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Abelian varieties over finite fields with a specified characteristic polynomial modulo ℓ

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RÉSUMÉ. Nous estimons la fraction des classes d'isogénie des variétés abeliennes sur un corps fini qui possèdent un polynôme caractéristique donné $P(T)$ modulo ℓ. Comme application nous trouvons la proportion des classes d'isogénie des variétés abeliennes qui possèdent un point rationnel d'ordre ℓ.

ABSTRACT. We estimate the fraction of isogeny classes of abelian varieties over a finite field which have a given characteristic polynomial $P(T)$ modulo ℓ. As an application we find the proportion of isogeny classes of abelian varieties with a rational point of order ℓ.

1. Introduction

Let $\mathbf{F}$ be a finite field of characteristic p and order q, and ℓ a prime not equal to p. Let

$$P(T) = (T^{2g} + q^g) + a_1(T^{2g-1} + q^{g-1}T) + \cdots + a_{g-1}(T^{g+1} + qT^{g-1}) + a_g T^g$$

be a polynomial. The goal of this paper is to estimate the number of isogeny classes of abelian varieties over $\mathbf{F}$ of dimension g for which the characteristic polynomial for the action of Frobenius is congruent to $P(T)$ modulo ℓ.

The initial motivation for this problem came from the following question, posed in [4] and related to the Fontaine-Mazur Conjecture for number fields:

Question 1. *Let k be a function field over a finite field $\mathbf{F}$ of characteristic p and order q, and ℓ a prime not equal to p. Let $K = k\mathbf{F}_{\ell^\infty}$ be obtained from k by taking the maximal ℓ-extension of the constant field. If M is an unramified ℓ-adic analytic ℓ-extension of k, and M does not contain K, must M be a finite extension of k?*

In general the answer to Question 1 is no, with examples due to Ihara ([5]) and to Frey, Kani, and Völklein ([3]). However, the following theorems were proved in [4]:

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Theorem 1 (Theorem 2 of [4]). *Let k_0 be a function field over a finite field of characteristic p, and let k be a constant field extension. Let ℓ be a prime not equal to p. If ℓ does not divide the class number $P(1)$ of k_0, then any everywhere unramified powerful (*a fortiori *uniform) pro-ℓ extension of k, Galois over k_0, with no constant field extension, is finite.*

Theorem 2 (See Corollary 4.11 of [4]). *Let k_0 be a function field over a finite field of characteristic p, and let k be a constant field extension. Let ℓ be a prime not equal to p. Let $P(T)$ be the characteristic polynomial of Frobenius for the Jacobian of the curve associated with k_0. Suppose that the distinct roots of $P(T)$ modulo ℓ (possibly in some extension of $\mathbf{Z}/\ell\mathbf{Z}$) consist of $\lambda_0, \lambda_1, \ldots, \lambda_n$ such that for all $i \neq j$, $\lambda_i\lambda_j \neq 1$. Suppose further that if any $\lambda_i = 1$, λ_i is at most a double root of $P(T)$ modulo ℓ, and if any $\lambda_i = -1$, λ_i is only a simple root of $P(T)$ modulo ℓ. Then there are no unramified infinite powerful pro-ℓ extensions of k_n, Galois over k_0, with no constant field extension.*

In the paper [1], Jeffrey Achter and the author address the question of how many function fields are associated with a given $P(T)$ modulo ℓ, and thus how many fall under the purview of Theorem 1 and Theorem 2. In this paper we will address the different but related question of how many isogeny classes of abelian varieties have a given characteristic polynomial $P(T)$ modulo ℓ. As an application we find the proportion of isogeny classes of abelian varieties with a rational point of order ℓ.

