$ | $2^{4a}$ |
| $\mathbf{Spin}(8a+4),\ a \geq 1$ | $2^{4a}$ |
| $\mathbf{Spin}(8a+5),\ a \geq 1$ | $2^{4a+1}$ |
| $\mathbf{Sp}(a),\ a \geq 3$ | $\frac{a}{3}(2a^2 - 3a - 2)$ |
| $\mathbf{Sp}(2)$ | 8 |
| $\mathbf{Sp}(1)$ | 2 |
| $\mathbf{E}_7$ | 28 |

It is now easy to see that (8) implies $K_1 = \mathbf{Sp}(1)$ and $p = 2$, and then the only admissible case is $G_1 = \mathbf{Sp}(1) \times \mathbf{Sp}(q)$, where $\mathbf{Sp}(1) \subset \mathbf{Sp}(2)$ via the irreducible representation of degree 4.

**3.2.2.** $G = \rho(G_1)$**, where** $G_1 = \mathbf{Sp}(p) \times K$**, and** $K \subset \mathbf{Sp}(q)$ **is a maximal connected subgroup.**

According to Theorem 2.3, we need to consider two cases.

**(a)** $K = \mu(K_1)$**, where** $K_1$ **is simple and** $\mu$ **is a complex irreducible representation of degree** $2q$ **of quaternionic type.** We may assume $p < q$.

Remark 2.7(a) gives that $4p^2 < 4pq \leq 2p^2 + 2p + r$, where $r = \dim K_1 + \text{rk } K_1$ (note that $r \geq 4$). This implies that

$$p < \frac{1 + \sqrt{1 + 2r}}{2} \quad \text{and} \quad 2p^2 + 2p(1 - 2q) + r \geq 0.$$

¿From this we get that

$$q \leq \frac{r}{4} + 1, \tag{9}$$

and

$$\text{if } q \geq \tfrac{1+\sqrt{1+2r}}{2}, \text{ then } p \leq q - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(2q-1)^2 - 2r}. \tag{10}$$

Running through the compact simple groups $K_1$ that admit quaternionic representations (see table in section 3.2.1) and using (9) and (10), we get the following admissible cases: $K_1 = \mathbf{Sp}(1)$, $p = 1$, $q = 2$; $K_1 = \mathbf{Sp}(3)$, $p = 1$, $q = 7$; $K_1 = \mathbf{Spin}(11)$, $p = 1$, $q = 16$; $K_1 = \mathbf{Spin}(12)$, $p = 1$, $q = 16$; $K_1 = \mathbf{SU}(6)$, $p = 1$, $q = 10$; $K_1 = \mathbf{E}_7$, $p = 1$, $q = 28$. All cases but that of $K_1 = \mathbf{Spin}(11)$ come from isotropy representations of symmetric spaces.

**(b) $K = \mu(\mathbf{SO}(k) \times \mathbf{Sp}(l))$, where $q = kl$, and $\mu$ is the real tensor product of the vector representations.** We postpone this case to section 3.4.

**3.2.3. $G = \{(x, \sigma(x)) : x \in \mathbf{Sp}(p)\}$, where $p = q$ and $\sigma$ is an automorphism of $\mathbf{Sp}(p)$.**

Here Remark 2.7(a) immediately implies that $(G, V)$ can have vanishing homogeneity rank only if $p = 1$, so this case is out.

**3.3. The case of $\rho(G_1)$.**

Here $G_1$ is a compact simple Lie group and $\rho$ is an absolutely irreducible representation of $G_1$ of degree $2n$. Remark 2.7(a) says that $2n \leq \dim G_1 + \text{rk } G_1$. In particular, this implies that $2 \dim G_1 \geq 2n - 2$, so we can use Lemma 2.6 in [10] to deduce that $(G, V)$ is orbit equivalent to the isotropy representation of a symmetric space.

**3.4. The case of $\rho(\mathbf{SO}(m) \times \mathbf{Sp}(p) \times \mathbf{Sp}(q))$, where $\rho$ is the real and quaternionic tensor products of the vector representations.**

Here $2n = 4mpq$, $m \geq 3$ and $p \leq q$. By direct computation or using Theorem 1.1 in [6], we see that:

(i) if $m \geq 4pq+2$, then the connected principal isotropy is given by $\mathbf{SO}(m - 4pq)$;

(ii) if $q \geq mp+1$, then the connected principal isotropy is given by $\mathbf{Sp}(q - mp)$;

(iii) in all other cases the connected principal isotropy is trivial.

