Journal de Théorie des Nombres
de Bordeaux **16** (2004), 293–333

Towards explicit description of ramification filtration in the 2-dimensional case

par Victor ABRASHKIN

Résumé. Le résultat principal de cet article est une description explicite de la structure des sous-groupes de ramification du groupe de Galois d'un corps local de dimension 2 modulo son sous-groupe des commutateurs d'ordre ≥ 3. Ce résultat joue un role clé dans la preuve par l'auteur d'un analogue de la conjecture de Grothendieck pour les corps de dimension supérieure, cf. Proc. Steklov Math. Institute, vol. 241, 2003, pp. 2-34.

Abstract. The principal result of this paper is an explicit description of the structure of ramification subgroups of the Galois group of 2-dimensional local field modulo its subgroup of commutators of order ≥ 3. This result plays a clue role in the author's proof of an analogue of the Grothendieck Conjecture for higher dimensional local fields, cf. Proc. Steklov Math. Institute, vol. 241, 2003, pp. 2-34.

0. Introduction

Let K be a 1-dimensional local field, i.e. K is a complete discrete valuation field with finite residue field. Let $\Gamma = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ be the absolute Galois group of K. The classical ramification theory, cf. [8], provides Γ with a decreasing filtration by ramification subgroups $\Gamma^{(v)}$, where $v \geq 0$ (the first term of this filtration $\Gamma^{(0)}$ is the inertia subgroup of Γ). This additional structure on Γ carries as much information about the category of local 1-dimensional fields as one can imagine: the study of such local fields can be completely reduced to the study of their Galois groups together with ramification filtration, cf. [6, 3]. The Mochizuki method is a very elegant application of the theory of Hodge-Tate decompositions, but his method works only in the case of 1-dimensional local fields of characteristic 0 and it seems it cannot be applied to other local fields. The author's method is based on an explicit description of ramification filtration for maximal p-extensions of local 1-dimensional fields of characteristic p with Galois groups of nilpotent class 2 (where p is a prime number ≥ 3). This information is sufficient to establish the above strong property of ramification filtration in the case of local fields of finite characteristic and can be applied to the characteristic 0 case via the field-of-norms functor.

Let now K be a 2-dimensional local field, i.e. K is a complete discrete valuation field with residue field $K^{(1)}$, which is again a complete discrete valuation field and has a finite residue field. Recently I.Zhukov [9] proposed an idea how to construct a higher ramification theory of such fields, which depends on the choice of a subfield of "1-dimensional constants" K_{c} in K (i.e. K_{c} is a 1-dimensional local field which is contained in K and is algebraically closed in K). We interpret this idea to obtain the ramification filtration of the group $\Gamma = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ consisting of ramification subgroups $\Gamma^{(v)}$, where v runs over the ordered set $J = J_1 \cup J_2$ with

$$J_1 = \{(v, \mathrm{c}) \in \mathbb{Q} \times \{\mathrm{c}\} \mid v \geq 0\}\,, \quad J_2 = \left\{j \in \mathbb{Q}^2 \mid j \geq (0,0)\right\}.$$

Notice that the orderings on J_1 and J_2 are induced, respectively, by the natural ordering on $\mathbb{Q}$ and the lexicographical ordering on $\mathbb{Q}^2$, and by definition any element from J_1 is less than any element of J_2. We notice also that the beginning of the above filtration $\{\Gamma^{(j)}\}_{j\in J_1}$ comes, in fact, from the classical "1-dimensional" ramification filtration of the group $\Gamma_{\mathrm{c}} = \mathrm{Gal}(K_{\mathrm{c,sep}}/K_{\mathrm{c}})$ and its "2-dimensional" part $\{\Gamma^{(v)}\}_{j\in J_2}$ gives a filtration of the group $\widetilde{\Gamma} = \mathrm{Gal}(K_{\mathrm{sep}}/KK_{\mathrm{c,sep}})$. Notice also that the beginning of the "J_2-part" of our filtration, which corresponds to the indices from the set $\{(0, v) \mid v \in Q_{\geq 0}\} \subset J_2$ comes, in fact, from the classical ramification filtration of the absolute Galois group of the first residue field $K^{(1)}$ of K.

In this paper we give an explicit description of the image of the ramification filtration $\{\Gamma^{(j)}\}_{j\in J}$ in the maximal quotient of Γ, which is a pro-p-group of nilpotent class 2, when K has a finite characteristic p. Our method is, in fact, a generalisation of methods from [1, 2], where the ramification filtration of the Galois group of the maximal p-extension of 1-dimensional local field of characteristic p modulo its subgroup of commutators of order $\geq p$ was described. Despite of the fact that we consider here only the case of local fields of dimension 2, our method admits a direct generalisation to the case of local fields of arbitrary dimension $n \geq 2$.

In a forthcoming paper we shall prove that the additional structure on Γ given by its ramification filtration $\{\Gamma^{(j)}\}_{j\in J}$ with another additional structure given by the special topology on each abelian sub-quotient of Γ (which was introduced in [5] and [7]) does not reconstruct completely (from the point of view of the theory of categories) the field K but only its composite with the maximal inseparable extension of K_{c}. The explanation of this phenomenon can be found in the definition of the "2-dimensional" part of the ramification filtration: this part is defined, in fact, over an algebraic closure of the field of 1-dimensional constants.

The paper was written during the author's visits of the University of Limoges (LACO UPRESA) in the autumn of 1999 and of the Universities of Nottingham and Cambridge in the beginning of 2000 year. The author expresses his gratitude to these organisations for hospitality and, especially, to Professors J.Coates, F.Laubie and I.Fesenko for fruitful discussions.

1. Preliminaries: Artin-Schreier theory for 2-dimensional local fields

1.1. Basic agreements. Let K be a 2-dimensional complete discrete valuation field of finite characteristic $p>0$. In other words, K is a complete field with respect to a discrete valuation v_1 and the corresponding residue field $K^{(1)}$ is complete with respect to a discrete valuation $\bar{v}_2$ with finite residue field $k\simeq\mathbb{F}_{p^{N_0}}$, $N_0\in\mathbb{N}$. Fix a field embedding $s:K^{(1)}\longrightarrow K$, which is a section of the natural projection from the valuation ring O_K onto $K^{(1)}$. Fix also a choice of uniformising elements $t_0\in K$ and $\bar{\tau}_0\in K^{(1)}$. Then $K=s(K^{(1)})((t_0))$ and $K^{(1)}=k((\bar{\tau}_0))$ (note that k is canonically identified with subfields in $K^{(1)}$ and K). We assume also that an algebraic closure K_{alg} of K is chosen, denote by K_{sep} the separable closure of K in K_{alg}, set $\Gamma=\mathrm{Gal}(K_{\mathrm{sep}}/K)$, and use the notation $\tau_0=s(\bar{\tau}_0)$.

1.2. ***P*-topology.** Consider the set $\mathcal{P}$ of collections $\omega=\{J_i(\omega)\}_{i\in\mathbb{Z}}$, where for some $I(\omega)\in\mathbb{Z}$, one has $J_i(\omega)\in\mathbb{Z}$ if $i\le I(\omega)$, and $J_i(\omega)=-\infty$ if $i>I(\omega)$. For any $\omega=\{J_i(\omega)\}_{i\in\mathbb{Z}}\in\mathcal{P}$, consider the set $A(\omega)\subset K$ consisting of elements written in the form $\sum_{i\in\mathbb{Z}}s(b_i)t_0^i$, where all $b_i\in K^{(1)}$, for a sufficiently small i one has $b_i=0$, and $b_i\in\bar{\tau}_0^{J_i(\omega)}O_{K^{(1)}}$ if $J_i(\omega)\ne-\infty$.

The family $\{A(\omega)\mid\omega\in\mathcal{P}\}$ when taken as a basis of zero neighbourhoods determines a topology of K. We shall denote this topology by $P_K(s,t_0)$ because its definition depends on the choice of the section s and the uniformiser t_0. In this topology $s(b_i)t_0^i\to0$ for $i\to+\infty$, where $\{b_i\}$ is an arbitrary sequence in $K^{(1)}$. Besides, for any $a\in\mathbb{Z}$, we have $\tau_0^jt_0^a\to0$ if $j\to+\infty$ and, therefore, s is a continuous embedding of $K^{(1)}$ into K (with respect to the valuation topology on $K^{(1)}$ and the $P_K(s,t_0)$-topology on K). It is known, cf. [5], if $t_1\in K$ is another uniformiser and s_1 is another section from $K^{(1)}$ to K, then the topologies $P_K(s,t_0)$ and $P_K(s_1,t_1)$ are equivalent. Therefore, we can use the notation P_K for any of these topologies. The family of topologies P_E for all extensions E of K in K_{alg} is compatible, cf. [7, 5]. This gives finally the topology on K_{alg} and this topology (as well as its restriction to any subfield of K_{alg}) can be denoted just by P.

1.3. Artin-Schreier theory. Let σ be the Frobenius morphism of K. Denote by Γ_1^{ab} the maximal abelian quotient of exponent p of Γ. Consider the Artin-Schreier pairing

$$\xi_1 : K/(\sigma - \mathrm{id})K \otimes_{\mathbb{F}_p} \Gamma_1^{\mathrm{ab}} \longrightarrow \mathbb{F}_p.$$

This pairing is a perfect duality of topological $\mathbb{F}_p$-modules, where $K/(\sigma - \mathrm{id})K$ is provided with discrete topology, and Γ_1^{ab} has the pro-finite topology of projective limit $\Gamma_1^{\mathrm{ab}} = \varprojlim \Gamma_{E/K}$, where E/K runs over the family of all finite extensions in K_{alg} with abelian Galois group of exponent p.

Consider the set $\mathbb{Z}^2$ with lexicographical ordering, where the advantage is given to the first coordinate. Set

$$\mathcal{A}_2 = \{(i,j) \in \mathbb{Z}^2 \mid (i,j) > (0,0), j \neq 0, (i,j,p) = 1\},$$

$\mathcal{A}_1 = \{(i,0) \mid i > 0, (i,p) = 1\}, \mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_1$ and $\mathcal{A}^0 = \mathcal{A} \cup \{(0,0)\}$.

Consider $K/(\sigma - \mathrm{id})K$ with topology induced by the P-topology of K (in this and another cases any topology induced by the P-topology will be also called the P-topology). Choose a basis $\{\alpha_r \mid 1 \leq r \leq N_0\}$ of the $\mathbb{F}_p$-module k and an element $\alpha_0 \in k$ such that $\mathrm{Tr}_{k/\mathbb{F}_p} \alpha_0 = 1$. Then the system of elements

$$\left\{\alpha_r \tau_0^{-j} t_0^{-i} \mid (i,j) \in \mathcal{A}, 1 \leq r \leq N_0\right\} \cup \{\alpha_0\}, \tag{1}$$

gives a P-topological basis of the $\mathbb{F}_p$ module $K/(\sigma - \mathrm{id})K$.