We have chosen the following way to address these questions, starting with the application to rational points. Fix distinct primes p and ℓ. For each r, let $\mathbf{F}_{p^r}$ be the finite field with p^r elements. By the work of Tate and Honda, two abelian varieties are isogenous if and only if they have the same zeta function. Thus to each isogeny class of abelian varieties defined over $\mathbf{F}_{p^r}$ we associate the unique polynomial $P(T)$ (the *Weil polynomial* or *Weil q-polynomial*) which is the characteristic polynomial for the action of Frobenius and the reciprocal of the numerator of the zeta function of any variety in the isogeny class. Then ℓ divides $P(1)$ if and only if each abelian variety in the class has an $\mathbf{F}_{p^r}$-rational point of order ℓ. For each g, there are finitely many isogeny classes of abelian varieties with dimension g. Let $d_{r,g}$ be the fraction of isogeny classes of dimension g over $\mathbf{F}_{p^r}$ for which ℓ divides $P(1)$. Then

Theorem 3. *For fixed g,*

$$\lim_{r\to\infty} d_{r,g} = \frac{1}{\ell}.$$

This result and the other major result of this paper could also be obtained using the techniques of [1]. The proofs given here are perhaps more elementary, and also give some access to the number of isogeny classes and not merely the proportion satisfying each condition.

2. Lattices

The proof of Theorem 3 relies on the method of counting abelian varieties introduced by DiPippo and Howe in [2]. Let $q = p^r$ and $I(q, g)$ be the number of isogeny classes of g-dimensional abelian varieties over $\mathbf{F}_q$. Let $P(T)$ be as before. If $P(T)$ is associated to the isogeny class of a g-dimensional abelian variety then $P(T)$ has degree $2g$. Write

$$P(T) = \prod_{j=1}^{2g} (T - \alpha_j).$$

Then $P(T)$ has the property that $|\alpha_j| = q^{1/2}$, and the real roots, if any, have even multiplicity. Note that since the possible Weil polynomials $P(T)$ for a given g are monic integral polynomials of fixed degree and have roots (and therefore coefficients) of bounded size, there are only finitely many of them. Thus $I(q, g)$ is finite.

If we write

$$P(T) = \\ (T^{2g} + q^g) + a_1(T^{2g-1} + q^{g-1}T) + \cdots + a_{g-1}(T^{g+1} + qT^{g-1}) + a_g T^g$$

and let $Q(T) = P(q^{1/2}T)/q^g$, then $P(T)$ is associated with another polynomial

$$Q(T) = (T^{2g} + 1) + b_1(T^{2g-1} + T) + \cdots + b_{g-1}(T^{g+1} + T^{g-1}) + b_g T^g.$$

Let V_g be the set of vectors $\mathbf{b} = (b_1, \ldots, b_g)$ in $\mathbf{R}^g$ such that all of the complex roots of $Q(T)$ lie on the unit circle and all real roots occur with even multiplicity. Let $\mathbf{e}_1, \ldots, \mathbf{e}_g$ be the standard basis vectors of $\mathbf{R}^g$ and let Λ_q be the lattice generated by the vectors $q^{-i/2}\mathbf{e}_i$. DiPippo and Howe explain that if $P(T)$ is the Weil polynomial of an isogeny class then the coefficients a_i are such that $(a_1 q^{-1/2}, \ldots, a_g q^{-g/2}) \in \Lambda_q \cap V_g$. Further, let Λ'_q be the lattice generated by the vectors $q^{-1/2}\mathbf{e}_1, \ldots, q^{-(g-1)/2}\mathbf{e}_{g-1}$ and $pq^{-g/2}\mathbf{e}_g$. Then all of the polynomials $P(T)$ with coefficients a_i such that $(a_1 q^{-1/2}, \ldots, a_g q^{-g/2}) \in (\Lambda_q \setminus \Lambda'_q) \cap V_g$ are exactly the Weil polynomials of isogeny classes of ordinary varieties. Finally, let Λ''_q be the lattice generated by the vectors $q^{-1/2}\mathbf{e}_1, \ldots, q^{-(g-1)/2}\mathbf{e}_{g-1}$ and $sq^{-g/2}\mathbf{e}_g$, where s is the smallest power of p such that q divides s^2. Then the set of polynomials $P(T)$ with coefficients a_i such that $(a_1 q^{-1/2}, \ldots, a_g q^{-g/2}) \in \Lambda''_q \cap V_g$ contains (perhaps properly) the set of Weil polynomials of isogeny classes of non-ordinary varieties.