In case (i) the condition of vanishing homogeneity rank reads

$$4p^2q^2 = p^2 + q^2 + p + q \leq 2p^2 + 2q^2,$$

and this implies $p = q = 1$. In case (ii) we have

$$4p^2(1 - m^2) + 4p(1 + m) + m^2 - m + 2\left[\frac{m}{2}\right] = 0.$$

If $m=2l$, then we have $p^2(1-4l^2)+p(1+2l)+l^2=0$, which implies that $1+2l$ divides $l^2$, impossible. If $m=2l+1$, then we have $(4p^2-1)l=2p$, which is impossible. In case (iii), we have the equation

$$8mpq = m^2 - m + 2\left[\frac{m}{2}\right] + 4p^2 + 4p + 4q^2 + 4q.$$

If $m=2l$, this reads

$$l^2 - 4pql + p^2 + q^2 + p + q = 0, \tag{11}$$

subject to the constraints

$$\frac{q}{2p} \leq l \leq 2pq, \quad p \leq q, \quad l \geq 2,$$

while if $m=2l+1$, we have

$$l^2 - (4pq-1)l + p^2 + q^2 - 2pq + p + q = 0, \tag{12}$$

subject to the constraints

$$\frac{q}{2p} - \frac{1}{2} \leq l \leq 2pq, \quad p \leq q, \quad l \geq 1.$$

Consider first equation (11). It can be solved in $l$ to yield $l=2pq\pm\sqrt{\Delta}$, where $\Delta = 4p^2q^2-p^2-q^2-p-q$. If $l=2pq+\sqrt{\Delta}$, using the fact that $l\leq 2pq$ we have $\Delta=0$ and then $l=2pq\leq p+q$, which gives $p=q=1$, and then $l=2$, $m=4$. If $l=2pq-\sqrt{\Delta}$, then $\frac{q}{2p}\leq l$ implies that

$$q^2(4p^2-1) - 4p^2q - 4p^2(p^2+p) \leq 0,$$

and therefore

$$p \leq q \leq \frac{2p^2 + 2p\sqrt{p(4p^3+4p^2-1)}}{4p^2-1} < p+2.$$

It then follows that we only need to consider the possibilities $q=p$ and $q=p+1$. If $q=p$, then we have

$$2 \leq l = 2p^2 - \sqrt{4p^4 - 2p^2 - 2p} \leq 2,$$

so that $l=2$, $p=q=1$. If $q=p+1$, then

$$2 \leq l = 2p^2 + 2p - \sqrt{4p^4 + 8p^3 + 2p^2 - 4p - 2} < 2,$$

which is impossible.

Next we consider equation (12). Here it is useful to note that Remark 2.7(b) applied to $G=\mathbf{SO}(m)\times\mathbf{Sp}(p)\times\mathbf{Sp}(q)$ viewed as a subgroup of $\mathbf{SO}(m)\times\mathbf{SO}(4pq)$ gives the extra condition

$$\left[\frac{m}{2}\right] \leq p+q.$$

Reasoning as above, we get that $p=q=1$ and $m=3,\ 5$.

**3.5. The examples and their subgroups.**

In this section we show that all candidates $G \subset \mathbf{SO}(2n)$ found in the previous sections are actually examples of groups acting absolutely irreducibly on $V = \mathbf{R}^{2n}$ with vanishing homogeneity rank, and these groups do not admit subgroups with the same property. This will complete the proof of Theorem 1.1.

We first examine the representations that are not orbit-equivalent to isotropy representations of non-Hermitian symmetric spaces of inner type. We have three candidates:

1. $G = \mathbf{Sp}(1) \times \mathbf{Sp}(q)$ $(q \geq 2)$ acting on $V = S^3(\mathbf{C}^2) \otimes_{\mathbf{H}} \mathbf{C}^{2q} \cong \mathbf{R}^{8q}$;
2. $G = \mathbf{SO}(4) \times \mathbf{Spin}(7)$ acting on $V = \mathbf{R}^4 \otimes \mathbf{R}^8 \cong \mathbf{R}^{32}$, where $\mathbf{Spin}(7)$ acts on $\mathbf{R}^8$ via the real spin representation;
3. $G = \mathbf{Sp}(1) \times \mathbf{Spin}(11)$ acting on $V = \mathbf{C}^2 \otimes_{\mathbf{H}} \mathbf{C}^{32} \cong \mathbf{R}^{64}$, where $\mathbf{Spin}(11)$ acts on $\mathbf{C}^{32}$ via the complex spin representation.