Let Ω be the set of collections $\omega = \{J_i(\omega)\}_{0 \leq i \leq I(\omega)}$, where $I(\omega) \in \mathbb{Z}_{\geq 0}$ and $J_i(\omega) \in \mathbb{N}$ for all $0 \leq i \leq I(\omega)$. Set

$$\mathcal{A}^0(\omega) = \left\{(i,j) \in \mathcal{A}^0 \mid 0 \leq i \leq I(\omega), j \leq J_i(\omega)\right\}$$

and $\mathcal{A}(\omega) = \mathcal{A}^0(\omega) \cap \mathcal{A} = \mathcal{A}^0(\omega) \setminus \{(0,0)\}$ (notice that $(0,0) \in \mathcal{A}^0(\omega)$). Denote by $U_1(\omega)$ the $\mathbb{F}_p$-submodule of $K/(\sigma - \mathrm{id})K$ generated by the images of elements of the set

$$\left\{\alpha_r \tau_0^{-j} t_0^{-i} \mid (i,j) \in \mathcal{A}(\omega), 1 \leq r \leq N_0\right\} \cup \{\alpha_0\}, \tag{2}$$

where $\tau_0 = s(\bar{\tau}_0)$. This is a basis of the system of compact $\mathbb{F}_p$-submodules in $K/(\sigma - \mathrm{id})K$ with respect to P-topology.

Let

$$G_1 = \left\{D_{(i,j)}^{(r)} \mid (i,j) \in \mathcal{A}, 1 \leq r \leq N_0\right\} \cup \{D_{(0,0)}\}$$

be the system of elements of Γ_1^{ab} dual to the system of elements (1) with respect to the pairing ξ_1. For $\omega \in \Omega$, set

$$G_1(\omega) = \left\{D_{(i,j)}^{(r)} \in G_1 \mid (i,j) \in \mathcal{A}(\omega), 1 \leq r \leq N_0\right\} \cup \{D_{(0,0)}\}.$$

Denote by $\mathcal{M}_1^f$ (resp., $\mathcal{M}_1^f(\omega)$) the $\mathbb{F}_p$-submodule in Γ_1^{ab} generated by elements of G_1 (resp., $G_1(\omega)$). Notice that G_1 (resp., $G_1(\omega)$) is an $\mathbb{F}_p$-basis of $\mathcal{M}_1^f$ (resp., $\mathcal{M}_1^f(\omega)$).

For any $\omega \in \Omega$, set $\Gamma_1(\omega)^{\mathrm{ab}} = \mathrm{Hom}(U_1(\omega), \mathbb{F}_p)$, then

$$\Gamma_1^{\mathrm{ab}} = \varprojlim_{\omega \in \Omega} \Gamma_1(\omega)^{\mathrm{ab}}.$$

We shall use the identification of elements $D^{(r)}_{(i,j)}$ where $(i,j) \in \mathcal{A}(\omega)$, $1 \le r \le N_0$, and $D_{(0,0)}$ with their images in $\Gamma_1(\omega)^{\mathrm{ab}}$. Then

$$\mathcal{M}_1^f(\omega) = \mathrm{Hom}_{P\text{-top}}(U_1(\omega), \mathbb{F}_p) \subset \Gamma_1^{\mathrm{ab}}$$

and $\Gamma_1(\omega)^{\mathrm{ab}}$ is identified with the completion of $\mathcal{M}_1^f(\omega)$ in the topology given by the system of zero neibourghoods consisting of all $\mathbb{F}_p$-submodules of finite index. Denote by $\mathcal{M}_1^{pf}(\omega)$ the completion of $\mathcal{M}_1^f(\omega)$ in the topology given by the system of zero neibourghoods consisting of $\mathbb{F}_p$-submodules, which contain almost all elements of the set $G_1(\omega)$. Then $\mathcal{M}_1^{pf}(\omega)$ is the set of all formal $\mathbb{F}_p$-linear combinations

$$\sum_{\substack{(i,j) \in \mathcal{A}(\omega) \\ 1 \le r \le N_0}} \alpha^{(r)}_{(i,j)} D^{(r)}_{(i,j)} + \alpha_{(0,0)} D_{(0,0)}$$

and we have natural embeddings $\mathcal{M}_1^f(\omega) \subset \Gamma_1(\omega)^{\mathrm{ab}} \subset \mathcal{M}_1^{pf}(\omega)$.

We notice that Γ_1^{ab} is the completion of $\mathcal{M}_1^f$ in the topology given by the basis of zero neibourghoods of the form $V_1 \oplus V_2$, where for some $\omega \in \Omega$, V_1 is generated by elements $D^{(r)}_{(i,j)}$ with $1 \le r \le N_0$ and $(i,j) \notin \mathcal{A}^0(\omega)$, and $\mathbb{F}_p$-module V_2 has a finite index in $\mathcal{M}_1^f(\omega)$. Denote by $\mathcal{M}_1^{pf}$ the completion of $\mathcal{M}_1^f$ in the topology given by the system of neibourghoods consisting of submodules containing almost all elements of the set G_1. Then $\mathcal{M}_1^{pf}$ is the set of all $\mathbb{F}_p$-linear combinations of elements from G_1, and we have natural embeddings $\mathcal{M}_1^f \subset \Gamma_1^{\mathrm{ab}} \subset \mathcal{M}_1^{pf}$.

1.4. Witt theory. Choose a p-basis $\{a_i \mid i \in I\}$ of K. Then for any $M \in \mathbb{N}$ and a field E such that $K \subset E \subset K_{\mathrm{sep}}$, one can construct a lifting $O_M(E)$ of E modulo p^M, that is a fully faithful $\mathbb{Z}/p^M\mathbb{Z}$-algebra $O_M(E)$ such that $O_M(E) \otimes_{\mathbb{Z}/p^M\mathbb{Z}} \mathbb{F}_p = E$. These liftings can be given explicitly in the form

$$O_M(E) = W_M(\sigma^{M-1}E)\,[\{[a_i] \mid i \in I\}]\,,$$

where $[a_i] = (a_i, 0, \dots, 0) \in W_M(E)$. The liftings $O_M(E)$ depend functorially on E and behave naturally with respect to the actions of the Galois group Γ and the Frobenius morphism σ.

For any $M \in \mathbb{N}$, consider the continuous Witt pairing modulo p^M

$$\xi_M : O_M(K)/(\sigma - \mathrm{id})O_M(K) \otimes_{\mathbb{Z}/p^M\mathbb{Z}} \Gamma_M^{\mathrm{ab}} \longrightarrow \mathbb{Z}/p^M\mathbb{Z},$$

where Γ_M^{ab} is the maximal abelian quotient of Γ of exponent p^M considered with its natural topology, and the first term of tensor product is provided with discrete topology. These pairings are compatible for different M and induce the continuous pairing

$$O(K)/(\sigma-\mathrm{id})O(K)\otimes_{\mathbb{Z}_p}\Gamma(p)^{\mathrm{ab}}\longrightarrow\mathbb{Z}_p,$$

where $O(K)=\varprojlim O_M(K)$ and $\Gamma(p)^{\mathrm{ab}}$ is the maximal abelian quotient of the Galois group $\Gamma(p)$ of the maximal p-extension of K in K_{sep}.

Now we specify the above arguments for the local field K of dimension 2 given in the notation of n.1.1. Clearly, the elements t_0 and τ_0 give a p-basis of K, i.e. the system of elements

$$\{\tau_0^b t_0^a \mid 0\le a,b<p\}$$

is a basis of the K^p-module K. So, for any $M\in\mathbb{N}$ and $K\subset E\subset K_{\mathrm{sep}}$, we can consider the system of liftings modulo p^M

$$O_M(E)=W_M(\sigma^{M-1}E)[t,\tau],\tag{3}$$

where $t=[t_0]$, $\tau=[\tau_0]$ are the Teichmuller representatives.

Choose a basis $\{\alpha_r\mid 1\le r\le N_0\}$ of the $\mathbb{Z}_p$-module $W(k)$ and its element α_0 with the absolute trace 1. We agree to use the same notation for residues modulo p^M of the above elements α_r, $0\le r\le N_0$. Then the system of elements

$$\left\{\alpha_r\tau^{-j}t^{-i}\mid(i,j)\in\mathcal{A},1\le r\le N_0\right\}\cup\{\alpha_0\}\tag{4}$$

gives a P-topological $\mathbb{Z}/p^M\mathbb{Z}$-basis of $O_M(K)/(\sigma-\mathrm{id})O_M(K)$.

For $\omega\in\Omega$, denote by $U_M(\omega)$ the P-topological closure of the $\mathbb{Z}/p^M\mathbb{Z}$-submodule of $O_M(K)/(\sigma-\mathrm{id})O_M(K)$ generated by the images of elements of the set

$$\left\{\alpha_r\tau^{-j}t^{-i}\mid(i,j)\in\mathcal{A}(\omega),1\le r\le N_0\right\}\cup\{\alpha_0\}.\tag{5}$$

This is a basis of the system of compact (with respect to the P-topology) submodules of $O_M(K)/(\sigma-\mathrm{id})O_M(K)$ (i.e. any its compact submodule is contained in some $U_M(\omega)$). As earlier, we introduce the system of elements of Γ_M^{ab}

$$G_M=\left\{D_{(i,j)}^{(r)}\mid(i,j)\in\mathcal{A},1\le r\le N_0\right\}\cup\{D_{(0,0)}\},$$

which is dual to the system (4) with respect to the pairing ξ_M. Similarly to subsection 1.3 introduce the $\mathbb{Z}/p^M\mathbb{Z}$-modules $\mathcal{M}_M^f$, $\mathcal{M}_M^{pf}$ and for any $\omega\in\Omega$, the subset $G_M(\omega)\subset G_M$ and the $\mathbb{Z}/p^M\mathbb{Z}$-submodules $\mathcal{M}_M^f(\omega)$, $\Gamma_M(\omega)^{\mathrm{ab}}$ and $\mathcal{M}_M^{pf}(\omega)$ such that

$$\mathcal{M}_M^f\subset\Gamma_M^{\mathrm{ab}}\subset\mathcal{M}_M^{pf},\quad\mathcal{M}_M^f(\omega)\subset\Gamma_M(\omega)^{\mathrm{ab}}\subset\mathcal{M}_M^{pf}(\omega),$$

$\Gamma_M^{\mathrm{ab}}=\varprojlim\Gamma_M(\omega)^{\mathrm{ab}}$, and $\mathrm{Hom}_{P\text{-top}}(U_M(\omega),\mathbb{F}_p)=\mathcal{M}_M^f(\omega)$.

Apply the pairing ξ_M to define the P-topology on Γ_M^{ab}. By definition, the basis of zero neibourghoods of Γ_M^{ab} consists of annihilators $U_M(\omega)^D$ of compact submodules $U_M(\omega)$, $\omega\in\Omega$, with respect to the pairing ξ_M.

We note that

$$U_M(\omega)^D = \mathrm{Ker}\left(\Gamma_M^{\mathrm{ab}} \longrightarrow \Gamma_M(\omega)^{\mathrm{ab}}\right).$$

Finally, we obtain the P-topology on $\Gamma(p)^{\mathrm{ab}} = \varprojlim\limits_{M,\omega} \Gamma_M(\omega)^{\mathrm{ab}}$ and note that the identity map $\mathrm{id} : \Gamma(p)^{\mathrm{ab}}_{P\text{-top}} \longrightarrow \Gamma(p)^{\mathrm{ab}}$ is continuous. Equivalently, if E/K is a finite abelian extension, then there is an $M \in \mathbb{N}$ and an $\omega \in \Omega$ such that the canonical projection $\Gamma(p)^{\mathrm{ab}} \longrightarrow \Gamma_{E/K}$ factors through the canonical projection $\Gamma(p)^{\mathrm{ab}} \longrightarrow \Gamma_M(\omega)^{\mathrm{ab}}$.