These facts are relevant because of Proposition 2.3.1 of [2]. In a slightly generalized form, the proposition says:

Proposition 2.1 (see 2.3.1 of [2]). *Let $n > 0$ be an integer and let $\Lambda \subseteq \mathbf{R}^n$ be a rectilinear lattice (possibly shifted) with mesh d at most D. Then we have*

$$\left| \#(\Lambda \cap V_g) - \frac{\text{volume } V_n}{\text{covolume } \Lambda} \right| \leq c(n, D) \frac{d}{\text{covolume } \Lambda}$$

for some constant $c(n, D)$ depending only on n and D which can be explicitly computed. (We will not need the explicit computation in this paper.)

Let v_n be the volume of V_n; Proposition 2.2.1 of [2] calculates it explicitly but we will not need that here. Let $r(q) = 1 - 1/p$. The lattice Λ_q has covolume $q^{-g(g+1)/4}$ and mesh $q^{-1/2}$. The lattice Λ'_q has covolume $pq^{-g(g+1)/4}$, and it has mesh $q^{-1/2}$ unless $g = 2$ and $q = p$, in which case it has mesh 1. Lastly, the lattice Λ''_q has covolume $sq^{-g(g+1)/4}$ and its mesh is at most 1. It is then an easy consequence of the proposition that

$$\begin{aligned} v_g r(q) q^{g(g+1)/4} - 2c(g,1) q^{g(g+1)/4 - 1/2} & \\ \leq I(q, g) & \\ \leq v_g r(q) q^{g(g+1)/4} + (v_g + 3c(g,1))\, q^{g(g+1)/4 - 1/2}. & \end{aligned}$$

(See [2] for details.)

Now let $I_\ell(q, g)$ be the number of isogeny classes of g-dimensional abelian varieties over $\mathbf{F}_q$ such that ℓ divides $P(1)$. Using the above notation we have

$$P(1) = (1 + q^g) + a_1(1 + q^{g-1}) + \cdots + a_{g-1}(1 + q) + a_g.$$

Then

$$I_\ell(q, g) = \sum_{\substack{(1+q^g) + m_1(1+q^{g-1}) + \cdots + m_{g-1}(1+q) + m_g \equiv 0 \pmod{\ell} \\ 0 \leq m_i < \ell}} I_{m_1, \ldots, m_g}(q, g)$$

where $I_{m_1,\ldots,m_g}(q, g)$ is the number of isogeny classes of g-dimensional abelian varieties over $\mathbf{F}_q$ such that $a_i \equiv m_i$ modulo ℓ. There are exactly ℓ^{g-1} terms on the right hand side of this expression.

Now let $\Lambda_{m_1,\ldots,m_g}$ be the lattice generated by the vectors $\ell q^{-i/2}\mathbf{e}_i$ and then shifted by $\sum_i m_i q^{-i/2}\mathbf{e}_i$, and let $\Lambda'_{m_1,\ldots,m_g} = \Lambda_{m_1,\ldots,m_g} \cap \Lambda'_q$ and $\Lambda''_{m_1,\ldots,m_g} = \Lambda_{m_1,\ldots,m_g} \cap \Lambda''_q$. Then $\Lambda_{m_1,\ldots,m_g}$ has covolume $\ell^g q^{-g(g+1)/4}$ and mesh $\ell q^{-1/2}$; $\Lambda'_{m_1,\ldots,m_g}$ has covolume $\ell^g pq^{-g(g+1)/4}$, and it has mesh $\ell q^{-1/2}$ unless $g = 2$ and $q = p$, in which case it has mesh ℓ; and $\Lambda''_{m_1,\ldots,m_g}$ has covolume $\ell^g sq^{-g(g+1)/4}$ and mesh at most ℓ.