We now show that in each case the representation has vanishing homogeneity rank. Indeed, in case 1 we have that a connected principal isotropy is given by $\mathbf{Sp}(q-2)$ (see [5], Proposition 7.12), therefore the cohomogeneity is three and the homogeneity rank vanishes. In case 2, a connected isotropy subgroup is trivial. This can be seen by selecting a pure tensor $v \otimes w$ with $v \in \mathbf{R}^4$ and $w \in \mathbf{R}^8$ and computing the connected isotropy, which is $\mathbf{SO}(3) \times \mathbf{G}_2$; then the slice representation is given by $\mathbf{R} \oplus \mathbf{R}^3 \otimes \mathbf{R}^7$; starting again with this new representation, we eventually come up with a trivial isotropy. Therefore the cohomogeneity is five and the homogeneity rank vanishes. In case 3, we also have trivial connected principal isotropy and vanishing homogeneity rank. Indeed, if $v \in \mathbf{C}^{32}$ is a highest weight vector for the spin representation of $\mathbf{Spin}(11)$, then the subgroup $H \subset \mathbf{Spin}(11)$ defined by $H = \{g \in \mathbf{Spin}(11) : \ g \cdot v \in \mathbf{C}^* \cdot v\}$ is given by $\mathbf{U}(5)$. Now if $p : \mathbf{Sp}(1) \times \mathbf{Spin}(11) \to \mathbf{Spin}(11)$ is the projection, then

$$p((\mathbf{Sp}(1) \times \mathbf{Spin}(11))_v) = \{g \in \mathbf{Spin}(11) : \ g \cdot v \in \mathbf{Sp}(1) \cdot v\} \supset H.$$

Since $H$ is maximal in $\mathbf{Spin}(11)$, we get that $(\mathbf{Sp}(1) \times \mathbf{Spin}(11))_v$ is given by $\mathbf{T}^1 \cdot \mathbf{SU}(5)$, where $\mathbf{T}^1$ sits diagonally in the product of a suitable maximal torus in $\mathbf{Sp}(1)$ and the center of $H$. ¿From this we see that the slice representation at $v$ is given by $\mathbf{R} \oplus \mathbf{C}^5 \oplus \Lambda^2\mathbf{C}^5$ and the connected principal isotropy is trivial. The cohomogeneity is six and the homogeneity rank vanishes.

We now examine subgroups of the previous examples. In case 1, a maximal subgroup of $G$ leaving no complex structure on $V$ invariant is of the form $G' = \mathbf{Sp}(1) \times K$, where $K \subset \mathbf{Sp}(q)$ is maximal. Since $\mathbf{Sp}(2) \times K$ does not have vanishing homogeneity rank on $V$ by the results of section 3.2.2, and $G' \subset \mathbf{Sp}(2) \times K$, we have that $G'$ does not have vanishing homogeneity rank on $V$. In cases 2 and 3, $G$ admits no proper subgroups acting with vanishing homogeneity rank because the connected principal isotropy is trivial and then we may apply Remark 2.7(b).

We finally consider the representations $(G, V)$ that are orbit-equivalent to isotropy representations of non-Hermitian symmetric spaces of inner type, and we classify the subgroups $G' \subset G$ which still act absolutely irreducibly on $V$ with vanishing homogeneity rank. In the following table we list the representations $\rho$

which need to be examined; we denote by $c$ the cohomogeneity of $\rho$, by $d$ the dimension of $V$, and by $[[W]]$ a real form of the $G$-module $W$.

| *Case* | $G$ | $\rho$ | $c$ | $d$ | $\dim G_{\mathrm{princ}}$ |
|---|---|---|---|---|---|
| 1 | $\mathbf{Sp}(1)\cdot\mathbf{SU}(6)$ | $\mathbf{C}^2\otimes_{\mathbf{H}}\Lambda^3\mathbf{C}^6$ | 4 | 40 | 2 |
| 2 | $\mathbf{Sp}(1)\cdot\mathbf{Spin}(12)$ | $\mathbf{C}^2\otimes_{\mathbf{H}}$ (half-spin) | 4 | 64 | 9 |
| 3 | $\mathbf{Sp}(1)\cdot\mathbf{E}_7$ | $\mathbf{C}^2\otimes_{\mathbf{H}}\mathbf{C}^{56}$ | 4 | 112 | 28 |
| 4 | $\mathbf{Sp}(1)\cdot\mathbf{Sp}(3)$ | $\mathbf{C}^2\otimes_{\mathbf{H}}\mathbf{C}^{14}$ | 4 | 28 | 0 |
| 5 | $\mathbf{SO}(4)$ | $S^3(\mathbf{C}^2)\otimes_{\mathbf{H}}\mathbf{C}^2$ | 2 | 8 | 0 |
| 6 | $\mathbf{Spin}(16)$ | half-spin | 8 | 128 | 0 |
| 7 | $\mathbf{SU}(8)$ | $[[\Lambda^4\mathbf{C}^8]]$ | 7 | 70 | 0 |
| 8 | $\mathbf{SO}(3)\times\mathbf{Spin}(7)$ | $\mathbf{R}^3\otimes\mathbf{R}^8$ | 3 | 24 | 3 |
| 9 | $\mathbf{Spin}(7)$ | spin | 1 | 8 | 14 |
| 10 | $\mathbf{Spin}(9)$ | spin | 1 | 16 | 21 |