1.5. Nilpotent Artin-Schreier theory. For any Lie algebra L over $\mathbb{Z}_p$ of nilpotent class $< p$, we agree to denote by $G(L)$ the group of elements of L with the law of composition given by the Campbell-Hausdorff formula

$$(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1, l_2] + \dots .$$

Consider the system of liftings (3) from n.1.4 and set $O(E) = \varprojlim O_M(E)$, where $K \subset E \subset K_{\mathrm{sep}}$. If L is a finite Lie algebra of nilpotent class $< p$ set $L_E = L \otimes_{\mathbb{Z}_p} O(E)$. Then the nilpotent Artin-Schreier theory from [1] is presented by the following statements:

a) for any $e \in G(L_K)$, there is an $f \in G(L_{K_{\mathrm{sep}}})$ such that $\sigma f = f \circ e$;

b) the correspondence $\tau \mapsto (\tau f) \circ (-f)$ gives the continuous group homomorphism $\psi_{f,e} : \Gamma \longrightarrow G(L)$;

c) if $e_1 \in G(L_K)$ and $f_1 \in G(L_{K_{\mathrm{sep}}})$ is such that $\sigma f_1 = f_1 \circ e_1$, then the homomorphisms $\psi_{f,e}$ and ψ_{f_1,e_1} are conjugated if and only if $e = c \circ e_1 \circ (-\sigma c)$ for some $c \in G(L_K)$;

d) for any group homomorphism $\psi : \Gamma \longrightarrow G(L)$ there are $e \in G(L_K)$ and $f \in G(L_{K_{\mathrm{sep}}})$ such that $\psi = \psi_{f,e}$.

In order to apply the above theory to study Γ we need its pro-finite version. Identify $\Gamma(p)^{\mathrm{ab}}$ with the projective limit of Galois groups $\Gamma_{E/K}$ of finite abelian p-extensions E/K in K_{sep}. With this notation denote by $\mathcal{L}(E)$ the maximal quotient of nilpotent class $< p$ of the Lie $\mathbb{Z}_p$-algebra $\widetilde{\mathcal{L}}(E)$ generated freely by the $\mathbb{Z}_p$-module $\Gamma_{E/K}$. Then $\widetilde{\mathcal{L}} = \varprojlim \widetilde{\mathcal{L}}(E)$ is a topological free Lie algebra over $\mathbb{Z}_p$ with topological module of generators $\Gamma(p)^{\mathrm{ab}}$ and $\mathcal{L} = \varprojlim \mathcal{L}(E)$ is the maximal quotient of $\widetilde{\mathcal{L}}$ of nilpotent class $< p$ in the category of topological Lie algebras.

Define the "diagonal element" $\tilde{e} \in O(K)/(\sigma - \mathrm{id})O(K) \hat{\otimes}_{\mathbb{Z}_p} \Gamma(p)^{\mathrm{ab}}$ as the element coming from the identity endomorphism with respect to the identification

$$O(K)/(\sigma - \mathrm{id})O(K) \hat{\otimes}_{\mathbb{Z}_p} \Gamma(p)^{\mathrm{ab}} = \mathrm{End}_{\mathrm{cont}}(O(K)/(\sigma - \mathrm{id})O(K))$$

induced by the Witt pairing (here $O(K)$ is considered with the p-adic topology). Denote by s the unique section of the natural projection from $O(K)$

to $O(K)/(\sigma-\mathrm{id})O(K)$ with values in the P-topological closed submodule of $O(K)$ generated by elements of the set (4). Use the section s to obtain the element

$$e=(s\otimes\mathrm{id})(\tilde{e})\in O(K)\hat{\otimes}\Gamma(p)^{\mathrm{ab}}\subset\mathcal{L}_K:=O(K)\hat{\otimes}\mathcal{L}$$

such that $e\mapsto\tilde{e}$ by the natural projection

$$\mathcal{L}_K\longrightarrow\mathcal{L}_K^{\mathrm{ab}}\,\mathrm{mod}(\sigma-\mathrm{id})\mathcal{L}_K^{\mathrm{ab}}=O(K)/(\sigma-\mathrm{id})O(K)\hat{\otimes}\Gamma(p)^{\mathrm{ab}}.$$

For any finite abelian p-extension E/K in K_{sep}, denote by e_E the projection of e to $\mathcal{L}_K(E)=\mathcal{L}(E)\otimes_{\mathbb{Z}_p}O(K)$, and choose a compatible on E system of $f_E\in\mathcal{L}(E)_{\mathrm{sep}}=\mathcal{L}(E)\otimes O(K_{\mathrm{sep}})$ such that $\sigma f_E=f_E\circ e_E$. Then the correspondences $\tau\mapsto\tau f_E\circ(-f_E)$ give a compatible system of group homomorphisms $\psi_E:\Gamma(p)\longrightarrow G(\mathcal{L}(E))$ and the continuous homomorphism

$$\psi=\varprojlim\psi_E:\Gamma(p)\longrightarrow G(\mathcal{L})$$

induces the identity morphism of the corresponding maximal abelian quotients. Therefore, $\bar{\psi}=\psi\,\mathrm{mod}\,C_p(\Gamma(p))$ gives identification of p-groups $\Gamma(p)\,\mathrm{mod}\,C_p(\Gamma(p))$ and $G(\mathcal{L})$, where $C_p(\Gamma(p))$ is the closure of the subgroup of $\Gamma(p)$ generated by commutators of order $\geq p$. Of course, if $f=\varprojlim f_E\in G(\mathcal{L}_{\mathrm{sep}})$, then $\sigma f=f\circ e$ and $\psi(g)=(gf)\circ(-f)$ for any $g\in\Gamma$. Clearly, the conjugacy class of the identification $\bar{\psi}$ depends only on the choice of uniformisers t_0 and τ_0 and the element $\alpha_0\in W(k)$.

For $\omega\in\Omega$ and $M\in\mathbb{N}$, denote by $\mathcal{L}_M(\omega)$ the maximal quotient of nilpotent class $<p$ of the free Lee $\mathbb{Z}/p^M\mathbb{Z}$-algebra $\widetilde{\mathcal{L}}_M(\omega)$ with topological module of generators $\Gamma_M(\omega)^{\mathrm{ab}}$. We use the natural projections $\Gamma(p)^{\mathrm{ab}}\longrightarrow\Gamma_M(\omega)^{\mathrm{ab}}$ to construct the projections of Lie algebras $\mathcal{L}\longrightarrow\mathcal{L}_M(\omega)$ and induced morphisms of topological groups

$$\psi_M(\omega):\Gamma(p)\longrightarrow G(\mathcal{L}_M(\omega)).$$

Clearly, the topology on the group $G(\mathcal{L}_M(\omega))$ is given by the basis of neighbourhoods of the neutral element consisting of all subgroups of finite index.

Consider $\mathbb{Z}/p^M\mathbb{Z}$-modules $\mathcal{M}_M^f(\omega)$ and $\mathcal{M}_M^{pf}(\omega)$ from n.1.4. Denote by $\mathcal{L}_M^f(\omega)$ the maximal quotient of nilpotent class $<p$ of a free Lie algebra over $\mathbb{Z}/p^M\mathbb{Z}$ generated by $\mathcal{M}_M^f(\omega)$, and by $\mathcal{L}_M^{pf}(\omega)$ the similar object constructed for the topological $\mathbb{Z}/p^M\mathbb{Z}$-module $\mathcal{M}_M^{pf}(\omega)$. Clearly, $\mathcal{L}_M^{pf}(\omega)$ is identified with the projective limit of Lie sub-algebras of $\mathcal{L}_M(\omega)$ generated by all finite subsystems of its system of generators

$$\left\{D_{(i,j)}^{(r)}\mid 1\leq r\leq N_0,(i,j)\in\mathcal{A}(\omega)\right\}\cup\left\{D_{(0,0)}^{(0)}\right\}.\tag{6}$$

Besides, we have the natural inclusions

$$\mathcal{L}_M^f(\omega)\subset\mathcal{L}_M(\omega)\subset\mathcal{L}_M^{pf}(\omega),$$

where $\mathcal{L}_M(\omega)$ is identified with the completion of $\mathcal{L}^f_M(\omega)$ in the topology defined by all its Lie sub-algebras of finite index. Let

$$e_M(\omega) = \sum_{\substack{(i,j)\in\mathcal{A}(\omega)\\ 1\leq r\leq N_0}} \alpha_r\tau^{-j}t^{-i}D^{(r)}_{(i,j)} + \alpha_0 D^{(0)}_{(0,0)} \in O_M(K)\hat{\otimes}\mathcal{M}^{pf}_M(\omega).$$

Lemma 1.1. *There exists $f_M(\omega) \in G(\mathcal{L}_M(\omega)_{\mathrm{sep}})$ such that $\sigma f_M(\omega) = f_M(\omega)\circ e_M(\omega)$ (and therefore $e_M(\omega) \in O_M(K)\hat{\otimes}\Gamma^{\mathrm{ab}}_M(\omega)$) and for any $g\in\Gamma(p)$,*

$$\psi_M(\omega)(g) = (gf_M(\omega))\circ(-f_M(\omega)).$$

Proof. Denote by $e'_M(\omega)$ the image of e in $O_M(K)\hat{\otimes}\Gamma_M(\omega)^{\mathrm{ab}}$. Let U_0 be an open submodule of $\mathcal{M}^{pf}_M(\omega)$ and $U'_0 = U_0\cap\Gamma_M(\omega)^{\mathrm{ab}}$. Set $e_0 = e_M(\omega)\bmod O_M(K)\hat{\otimes}U_0$ and $e'_0 = e'_M(\omega)\bmod O_M(K)\hat{\otimes}U'_0$. Then

$$e_0, e'_0 \in V := O_M(K)\otimes\Gamma_M(\omega)^{\mathrm{ab}}/U'_0 = O_M(K)\otimes\mathcal{M}^{pf}_M(\omega)/U_0.$$

The residues $e_0 \bmod(\sigma-\mathrm{id})V$ and $e'_0\bmod(\sigma-\mathrm{id})V$ coincide because the both appear as the images of the "diagonal element" for the Witt pairing. But e_0 and e'_0 are obtained from the above residues by the same section $V\bmod(\sigma-\mathrm{id})V \longrightarrow V$, therefore,

$$e'_M(\omega)\equiv e_M(\omega)\bmod O_M(K)\hat{\otimes}U_0.$$

Because intersection of all open submodules U_0 of $\mathcal{M}_M(\omega)$ is 0, one has $e'_M(\omega) = e_M(\omega)$ and we can take as $f_M(\omega)$ the image of $f\in G(\mathcal{L}_{\mathrm{sep}})$ under the natural projection $G(\mathcal{L}_{\mathrm{sep}}) \longrightarrow G(\mathcal{L}_M(\omega)_{\mathrm{sep}})$. The lemma is proved. $\square$

By the above lemma we have an explicit construction of all group morphisms $\psi_M(\omega)$ with $M\in\mathbb{N}$ and $\omega\in\Omega$. Their knowledge is equivalent to the knowledge of the identification $\bar{\psi}\bmod C_p(\Gamma(p))$, because of the equality

$$\psi = \varprojlim_{M,\omega}\psi_M(\omega)$$

which is implied by the following lemma.