We can then prove:

Proposition 2.2.

$$\begin{aligned} v_g r(q) q^{g(g+1)/4} \ell^{-g} &- 2c(g,\ell) q^{g(g+1)/4-1/2} \ell^{1-g} \\ &\leq I_{m_1,\ldots,m_g}(q,g) \\ &\leq v_g r(q) q^{g(g+1)/4} \ell^{-g} + (v_g + 3c(g,\ell))\, q^{g(g+1)/4-1/2} \ell^{1-g}, \end{aligned}$$

and thus:

Proposition 2.3.

$$\begin{aligned} v_g r(q) q^{g(g+1)/4} \ell^{-1} &- 2c(g,\ell) q^{g(g+1)/4-1/2} \\ &\leq I_\ell(q,g) \\ &\leq v_g r(q) q^{g(g+1)/4} \ell^{-1} + (v_g + 3c(g,\ell))\, q^{g(g+1)/4-1/2}. \end{aligned}$$

Combining this with our earlier result, we get

$$\begin{aligned} &\frac{v_g r(q) q^{g(g+1)/4} \ell^{-1} - 2c(g,\ell) q^{g(g+1)/4-1/2}}{v_g r(q) q^{g(g+1)/4} + (v_g + 3c(g,1))\, q^{g(g+1)/4-1/2}} \\ &\qquad \leq \frac{I_\ell(q,g)}{I(q,g)} \\ &\qquad \leq \frac{v_g r(q) q^{g(g+1)/4} \ell^{-1} + (v_g + 3c(g,\ell))\, q^{g(g+1)/4-1/2}}{v_g r(q) q^{g(g+1)/4} - 2c(g,1) q^{g(g+1)/4-1/2}}. \end{aligned}$$

Thus we have:

Theorem 4. *For fixed g,*

$$\lim_{r\to\infty} \frac{I_\ell(p^r,g)}{I(p^r,g)} = \frac{1}{\ell}.$$

from which Theorem 3 follows immediately.

3. The general case

Obviously, an identical argument could be used to establish the fraction of isogeny classes of dimension g for which $P(x) \equiv y$ modulo ℓ for any x and y in $\mathbf{Z}$. More generally, we can establish the fraction of isogeny classes of dimension g for which $P(T) \equiv f(T)$ modulo ℓ for any given polynomial $f(T)$ of the correct form. Fix

$$f(T) = (T^{2g} + q^g) + m_1(T^{2g-1} + q^{g-1}T) + \cdots + m_{g-1}(T^{g+1} + qT^{g-1}) + m_g T^g.$$

For fixed p and ℓ, let $e_{r,g}$ be the fraction of isogeny classes of g-dimensional abelian varieties over $\mathbf{F}_{p^r}$ such that $P(T) \equiv f(T)$ modulo ℓ.

Theorem 5. *For fixed g,*

$$\lim_{r\to\infty} e_{r,g} = \frac{1}{\ell^g}.$$

Proof. We can follow the same argument as we did for Theorem 3. Let $J_\ell(q,g) = e_{r,g} I(q,g)$ be the number of isogeny classes of g-dimensional abelian varieties over $\mathbf{F}_{p^r} = \mathbf{F}_q$ such that $P(T) \equiv f(T)$ modulo ℓ. Then our bounds on $J_\ell(q,g) = I_{m_1,\ldots,m_g}(q,g)$ and $I(q,g)$ give us

$$\begin{aligned}
&\frac{v_g r(q) q^{g(g+1)/4} \ell^{-g} - 2c(g,\ell) q^{g(g+1)/4 - 1/2} \ell^{1-g}}{v_g r(q) q^{g(g+1)/4} + (v_g + 3c(g,1))\, q^{g(g+1)/4 - 1/2}} \\
&\qquad \leq \frac{J_\ell(q,g)}{I(q,g)} \\
&\qquad \leq \frac{v_g r(q) q^{g(g+1)/4} \ell^{-g} + (v_g + 3c(g,\ell))\, q^{g(g+1)/4 - 1/2} \ell^{1-g}}{v_g r(q) q^{g(g+1)/4} - 2c(g,1) q^{g(g+1)/4 - 1/2}}.
\end{aligned}$$

On taking the limit, the theorem follows. □

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