Cases 4 through 7 can be dealt with using Remark 2.7(b). We consider case 1. If $G'$ is a maximal subgroup of $G$, we may assume that $G'$ is of the form $G'=\mathbf{Sp}(1)\cdot G''$, where $G''$ is maximal in $\mathbf{SU}(6)$, since $G'$ does not leave any complex structure invariant. Now Remark 2.7(a) implies that $\dim G''+\mathrm{rk}\ G''\geq 36$, and $\mathrm{rk}\ G''\leq 5$ implies $\dim G''\geq 31$, so that $\dim\mathbf{SU}(6)/G''\leq 4$. If $G''$ is a proper subgroup of $\mathbf{SU}(6)$, then the left action of $\mathbf{SU}(6)$ on $\mathbf{SU}(6)/G''$ is almost effective because $\mathbf{SU}(6)$ is simple. Therefore $\dim\mathbf{SU}(6)$ is less than the dimension of the isometry group of $\mathbf{SU}(6)/G''$, which is at most $10$, but this is a contradiction. Hence $G''=\mathbf{SU}(6)$.

In case 2, again we can assume that $G'$ is of the form $G'=\mathbf{Sp}(1)\cdot G''$, where $G''$ is maximal in $\mathbf{Spin}(12)$. We have $\dim G''+\mathrm{rk}\ G''\geq 60$; since $G'$ is supposed to act absolutely irreducibly on $V$, its rank is not maximal by a theorem of Dynkin (see Theorem 7.1, p. 158 in [4]), and therefore $\dim G''\geq 55=\dim\mathbf{Spin}(11)$. It is known that a subgroup of $\mathbf{Spin}(n)$ of dimension greater or equal to $\dim\mathbf{Spin}(n-1)$ is conjugate to the standard $\mathbf{Spin}(n-1)\subset\mathbf{Spin}(n)$ if $n\neq 4,\ 8$ (see e.g. [9], p. 49). So, $G'=\mathbf{Sp}(1)\cdot\mathbf{Spin}(11)$, which is indeed an example with trivial connected principal isotropy by the discussion above.

In case 3, using the same argument as in case 2, we see that $G'=\mathbf{Sp}(1)\cdot G''$, where $G''$ is maximal in $\mathbf{E}_7$ and $\dim G''\geq 102$. Now a maximal subgroup of maximal rank of $\mathbf{E}_7$ has dimension at most 79 (see, for example, section 8.10 in [15]), whereas one sees by direct enumeration that an arbitrary compact Lie group of rank at most 6 has dimension at most $78$. This shows that there is no such proper subgroup.

In case 8, a maximal subgroup $G'$ acting absolutely irreducibly on $V$ must be of the form $G'=\mathbf{SO}(3)\times K$, where $K\subset\mathbf{Spin}(7)$ is maximal; arguing as above, we see that $\dim K\geq 18$, so that $\dim(\mathbf{Spin}(7)/K)\leq 3$ and this is impossible, because $\mathbf{Spin}(7)$ is simple.

In case 9, let $K\subset\mathbf{Spin}(7)$ be a maximal subgroup acting absolutely irreducibly on $\mathbf{R}^8$. Since $K$ cannot have maximal rank as above, and using Theorem 2.3, we see that $K$ must be simple of rank at most two and it must admit an irreducible representation of degree 7 and of real type. Moreover, by Remark 2.7(a), we have $\dim K\geq 6$, hence $\mathrm{rk}\ K=2$, and a direct inspection of all such simple groups shows that none of them but $\mathbf{G}_2$ admits an irreducible representation of degree $7$. But $\mathbf{G}_2$ does not admit an irreducible representation of degree $8$.

In case 10, we consider a maximal subgroup $K$ of $\mathbf{Spin}(9)$ acting absolutely irreducibly on $\mathbf{R}^{16}$. This means that $\operatorname{rk} K \leq 3$ and $\dim K \geq 13$. Looking at the list of all maximal subgroups of $\mathbf{Spin}(9)$, we see that we can suppose $K$ to be simple and to act irreducibly on $\mathbf{R}^9$, via the embedding $K \subset \mathbf{Spin}(9)$. Therefore $K$ must be one of $\mathbf{G}_2$, $\mathbf{SU}(4)$, $\mathbf{Spin}(7)$ or $\mathbf{Sp}(3)$, but we immediately see that none of these groups admits an irreducible representation of degree $9$ and of real type.

This finishes the proof of Theorem 1.1.

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Claudio Gorodski
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão, 1010
São Paulo, SP 05508-090
Brazil
gorodski@ime.usp.br

Fabio Podestà
Dipartimento di Matematica e Appl.
per l'Architettura
Piazza Ghiberti 27
50100 Florence
Italy
podesta@math.unifi.it



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