Lemma 1.2. *Let L be a finite (discrete) Lie algebra over $\mathbb{Z}_p$ and let $\phi : \Gamma(p)\longrightarrow G(L)$ be a continuous group morphism. Then there are $M\in\mathbb{N}$, $\omega\in\Omega$ and a continuous group morphism*

$$\phi_M(\omega) : G(\mathcal{L}_M(\omega)) \longrightarrow G(L)$$

such that $\phi(\omega) = \psi_M(\omega)\circ\phi_M(\omega)$.

Proof. Let $e\in G(L_K)$ and $f\in G(L_{\mathrm{sep}})$ be such that $\sigma f = f\circ e$ and for any $g\in\Gamma(p)$, it holds

$$\phi(g) = (gf)\circ(-f).$$

One can easily prove the existence of $c \in G(L_K)$ such that for $e_1 = (-c) \circ e \circ (\sigma c)$, one has

$$e_1 = \sum_{(a,b)\in\mathcal{A}^0} \tau^{-b} t^{-a} l_{(a,b),0},$$

where all $l_{(a,b),0} \in L_k = L \otimes W(k)$ and $l_{(0,0),0} = \alpha_0 l_{(0,0)}$ for some $l_{(0,0)} \in L$. If $f_1 = f \circ c$ then $\sigma f_1 = f_1 \circ e_1$ and for any $g \in \Gamma$,

$$\phi(g) = (gf_1) \circ (-f_1).$$

Let $h_1, \dots, h_u \in L$ be such that for some $m_i \in \mathbb{Z}_{\geq 0}$ with $1 \leq i \leq u$,

$$L = \oplus_{1\leq i\leq u} h_i \mathbb{Z}/p^{m_i}\mathbb{Z}.$$

If

$$l_{(a,b),0} = \sum_{1\leq i\leq u} \alpha_{(a,b),i} h_i,$$

where all coefficients $\alpha_{(a,b),i} \in W(k)$, then

$$e_1 = \sum_{1\leq i\leq u} A_i h_i,$$

where all coefficients

$$A_i = \sum_{(a,b)\in\mathcal{A}^0} \alpha_{(a,b),i} \tau^{-b} t^{-a} \in O_M(K)$$

with $M = \max\{m_i \mid 1 \leq i \leq u\}$. Clearly, there exists $\omega \in \Omega$ such that $\alpha_{(a,b),i} = 0$ for all $1 \leq i \leq u$ and $(a,b) \notin \mathcal{A}^0(\omega)$.

Let $\beta_1, \dots, \beta_{N_0}$ be the dual $W(\mathbb{F}_p)$-basis of $W(k)$ for the basis $\alpha_1, \dots, \alpha_{N_0}$ from n.1.4. Consider the morphism of Lie algebras $\phi'_M(\omega) : \mathcal{L}_M(\omega) \longrightarrow L$ uniquely determined by the correspondences

$$D_{(0,0)} \mapsto l_{(0,0)}, \quad D^{(r)}_{(a,b)} \mapsto \sum_{0\leq n<N_0} \sigma^n(\beta_r l_{(a,b),0}),$$

for all $(a,b) \in \mathcal{A}(\omega)$ and $1 \leq r \leq N_0$.

Clearly, $\phi'_M(\omega)$ is a continuous morphism of Lie algebras, which transforms $e_M(\omega)$ to e_1. Let $f' \in G(L_{\mathrm{sep}})$ be the image of $f_M(\omega)$, then $\sigma f' = f' \circ e_1$. So, the composition

$$\phi' = \psi_M(\omega) \circ \phi'_M(\omega) : \Gamma(p) \longrightarrow G(L)$$

is given by the correspondence $\phi'(g) = (gf') \circ (-f')$ for all $g \in \Gamma(p)$.

Let $c_0 = f' \circ (-f) \in G(L_{\mathrm{sep}})$. Then $c_0 \in G(L_{\mathrm{sep}})|_{\sigma=\mathrm{id}} = G(L)$. Therefore, for any $g \in \Gamma(p)$,

$$\phi(g) = (gf) \circ (-f) = (-c_0) \circ (gf') \circ (-f') \circ c_0 = (-c_0) \circ \phi'(g) \circ c_0.$$

So, we can take $\phi_M(\omega)$ such that for any $l \in \mathcal{L}_M(\omega)$,

$$\phi_M(\omega)(l) = (-c_0) \circ \phi'_M(\omega)(l) \circ c_0.$$

The lemma is proved. □

2. 2-dimensional ramification theory

In this section we assume that K is a 2-dimensional complete discrete valuation field of characteristic p provided with an additional structure given by its subfield of 1-dimensional constants K_{c} and by a double valuation $v^{(0)} : K \longrightarrow \mathbb{Q}^2 \cup \{\infty\}$. By definition K_{c} is complete (with respect to the first valuation of K) discrete valuation subfield of K, which has finite residue field and is algebraically closed in K. As usually, we assume that an algebraic closure K_{alg} of K is chosen, denote by E_{sep} the separable closure of any subfield E of K_{alg} in K_{alg}, set $\Gamma_E = \mathrm{Gal}(E_{\mathrm{sep}}/E)$ and use the algebraic closure of K_{c} in E as its field of 1-dimensional constants E_{c}. We shall use the same symbol $v^{(0)}$ for a unique extension of $v^{(0)}$ to E. We notice that $\mathrm{pr}_1(v^{(0)}) : E \longrightarrow \mathbb{Q} \cup \{\infty\}$ gives the first valuation on E and $\mathrm{pr}_2(v^{(0)})$ is induced by the valuation of the first residue field $E^{(1)}$ of E. The condition $v^{(0)}(E^*) = \mathbb{Z}^2$ gives a natural choice of one valuation in the set of all equivalent valuations of the field E.

2.1. 2-dimensional ramification filtration of $\widetilde{\Gamma}_E \subset \Gamma_E$. Let E be a finite extension of K in K_{alg}. Consider a finite extension L of E in E_{sep} and set $\widetilde{\Gamma}_{L/E} = \mathrm{Gal}(L/EL_{\mathrm{c}})$ (we note that $L_{\mathrm{c}} = (EL_{\mathrm{c}})_{\mathrm{c}}$). If $\varprojlim_L \widetilde{\Gamma}_{L/E} := \widetilde{\Gamma}_E$ then we have the natural exact sequence of pro-finite groups

$$1 \longrightarrow \widetilde{\Gamma}_E \longrightarrow \Gamma_E \longrightarrow \Gamma_{E_{\mathrm{c}}} \longrightarrow 1. \tag{7}$$

The 2-dimensional ramification theory appears as a decreasing sequence of normal subgroups $\left\{\Gamma_E^{(j)}\right\}_{j\in J_2}$ of $\widetilde{\Gamma}_E$, where

$$J_2 = \left\{(a,b) \in \mathbb{Q}^2 \mid (a,b) \geq (0,0)\right\}.$$

Here $\mathbb{Q}^2$ is considered with lexicographical ordering (where the advantage is given to the first coordinate), in particular, $J_2 = (\{0\} \times \mathbb{Q}_{\geq 0}) \bigcup (\mathbb{Q}_{>0} \times \mathbb{Q})$. Similarly to the classical (1-dimensional) case, one has to introduce the filtration in lower numbering $\{\Gamma_{L/E,j}\}_{j\in J_2}$ for any finite Galois extension L/E. Apply the process of "eliminating wild ramification" from [4] to choose a finite extension $\widetilde{E}_{\mathrm{c}}$ of L_{c} in $K_{\mathrm{c,alg}}$ such that the extension $\widetilde{L} := L\widetilde{E}_{\mathrm{c}}$ over $\widetilde{E} := E\widetilde{E}_{\mathrm{c}}$ has relative ramification index 1. Then the corresponding extension of the (first) residue fields $\widetilde{L}^{(1)}/\widetilde{E}^{(1)}$ is a totally ramified (usually, inseparable) extension of complete discrete valuation fields of degree $[\widetilde{L} : \widetilde{E}]$.

If $\bar{\theta}$ is a uniformising element of $\widetilde{L}^{(1)}$ then $O_{\widetilde{L}^{(1)}} = O_{\widetilde{E}^{(1)}}[\bar{\theta}]$. Introduce the double valuation rings $\mathcal{O}_{\widetilde{E}} := \left\{l \in \widetilde{E} \mid v^{(0)}(l) \geq (0,0)\right\}$ and $\mathcal{O}_{\widetilde{L}} :=$

$\left\{l \in \widetilde{L} \mid v^{(0)}(l) \geq (0,0)\right\}$. Then $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{E}}[\theta]$ for any lifting θ of $\bar{\theta}$ to $\mathcal{O}_{\widetilde{L}}$. This property provides us with well-defined ramification filtration of $\Gamma_{\widetilde{L}/\widetilde{E}} \subset \Gamma_{L/E}$ in lower numbering

$$\Gamma_{L/E,j} = \left\{g \in \Gamma_{\widetilde{L}/\widetilde{E}} \mid v^{(0)}(g\theta - \theta) \geq v^{(0)}(\theta) + j\right\},$$

where j runs over the set J_2.

One can easily see that the above definition does not depend on the choices of $\widetilde{E}_{\mathrm{c}}$ and $\bar{\theta}$. The Herbrand function $\varphi^{(2)}_{L/E} : J_2 \longrightarrow J_2$ is defined similarly to the classical case: for any $(a,b) \in J^{(2)}$ take a partition

$$(0,0) = (a_0, b_0) < (a_1, b_1) < \cdots < (a_s, b_s) = (a,b),$$

such that the groups $\Gamma_{L/E,j}$ are of the same order g_i for all j between (a_{i-1}, b_{i-1}) and (a_i, b_i), where $1 \leq i \leq s$, and set

$$\varphi^{(2)}_{L/E}(a,b) = (g_1(a_1 - a_0) + \cdots + g_s(a_s - a_{s-1}), g_1(b_1 - b_0) + \cdots + g_s(b_s - b_{s-1})).$$

Let $E \subset L_1 \subset L$ be a tower of finite Galois extensions in E_{sep}. Then the above defined Herbrand function satisfies the composition property, i.e. for any $j \in J^{(2)}$, one has

$$\varphi^{(2)}_{L/E}(j) = \varphi^{(2)}_{L_1/E}\left(\varphi^{(2)}_{L/L_1}(j)\right). \tag{8}$$

This property can be proved as follows. Choose as earlier the finite extension $\widetilde{E}_{\mathrm{c}}$ of L_{c}, then all fields in the tower

$$\widetilde{L} \supset \widetilde{L}_1 \supset \widetilde{E},$$

where $\widetilde{L} = L\widetilde{E}_{\mathrm{c}}$, $\widetilde{L}_1 = L_1\widetilde{E}_{\mathrm{c}}$, $\widetilde{E} = E\widetilde{E}_{\mathrm{c}}$ (note that $\widetilde{L}_{\mathrm{c}} = \widetilde{L}_{1,\mathrm{c}} = \widetilde{E}_{\mathrm{c}}$), have the same uniformiser (with respect to the first valuation). If $\bar{\theta}$ is a uniformiser of the first residue field $\widetilde{L}^{(1)}$ of $\widetilde{L}$ and $\theta \in \mathcal{O}_{\widetilde{L}}$ is a lifting of $\bar{\theta}$, then $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{E}}[\theta]$ and $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{L}_1}[\theta]$. But we have also $\mathcal{O}_{\widetilde{L}_1} = \mathcal{O}_{\widetilde{E}}[N_{\widetilde{L}/\widetilde{L}_1}(\theta)]$ because $N_{\widetilde{L}/\widetilde{L}_1}(\bar{\theta})$ is uniformizing element of $\widetilde{L}_1^{(1)}$. Now one can relate the values of the Herbrand function in the formula (8) by classical 1-dimensional arguments from [8].

Similarly to classical case one can use the composition property (8) to extend the definition of the Herbrand function to the class of all (not necessarily Galois) finite separable extensions, introduce the upper numbering $\Gamma_{L/E,j} = \Gamma_{L/E}^{(\varphi^{(2)}_{L/E}(j))}$ and apply it to define the ramification filtration $\left\{\Gamma_E^{(j)}\right\}_{j \in J_2}$ of the subgroup $\widetilde{\Gamma}_E \subset \Gamma_E$.

2.2. Ramification filtration of Γ_E. The above definition of 2-dimensional ramification filtration works formally in the case of 1-dimensional complete discrete valuation fields K. Note that in this case there is a canonical choice of the field of 0-dimensional constants K_{c}, and we do not need to apply the process of eliminating wild ramification. This gives for any complete discrete valuation subfield $E \subset K_{\mathrm{alg}}$, the filtration $\left\{\Gamma_E^{(v)}\right\}_{v\geq 0}$ of the inertia subgroup $\widetilde{\Gamma}_E \subset \Gamma_E$. Note also that this filtration depends on the initial choice of the valuation $v^{(0)} : K \longrightarrow \mathbb{Q}\cup\{\infty\}$ and coincides with classical ramification filtration if $v^{(0)}(E^*) = \mathbb{Z}$.

Consider the 2-dimensional ramification filtration $\left\{\Gamma_E^{(j)}\right\}_{j\in J_2}$ and the above defined 1-dimensional ramification filtration $\left\{\Gamma_{E_{\mathrm{c}}}^{(v)}\right\}_{v\geq 0}$ for the (first) valuation $\mathrm{pr}_1(v^{(0)}) : K_{\mathrm{c}} \longrightarrow \mathbb{Q}\cup\{\infty\}$.

Let $J = J_1 \cup J_2$, where $J_1 = \{(v, \mathrm{c}) \mid v \geq 0\}$. Introduce the ordering on J by the use of natural orderings on J_1 and J_2, and by setting $j_1 < j_2$ for any $j_1 \in J_1$ and $j_2 \in J_2$. For any $j = (v, \mathrm{c}) \in \mathrm{J}_1$, set $\Gamma_E^{(j)} = \mathrm{pr}^{-1}\left(\Gamma_{E_{\mathrm{c}}}^{(v)}\right)$ where $\mathrm{pr} : \Gamma_E \longrightarrow \Gamma_{E_{\mathrm{c}}}$ is the natural projection. This gives the complete ramification filtration $\left\{\Gamma_E^{(j)}\right\}_{j\in J}$ of the group Γ_E. For any finite extension L/E, we denote by

$$\varphi_{L/E} : J \longrightarrow J$$

its Herbrand function given by the bijection $\varphi_{L/E}^{(2)} : J_2 \longrightarrow J_2$ from n.2.1 and its 1-dimensional analogue $\varphi_{L_{\mathrm{c}}/E_{\mathrm{c}}}^{(1)} : J_1 \longrightarrow J_1$ (which coincides with the classical Herbrand function if $v^{(0)}(E^*) = \mathbb{Z}^2$). We note also that the above filtration contains two pieces coming from the 1-dimensional theory and the both of them coincide with the classical filtration if $v^{(0)}(E^*) = \mathbb{Z}^2$. The first piece comes as the ramification filtration of $\Gamma_{E_{\mathrm{c}}}$ given by the groups $\Gamma_{E_{\mathrm{c}}}^{(v)} = \Gamma_E^{(v,\mathrm{c})}/\Gamma_E^{(0,0)}$ for all $v \geq 0$. The second piece comes from the ramification filtration of the first residue field $E^{(1)}$ of E. Here for any $v \geq 0$, $\Gamma_{E^{(1)}}^{(v)} = \Gamma_E^{(0,v)}/\Gamma_E^{(0,\infty)}$, where

$$\Gamma_E^{(0,\infty)} = \quad \text{the closure of } \bigcup\left\{\Gamma_E^{(a,b)} \mid (a,b) \in J^{(2)}, a > 0\right\}.$$

2.3. n-dimensional filtration. The above presentation of the 2-dimensional aspect of ramification theory can be generalised directly to the case of n-dimensional local fields. If K is an n-dimensional complete discrete valuation field, then we provide it with an additional structure by its $(n-1)$-dimensional subfield of “constants” K_{c} and an n-valuation $v^{(0)} : K \longrightarrow$

$\mathbb{Q}^n \cup \{\infty\}$. For any complete discrete valuation subfield E of K, the n-dimensional ramification filtration appears as the filtration $\left\{\Gamma_E^{(j)}\right\}_{j\in J^{(n)}}$ of the group $\widetilde{\Gamma}_E = \mathrm{Gal}\left(E_{\mathrm{sep}}/E_{\mathrm{c,sep}}\right)$ with indexes from the set

$$J_n = \{a \in \mathbb{Q}^n \mid a \geq (0, \dots, 0)\}$$

(where E_{c} is the algebraic closure of K_{c} in E). The process of eliminating wild ramification gives for any finite Galois extension L/E a finite extension $\widetilde{E}_{\mathrm{c}}$ of L_{c} such that for the corresponding fields $\widetilde{L} = L\widetilde{E}_{\mathrm{c}}$ and $\widetilde{E} = E\widetilde{E}_{\mathrm{c}}$, the ramification index of each residue field $\widetilde{L}^{(r)}$ of $\widetilde{L}$ with respect to the first $r \leq n-2$ valuations over the similar residue field $\widetilde{E}^{(r)}$ of $\widetilde{E}$ is equal to 1. Then one can use the lifting Θ of any uniformising element of the residue field $\widetilde{L}^{(n-1)}$ to the n-valuation ring $\mathcal{O}_{\widetilde{L}} = \left\{l \in \widetilde{L} \mid v^{(0)}(l) \geq (0, \dots, 0)\right\}$ to obtain the property

$$\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{E}}[\Theta].$$

This property provides us with a definition of ramification filtration of $\Gamma_{\widetilde{L}/\widetilde{E}} \subset \Gamma_{L/E}$ in lower numbering. Clearly, if L_1 is any field between E and L, and $\widetilde{L}_1 = L_1\widetilde{E}_{\mathrm{c}}$, then one has the property

$$\mathcal{O}_{\widetilde{L}_1} = \mathcal{O}_{\widetilde{E}}\left[N_{L/L_1}\Theta\right].$$

This provides us with the composition property for Herbrand function, and gives finally the definition of the ramification filtration $\{\Gamma_E^{(j)}\}_{j\in J_n}$ of $\widetilde{\Gamma}_E$ in upper numbering.

One can choose a subfield of $(n-2)$-dimensional constants $K_{\mathrm{cc}} \subset K_{\mathrm{c}}$ and apply the above arguments to obtain the ramification filtration of $\mathrm{Gal}(K_{\mathrm{c,sep}}/K_{\mathrm{cc,sep}})$. This procedure gives finally the ramification filtration of the whole group Γ_E, which depends on the choice of a decreasing sequence of fields of constants of dimensions $n-1$, $n-2$, ... , and 1.

3. Auxiliary facts

In this section K is a 2-dimensional complete discrete valuation field given in the notation of n.1.1. We assume that an additional structure on K is given by its subfield of 1-dimensional constants K_{c} and a double valuation $v^{(0)}$ such that $K_{\mathrm{c}} = k((t_0))$ and $v^{(0)}(K^*) = \mathbb{Z}^2$ (or, equivalently, $v^{(0)}(t_0) = (1,0)$ and $v^{(0)}(\tau_0) = (0,1)$). As in n.1.4 we use the construction of liftings of K and K_{sep}, which corresponds to the p-basis $\{t_0, \tau_0\}$ of K. We reserve the notation t and τ for the Teichmuller representatives of t_0 and τ_0, respectively. For any tower of field extensions $K \subset E \subset L \subset K_{\mathrm{alg}}$, we set

$$j(L/E) = \max\left\{j \in J \mid \Gamma_E^{(j)} \text{ acts non-trivially on } L\right\},$$

where $\Gamma_E^{(j)}$ is the ramification subgroup of $\Gamma_E = \mathrm{Gal}(E_{\mathrm{sep}}/E)$ with the upper index $j \in J$. Similarly to the 1-dimensional case, $j(L/E)$ is the value of the Herbrand function of the extension L/E in its maximal " edge point". Then the composition property (8) from n.2.1 gives for arbitrary tower of finite extensions $E \subset L_1 \subset L$,

$$j(L/E) = \max\left\{j(L_1/E), \varphi_{L_1/E}(j(L/L_1))\right\}. \tag{9}$$

If $\alpha \in W(k)$, then as usually

$$E(\alpha, X) = \exp\left(\alpha X + \sigma(\alpha)X^p/p + \dots + \sigma^n(\alpha)X^{p^n}/p^n + \dots\right) \in W(k)[[X]].$$

3.1. Artin-Schreier extensions. Let $L = K(X)$, where

$$X^p - X = \alpha_0 \tau_0^{-b} t_0^{-a}, \tag{10}$$

with $\alpha_0 \in k^*$ and $(a, b) \in \mathcal{A}$, i.e. $a, b \in \mathbb{Z}$, $a \geq 0$, $(a, b, p) = 1$.

Proposition 3.1. a) *If* $b = 0$*, then* $j(L/K) = (a, \mathrm{c})$*;*
b) *if* $b \neq 0$ *and* $v_p(b) = s \in \mathbb{Z}_{\geq 0}$*, then* $j(L/K) = (a/p^s, b/p^s)$*.*

Proof. The above examples can be found in [9]. The property a) is a well-known 1-dimensional fact. The property b) follows directly from definitions, we only note that one must take the extension $M_{\mathrm{c}} = K_{\mathrm{c}}(t_1)$, $t_1^{p^{s+1}} = t_0$, to kill the ramification of L/K and to rewrite the equation (10) in the form

$$\tau_1^p - t_1^{a(p-1)}\tau_1 = \alpha_1 \tau_0^{-b_1},$$

where $b_1 = b/p^s$, $\alpha_1^{p^s} = \alpha_0$ and $X = \left(\tau_1 t_1^{-a}\right)^{p^s}$. Then for any $j \in J$,

$$\varphi_{L/K}(j) = \begin{cases} pj, & \text{for } j \leq (a/p^{s+1}, b_1/p), \\ j + (1 - 1/p)(a/p^s, b_1), & \text{otherwise.} \end{cases}$$

So, $\Gamma_{L/K}^{(j)} = e$ if and only if $j > (a/p^s, b_1)$, that is $j(L/K) = (a/p^s, b/p^s)$. The proposition is proved. $\square$

3.2. The field $K(N^*, j^*)$. Let $N^* \in \mathbb{N}$, $q = p^{N^*}$ and let $j^* = (a^*, b^*) \in J_2$ be such that $A^* := a^*(q-1) \in \mathbb{N}[1/p]$, $B^* := b^*(q-1) \in \mathbb{Z}$ and $(B^*, p) = 1$. Set $s^* = \max\{0, -v_p(a^*)\}$ and introduce $t_{10}, t_{20} \in K_{\mathrm{alg}}$ such that $t_{10}^q = t_0$ and $t_{20}^{p^{s^*}} = t_{10}$.

Proposition 3.2. *There exists an extension* $K_0 = K(N^*, j^*)$ *of* K *in* K_{sep} *such that*
a) $K_{0,\mathrm{c}} = K_{\mathrm{c}}$ *and* $[K_0 : K] = q$*;*
b) *for any* $j \in J_2$*, one has*

$$\varphi^{(2)}_{K_0/K}(j) = \begin{cases} qj, & \textit{for } j \leq j^*/q, \\ (q-1)j^* + j, & \textit{otherwise.} \end{cases}$$

(what implies that $j(K_0/K)=j^*$);

c) *if $K_2':=K_0(t_{20})$, then its first residue field ${K'}_2^{(1)}$ equals $k((\tau_{10}))$, where*

$$\tau_{10}^q E(-1,\tau_{10}^{B^*}t_{10}^{A^*})=\tau_0$$

(here $t_{10}^{A^*}:=t_{20}^{p^{s^*}A^*}$ and E is an analogue of the Artin-Hasse exponential from the beginning of this section).

Proof. We only sketch the proof, which is similar to the proof of proposition of n.1.5 in the paper [2].

Let $t_1^{q-1}=t_0$, $\tau_1^{q-1}=\tau_0$, and $K_1=K(t_1,\tau_1)$. Let $L_1=K_1(U)$, where

$$U^q+b^*U=\tau_1^{-b^*(q-1)p^{s^*}}t_1^{-a^*(q-1)p^{s^*}}.$$

It is easy to see that $[L_1:K_1]=q$ and the "2-dimensional component" of the Herbrand function $\varphi^{(2)}_{L_1/K_1}$ is given by the expression from n.b) of our proposition. Then one can check the existence of the field K' such that $K\subset K'\subset L_1$, $[K':K]=q$ and $L_1=K'K_1$. We notice that $K'_{\mathrm{c}}=K_{\mathrm{c}}$ and one can assume that $K'=K(U^{q-1})$. Now the composition property of the Herbrand function implies that $\varphi^{(2)}_{L_1/K_1}=\varphi^{(2)}_{K'/K}$.

To verify the property c) of our proposition let us rewrite the above equation for U in the following form

$$\left(U_1t_2^{a^*(q-1)}\right)^q+b^*t_2^{a^*(q-1)^2}\left(U_1t_2^{a^*(q-1)}\right)=\tau_1^{-b^*(q-1)},$$

where $U_1^{p^{s^*}}=U$ and $t_2^q=t_1$ (notice that $t_2^{q-1}=t_{10}$). This implies the existence of $\tau_2\in L_1$ such that $U_1t_2^{a^*(q-1)}=\tau_2^{-b^*(q-1)}$, i.e. the last equation can be written in the form

$$\tau_2^{-b^*q(q-1)}\left(1+b^*t_2^{a^*(q-1)^2}\tau_2^{b^*(q-1)^2}\right)=\tau_1^{-b^*(q-1)}.$$

One can take τ_2 in this equality such that $\tau_2^{q-1}=\tau'_{10}\in K'$ and after taking the $-(1/b^*)$-th power of the both sides of that equality, we obtain

$${\tau'_{10}}^{q}\left(1+b^*t_{10}^{a^*(q-1)}{\tau'_{10}}^{b^*(q-1)}\right)^{-1/b^*}=\tau_0.$$

This gives the relation

$${\tau'_{10}}^{q}\left(1-{\tau'_{10}}^{b^*(q-1)}t_{10}^{a^*(q-1)}\right)=\tau_0+\tau_0{\tau'_{10}}^{2b^*(q-1)}t_{10}^{2a^*(q-1)}A,$$

where $A\in K'(t_{20})$ is such that $v^0(A)\geq(0,0)$. Then a suitable version of the Hensel Lemma gives the existence of $B\in K'(t_{20})$ such that $v^0(B)\geq(0,0)$ and the equality of n.c) of our proposition holds with

$$\tau_{10}=\tau'_{10}+B\tau_{10}^{b^*(q-1)+1}t_{10}^{a^*(q-1)}.$$

□

3.3. Relation between different liftings. Choose $j^*\in J_2$ and $N^*\in\mathbb{N}$, which satisfy the hypothesis from the beginning of n.3.2 and consider the corresponding field $K_0=K(N^*,j^*)$. Let $K'=K_0(t_{10})$, then K' is a purely inseparable extension of K_0 of degree q and $K'_{\mathrm{sep}}=K_{\mathrm{sep}}K'$. Clearly, $K'={K'}^{(1)}((t_{10}))$, where ${K'}^{(1)}=k((\tau_{10}))$.

Consider the field isomorphism $\eta:K\longrightarrow K'$, which is uniquely defined by the conditions $\eta|_k=\sigma^{-N^*}$, $\eta(t_0)=t_{10}$ and $\eta(\tau_0)=\tau_{10}$. Denote by η_{sep} an extension of η to a field isomorphism of K_{sep} and K'_{sep}.

For $M\geq 0$, denote by $O'_{M+1}(K')$ and $O'_{M+1}(K'_{\mathrm{sep}})$, respectively, the liftings modulo p^{M+1} of K' and K'_{sep} with respect to the p-basis $\{t_{10},\tau_{10}\}$ of K'. We reserve now the notation t_1 and τ_1 for the Teichmuller representatives of elements t_{10} and τ_{10}, respectively. Clearly, $\eta(O_{M+1}(K))=O'_{M+1}(K')$ and $\eta_{\mathrm{sep}}(O_{M+1}(K_{\mathrm{sep}}))=O'_{M+1}(K'_{\mathrm{sep}})$. On the other hand, by n.c) of Prop. 3.2, $K'_2:=K'(t_{20})=K_0(t_{20})$ is a separable extension of $K_2:=K(t_{20})$. We note that $K'_{2,\mathrm{sep}}=K'_{\mathrm{sep}}K'_2$ and $K_{2,\mathrm{sep}}=K_{\mathrm{sep}}K_2$. Denote by $O_{M+1}(K_2)$ and $O_{M+1}(K_{2,\mathrm{sep}})$, respectively, the liftings modulo p^M of K_2 and $K_{2,\mathrm{sep}}$ with respect to the p-basis $\{t_{20},\tau_0\}$ of K_2 (as earlier, t_2 and τ are the Teichmuller representatives of elements t_{20} and τ_0, respectively).

Clearly, one has the natural embeddings

$$O_{M+1}(K)\subset O_{M+1}(K_2),\quad O_{M+1}(K_{\mathrm{sep}})\subset O_{M+1}(K_{2,\mathrm{sep}}).$$

With respect to these embeddings we have $t=t_2^{qp^{s^*}}$. Denote by $O'_{M+1}(K'_2)$ and $O'_{M+1}(K'_{2,\mathrm{sep}})$ the liftings of K'_2 and $K'_{2,\mathrm{sep}}$ with respect to the p-basis $\{t_{20},\tau_{10}\}$ of K'_2 (as usually, t_2 and τ_1 are the Teichmuller representatives of elements t_{20} and τ_{10}, respectively). Clearly,

$$O'_{M+1}(K')\subset O'_{M+1}(K'_2),\quad O'_{M+1}(K'_{\mathrm{sep}})\subset O'_{M+1}(K'_{2,\mathrm{sep}}).$$

With respect to these embeddings we have $t_2^{p^{s^*}}=t_1$.

The first group of the above liftings can be related to the liftings of the second group by the following chain of embeddings

$$\sigma^M O_{M+1}(K_2)\subset W_{M+1}(\sigma^M K_2)\subset W_{M+1}(\sigma^M K'_2)\subset O_{M+1}(K'_2).$$

Similarly, one has the embedding $\sigma^M O_{M+1}(K_{2,\mathrm{sep}})\subset O'_{M+1}(K'_{2,\mathrm{sep}})$. The above embeddings correspond to the relation

$$\tau_1^{qp^M}E(-1,\tau_1^{B^*}t_1^{A^*})^{p^M}\equiv\tau^{p^M}\bmod p^{M+1},$$

which follows from the basic equation given by n.c) of Prop. 3.2.

3.4. A criterion. Let L be a finite Lie algebra over $\mathbb{Z}_p$ of nilpotent class $<p$ and let $M\in\mathbb{N}$ be such that $p^{M+1}L=0$. Consider the group homomorphism $\psi_0:\Gamma\longrightarrow G(L)$. By the nilpotent Artin-Schreier theory there exists an $e\in G(L_K)=L\otimes O_{M+1}(K)$ and an $f\in G(L_{\mathrm{sep}})=L\otimes O_{M+1}(K_{\mathrm{sep}})$ such

that $\sigma f = f \circ e$ and for any $g \in \Gamma$, $\psi_0(g) = (gf) \circ (-f)$. Let $K(f) = K_{\mathrm{sep}}^{\mathrm{Ker}\,\psi_0}$ be the field of definition of f over K. Note that for $j_0 \in J$, the ramification subgroup $\Gamma^{(j_0)} \subset \mathrm{Ker}\,\psi_0$ if and only if $j(f/K) := j(K(f)/K) < j_0$.

Consider $f_1 = \eta_{\mathrm{sep}}(f) \in L'_{\mathrm{sep}} := L \otimes O'_{M+1}(K'_{\mathrm{sep}})$. We use the embeddings from n.3.3

$$\sigma^M O_{M+1}(K_{\mathrm{sep}}) \subset \sigma^M O_{M+1}(K_{2,\mathrm{sep}}) \subset O'_{M+1}(K'_{2,\mathrm{sep}}),$$

and $O'_{M+1}(K'_{\mathrm{sep}}) \subset O'_{M+1}(K'_{2,\mathrm{sep}})$ to define $X \in L'_{2,\mathrm{sep}} := L \otimes O'_{M+1}(K'_{2,\mathrm{sep}})$ such that

$$\sigma^M f = \left(\sigma^{M+N^*} f_1\right) \circ X.$$

Let $\Gamma_2 = \mathrm{Gal}(K_{2,\mathrm{sep}}/K_2)$ and let $\{\Gamma_2^{(j)}\}_{j \in J}$ be the ramification filtration of Γ_2 related to the additional structure on K_2 given by the valuation $v^{(0)}|_{K_2}$ and the subfield of 1-dimensional constants $K_{2,\mathrm{c}} = K_{\mathrm{c}}(t_{20})$.

Let $K'_2(X)$ be the field of definition of X over K'_2. Set

$$j_2(X/K_2) = \max\left\{ j \in J \mid \Gamma_2^{(j)} \text{ acts non-trivially on } K'_2(X) \right\}.$$

Proposition 3.3. $j_2(X/K_2) = \max\{j^*, j(f/K)\}$.

Proof. One has the natural identification $\Gamma = \Gamma_2$, because $K_{2,\mathrm{sep}} = K_{\mathrm{sep}} K_2$ and K_2 is purely inseparable over K. With respect to this identification for any $j \in J_2$, we have $\Gamma^{(j)} = \Gamma_2^{(j)}$, because the extension K_2/K is induced by the extension of 1-dimensional constants $K_{2,\mathrm{c}}/K_{\mathrm{c}}$. This implies $\varphi^{(2)}_{K'_2/K_2} = \varphi^{(2)}_{K_0/K}$, $j_2(K'_2/K_2) = j^*$ and $j_2(X/K_2) \geq j^*$.

If $j(f/K) \in J_1$, then $K(f) \subset K_{\mathrm{c,sep}}$, $K'(f_1) \subset K'_{\mathrm{c,sep}}$ and, therefore, X is defined over $K'_{2,\mathrm{c,sep}}$ and $j_2(X/K_2) = j_2(K'_2/K_2) = j^*$, i.e. in this case the proposition is proved.

Now we can assume that $A = j(f/K) \in J_2$. Let $\Gamma' = \mathrm{Gal}(K'_{\mathrm{sep}}/K')$ and let $\{\Gamma'^{(j)}\}_{j \in J}$ be the ramification filtration corresponding to the valuation $v'^{(0)} = \eta(v^{(0)})$ and $K'_{\mathrm{c}} = K_{\mathrm{c}}(t_{10})$. Then $j'(f_1/K') = A$, where $j'(f_1/K')$ is defined similarly to $j(f/K)$ but with the use of the filtration $\{\Gamma'^{(j)}\}_{j \in J}$.

Because $K'_2 = K'(t_{20})$ is obtained from K' by extension of its field of constants, there is an equality

$$j'(f_1/K') = j'(f_1/K'_2) = A.$$

But the relation $v'^{(0)} = qv^{(0)}$ implies that

$$q^{-1}A = q^{-1} j'(f_1/K'_2) = j_2(f_1/K'_2).$$

This gives

$$j_2(K'_2(f_1)/K_2) = \max\left\{ j(K'_2/K_2), \varphi^{(2)}_{K'_2/K_2}(q^{-1}A) \right\}.$$

Because $j(K_2'/K_2) = j^*$ and $\varphi^{(2)}_{K_2'/K_2} = \varphi^{(2)}_{K_0/K}$, it remains to consider the following two cases:

— let $A = j(f/K) = j(K_2(f)/K_2) \leq j^*$, then $j_2(K_2'(f_1)/K_2) = j(K_2'/K_2) = j^*$ and, therefore, $j(K_2'(X)/K_2) \leq j^*$, i.e. $j(K_2'(X)/K_2) = j^*$;

— let $A \geq j^*$, then $j_2(K_2'(f_1)/K_2) = \varphi^{(2)}_{K_2'/K_2}(q^{-1}A) = j^* + (A - j^*)/q < A = j(K_2'(f)/K_2)$, what gives that $j_2(K_2'(X)/K_2) = j(K_2'(f)/K_2) = A$. The proposition is proved. $\square$

Corollary 3.4. *Suppose that $j_0 > j^*$. Then the following three conditions are equivalent:*

a) $j(f/K) < j_0$;

b) $j_2(X/K_2) < j_0$;

c) $j'(X/K') < qj^0 - (q-1)j^*$.

3.5. First applications of the above criterion. Corollary 3.4 can be applied to study ramification properties of the homomorphism ψ_0. This criterion has been already applied in the case of 1-dimensional local fields to describe the structure of the ramification filtration modulo commutators of order $\geq p$ [1, 2], and will be applied in section 5 to the description of 2-dimensional ramification filtration modulo 3rd commutators. It can be used to prove also the following two propositions.

Proposition 3.5. *Let $M \in \mathbb{Z}_{\geq 0}$ and let $f \in O_{M+1}(K_{\mathrm{sep}})$ be such that*

$$\sigma f - f = w\tau^{-b}t^{-a},$$

where $w \in W(k)^*$, $a \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{Z} \setminus \{0\}$, $(a, b, p) = 1$ and $v_p(b) = s$. Then

$$j(K(f)/K) = \left(p^{M-s}a, p^{M-s}b\right).$$

Proof. First, consider the case $s = v_p(b) = 0$. We are going to reduce the proof to the case $M = 0$, where the statement of our proposition has been already known by Prop. 3.1.

Choose $a^* = m^*/(q-1)$, where $(m^*, p) = 1$, $q = p^{N^*}$ for some $N^* \in \mathbb{N}$, and

$$\frac{q}{2(q-1)}p^M a < a^* < p^M a.$$

One can take, for example, $q = p$, $N^* = 1$ if $p \neq 2$, $q = 4$, $N^* = 2$ if $p = 2$, and $m^* = p^M a(q-1) - 1$.

Take $b^* = 1/(q-1)$, $j^* = (a^*, b^*)$ and consider the field $K_0 = K(N^*, j^*)$ and all related objects introduced in n. 3.4. Consider $f_1 \in O_{M+1}(K'_{\mathrm{sep}})$ such that

$$\sigma f_1 - f_1 = (\sigma^{-N^*}w)\tau_1^{-b}t_1^{-a},$$

then $X = \sigma^M f - \sigma^{M+N^*} f_1 \in O'_{M+1}(K'_{2,\mathrm{sep}})$ and satisfies the equation

$$\sigma X - X = (\sigma^M w)\tau_1^{-bp^M q} t_1^{-ap^M q}\left(E(b,\Theta^*)^{p^M} - 1\right),$$

where $\Theta^* = \tau_1^{b^*(q-1)} t_1^{a^*(q-1)}$. It can be easily seen that for some $h = h(T) \in \mathbb{Z}_p[[T]]$, one has

$$E(b,T)^{p^M} = 1 + bp^M T + T^2 h(T).$$

Therefore, $X = X_1 + X_2$, where

$$\sigma X_1 - X_1 = p^M b(\sigma^M w)\tau_1^{-bp^M q + b^*(q-1)} t_1^{-ap^M q + a^*(q-1)},$$

$$\sigma X_2 - X_2 = A_2 := \tau_1^{-bp^M q + 2b^*(q-1)} t_1^{-ap^M q + 2a^*(q-1)} h(\Theta^*).$$

By the choice of a^* we have the inequality $-ap^M q + 2a^*(q-1) > 0$, which implies $\lim_{n\to\infty} \sigma^n A_2 = 0$, $X_2 \in O'_{M+1}(K') \subset O'_{M+1}(K'_2)$ and $j'(X/K') = j'(X_1/K')$. But

$$j'(X_1/K') = (ap^M q - a^*(q-1), bp^M q - b^*(q-1))$$

by Prop.3.1. By Corollary 3.4 we conclude that $j(f/K) = (p^M a, p^M b)$, and the case $s = 0$ is considered.

Let $v_p(b) = s \geq 1$. Set $b = b'p^s$, $t'^{p^s} = t$, $L = K(t')$, $w' = \sigma^{-s} w$, and take $f \in O'_{M+1}(L_{\mathrm{sep}})$ such that $\sigma f' - f' = w'\tau^{-b'} t'^{-a'}$ and $\sigma^s f' = f$. Let $L_{\mathrm{c}} := k((t'))$ and consider the valuation $v_L^{(0)}$ of L such that $v_L^{(0)}(t') = (1,0)$ and $v_L^{(0)}(\tau) = (0,1)$. In the ramification theory, which corresponds to the valuation $v_L^{(0)}$ and the field of constants L_{c}, we have already known that $j_L(f'/L) = (p^M a, p^M b')$. But if $\alpha \in L_{\mathrm{alg}}$ and $v^{(0)}(\alpha) = (a', b')$, then $v_L^{(0)}(\alpha) = (p^s a', b')$. Because the field of constants of L is the same in the both ramification theories, one has

$$j(f'/L) = (p^{M-s} a, p^M b') = (p^{M-s} a, p^{M-s} b).$$

It remains only to note that $j(f'/L) = j(f/K)$. The proposition is proved. $\square$

Proposition 3.6. *Let $a \in \mathbb{N}$, $b, c \in \mathbb{Z} \setminus \{0\}$, $(a,b,p) = 1$, $s = v_p(b) \leq v_p(c)$ and $\alpha, \beta \in W(k)^*$. Let $f = f(\beta,\alpha,a,b), g = g(\alpha,a,b) \in O_{M+1}(K_{\mathrm{sep}})$ be such that*

$$\sigma g - g = \alpha\tau^{-b} t^{-a}, \quad \sigma f - f = \beta\tau^{-c} g.$$

Then $j(f/K) = (p^{M-s} a, p^{M-s}(b+c))$.

Proof. First, consider the case $v_p(b) = 0$ and $M = 0$. Let $L_1 = K(g)$, $L_2 = L_1(f)$. Let $t_1^p = t$ and $g t_1^a = \sigma^{-1}(\alpha)\tau_1^{-b}$. Then

$$\tau_1^{-bp}\left(1 - \frac{\sigma^{-1}(\alpha)}{\alpha}\tau_1^{b(p-1)} t_1^{a(p-1)}\right) = \tau^{-b},$$

and we can assume that

$$\tau \equiv \tau_1^p \left(1 + \frac{\sigma^{-1}(\alpha)}{b\alpha}\tau_1^{b(p-1)}t_1^{a(p-1)}\right) \operatorname{mod} t_1^{2a(p-1)}k((\tau_1))[[t_1]].$$

Therefore, $\tau^{-c}g = \sigma^{-1}(\alpha)\tau_1^{-cp-b}t_1^{-a} + A$, where $A \in L_1$ and $\lim_{n\to\infty} \sigma^n A = 0$. Therefore,

$$j(f/L_1) = j(f'/L_1) = \left(\frac{a}{p}, \frac{cp+b}{p}\right),$$

where $f'^p - f' = \tau_1^{-cp-b}t_1^{-a}$. This implies that

$$\begin{aligned} j(f/K) &= \max\left\{j(L_1/K), \varphi_{L_1/K}(j(f/L_1))\right\} \\ &= \max\left\{(a,b), \varphi_{L_1/K}\left(\frac{a}{p}, \frac{b}{p} + c\right)\right\} = (a, b+c). \end{aligned}$$

Consider now the case $s = 0$ and $M \in \mathbb{N}$. Set $a^* = p^M a$ and choose $b^* \in \mathbb{Z}$ such that $(b^*, p) = 1$ and $bp^M < b^* < bp^M + cp^M$. Consider the field $K_0 = K(1, (a^*, b^*))$ from n. 3.2 together with all related objects. Introduce $f_1, g_1 \in O'_{M+1}(K'_{\mathrm{sep}})$ such that

$$\sigma g_1 - g_1 = \sigma^{-1}(\alpha)\tau_1^{-b}t_1^{-a}, \quad \sigma f_1 - f_1 = \sigma^{-1}(\beta)\tau_1^{-c}g_1.$$

Set

$$Y = \sigma^M g - \sigma^{M+1}g_1, \quad X = \sigma^M f - \sigma^{M+1}f_1,$$

then

$$\sigma Y - Y = \sigma^M(\alpha)\tau_1^{-bp^{M+1}}t_1^{-ap^{M+1}}\left(E(b,\Theta^*)^{p^M} - 1\right),$$

$$\sigma X - X = \sigma^M(\beta)\tau_1^{-cp^{M+1}}E(c,\Theta^*)^{p^M}\sigma^M g - \sigma^M(\beta)\tau_1^{-cp^{M+1}}\sigma^{M+1}g_1.$$

In the notation from the proof of Prop. 3.5 we have

$$\begin{aligned} \sigma Y - Y = \sigma^M(\alpha)p^M b\tau_1^{-bp^{M+1}+b^*(p-1)}t_1^{-ap^M} &+ \\ \sigma^M(\alpha)\tau_1^{-bp^{M+1}}t_1^{-ap^{M+1}}&\Theta^{*2}h(\Theta^*). \end{aligned}$$

Therefore, $Y = p^M Y_1 + t_1^{ap^M(p-2)}A_1$, where $A_1 \in k((\tau_1))[[t_1]]$ and Y_1 is the element from $O'_{M+1}(K'_{\mathrm{sep}}) \operatorname{mod} p = K'_{\mathrm{sep}}$ such that

$$\sigma Y_1 - Y_1 = \sigma^M(\alpha)b\tau_1^{-bp^{M+1}+b^*(p-1)}t_1^{-ap^M}.$$

Therefore,

$$\sigma^M g = \sigma^{M+1}g_1 + p^M Y_1 + t_1^{ap^M(p-1)}A_1,$$

and for some $A_2 \in k((\tau_1))[[t_1]]$, one has

$$\begin{aligned} \sigma X - X = \sigma^M(\beta)\tau_1^{-cp^{M+1}}\left(E(c,\Theta^*)^{p^M} - 1\right)\sigma^{M+1}g_1 &+ \\ \sigma^M(\beta)p^M\tau_1^{-cp^{M+1}}Y_1 + t_1^{ap^M(p-2)}&A_2. \end{aligned}$$

One can check up that

$$\tau_1^{-cp^{M+1}}\left(E(c,\Theta^*)^{p^M}-1\right)\in t_1^{ap^M(p-1)}k((\tau_1))[[t_1]]$$

and

$$\lim_{s\to\infty}\sigma^s\left(t_1^{ap^M(p-1)}\sigma^{M+1}g_1\right)=0.$$

So, if $X_1\in K'_{\mathrm{sep}}$ is such that

$$\sigma X_1-X_1=\sigma^M(\beta)\tau_1^{-cp^{M+1}}Y_1,$$

then $j_2(X/K'_2)=j_2(X_1/K'_2)$. Now we can apply the case $M=0$ of our proposition to obtain that

$$j_2(X_1/K'_2)=\frac{1}{p}(ap^M,cp^{M+1}+bp^{M+1}-b^*(p-1)).$$

This gives immediately that $j_2(X_1/K_2)=(p^Ma,p^M(b+c))$. The case $s=0$ is considered.

The case of arbitrary $s\in\mathbb{Z}_{\geq 0}$ can be reduced now to the case $s=0$ in the same way as in Prop. 3.5. The proposition is proved. $\square$

4. Filtration of $\mathcal{L}_M^f(\omega)\bmod C_3(\mathcal{L}_M^f(\omega))$

In this section we fix $\omega=\{J_i(\omega)\}_{0\leq i\leq I(\omega)}\in\Omega$ and $M\in\mathbb{N}$. We set

$$J(\omega)=\max\{J_i(\omega)\mid 0\leq i\leq I(\omega)\}.$$

Clearly, the set

$$S_1(\omega)=\{p^na\mid (a,b)\in\mathcal{B}^0(\omega),n\in\mathbb{Z}\}$$

consists of non-negative rational numbers and has only one limit point 0. Therefore, for any $a\in\mathbb{Q}_{>0}$, we can define the positive rational number

$$\delta_1(\omega,a)=\min\{a-s\mid s\in S_1(\omega),s<a\}.$$

We also agree to use the notation

$$L^f:=\mathcal{L}_M(\omega)^f\bmod C_3\left(\mathcal{L}_M(\omega)^f\right),\quad L^{pf}:=\mathcal{L}_M(\omega)^{pf}\bmod C_3\left(\mathcal{L}_M(\omega)^{pf}\right),$$

where $\mathcal{L}_M(\omega)^f$ and $\mathcal{L}_M(\omega)^{pf}$ are Lie $W_M(\mathbb{F}_p)$-algebras from n. 1.5. We also set

$$\mathcal{L}_M(\omega)_k^f=\mathcal{L}_M(\omega)^f\otimes W_M(k),\qquad \mathcal{L}_M(\omega)_k^{pf}=\mathcal{L}_M(\omega)^{pf}\otimes W_M(k),$$

$$L_k^f=L^f\otimes W(k)\qquad\text{and}\qquad L_k^{pf}=L^{pf}\otimes W(k).$$

4.1. Special system of generators. Let

$$\mathcal{B}_2 = \{(a,b) \in J_2 \mid b \in \mathbb{Z}, (b,p) = 1, a \in \mathbb{Z}[1/p]\}\,.$$

This is a set of rational pairs which are either of the form $(0,b)$, where $b \in \mathbb{N}, (b,p) = 1$, or — (a,b), where $b \in \mathbb{Z}, (b,p) = 1$, $a > 0$ and $a \in \mathbb{Z}[1/p]$. Set $\mathcal{B}_1 = \mathcal{A}_1$, i.e. $\mathcal{B}_1$ is the family of pairs $(a,0)$ such that $a \in \mathbb{N}$ and $(a,p) = 1$. We also set $\mathcal{B} = \mathcal{B}_2 \cup \mathcal{B}_1$ and $\mathcal{B}^0 = \mathcal{B} \cup \{(0,0)\}$.

For $j = (a,b) \in \mathbb{Z}^2$, let

$$s(j) = s(a) = \max\{-v_p(a), 0\}.$$

Then the correspondence $(a,b) \mapsto (ap^s, bp^s)$, where $s = s(a,b)$, induces the bijection $f_2 : \mathcal{B}_2 \longrightarrow \mathcal{A}_2$ and the identical map $f_1 : \mathcal{B}_1 \longrightarrow \mathcal{A}_1$. One can set by definition $f_0 : (0,0) \mapsto (0,0)$ and apply these maps f_0, f_1, f_2 to obtain the bijections $f : \mathcal{B} \longrightarrow \mathcal{A}$ and $f^0 : \mathcal{B}^0 \longrightarrow \mathcal{A}^0$. Set $\mathcal{B}_2(\omega) = f_2^{-1}\mathcal{A}_2(\omega)$, $\mathcal{B}_1(\omega) = f_1^{-1}\mathcal{A}_1(\omega)$, $\mathcal{B}(\omega) = f^{-1}\mathcal{A}(\omega)$, and $\mathcal{B}^0(\omega) = f^{0^{-1}}(\mathcal{A}^0(\omega)$. If $(a,b) \in \mathcal{B}(\omega)$ and $f(a,b) = (i,j) \in \mathcal{A}(\omega)$, set in the notation of n.1.4

$$D_{(a,b),0} = \sum_{1 \le r \le N_0} \alpha_r D^{(r)}_{(i,j)} \in \mathcal{L}_M(\omega)^f_k.$$

We set also $D_{(0,0),0} = \alpha_0 D_{(0,0)}$ and for any $(a,b) \in \mathcal{B}^0(\omega)$ and $n \in \mathbb{Z}$, $D_{(a,b),n} = \sigma^n D_{(a,b),0}$. Clearly, $D_{(a,b),n+N_0} = D_{(a,b),n}$, so we can assume if necessary that $n \in \mathbb{Z} \operatorname{mod} N_0$.

It is easy to see that the family

$$\{D_{j,n} \mid j \in \mathcal{B}(\omega), n \in \mathbb{Z} \operatorname{mod} N_0\} \cup \left\{D_{(0,0)}\right\}$$

is the set of free generators of the $W_M(k)$-Lie algebra $\widetilde{\mathcal{L}}_M(\omega)^f \otimes W_M(k)$ (or the set of free generators modulo $\deg p$ of the Lie algebra $\mathcal{L}_M(\omega)^f \otimes W_M(k)$).

We shall agree to use the notation $D_{j,n}$ for all $j \in \mathcal{B}^0$, by setting $D_{j,n} = 0$ if $j \notin \mathcal{B}^0(\omega)$.

4.2. Elements $\mathcal{F}_\gamma(\omega)$, $\gamma \in J$. Define the elements $\mathcal{F}_\gamma(\omega) = \mathcal{F}_\gamma \in L_k^{pf} := L_k^{pf} \otimes W(k)$ for all $\gamma \in J$ as follows.

Let $\gamma = (\gamma_1, \mathrm{c}) \in \mathrm{J}_1$.

For $\gamma_1 \notin \mathbb{Z}$, set

$$\mathcal{F}_\gamma = -\sum_{n_1,n_2,j_1,j_2} p^{n_1} a_1 \left[D_{j_1 n_1}, D_{j_2 n_2}\right],$$

where the sum is taken under the restrictions $n_1 \in \mathbb{Z}_{\ge 0}$, $n_2 \in \mathbb{Z}$, $j_1 = (a_1, 0), j_2 = (a_2, 0) \in \mathcal{B}_1(\omega)$ and $p^{n_1} a_1 + p^{n_2} a_2 = \gamma_1$ (notice that $n_2 = v_p(\gamma_1) < 0$).

If $\gamma_1 \in \mathbb{Z}$, then $\gamma_1 = ap^m$, where $m \in \mathbb{Z}_{\geq 0}$, $a \in \mathbb{N}$ and $(a,p)=1$. In this case we set

$$\mathcal{F}_\gamma = ap^m D_{(a,0),m} - \sum_{n_1,n_2,j_1,j_2} \eta(n_1,n_2)p^{n_1}a_1 \left[D_{j_1n_1}, D_{j_2n_2}\right],$$

where the sum is taken under the same restrictions as in the case of non-integral γ_1 and

$$\eta(n_1,n_2) = \begin{cases} 1, & \text{if } n_1 > n_2, \\ 1/2, & \text{if } n_1 = n_2, \\ 0, & \text{if } n_1 < n_2. \end{cases}$$

Let $\gamma = (\gamma_1,\gamma_2) \in J_2$.

If $\gamma_2 \notin \mathbb{Z}$, set

$$\mathcal{F}_\gamma = -\sum_{n_1,n_2,j_1,j_2} \eta(n_1-s_1,n_2-s_2)p^{n_1}b_1 \left[D_{j_1n_1}, D_{j_2n_2}\right],$$

where the sum is taken under the restrictions $n_1 \in \mathbb{Z}_{\geq 0}$, $n_2 \in \mathbb{Z}$, $j_1 = (a_1,b_1), j_2 = (a_2,b_2) \in \mathcal{B}(\omega)$, $p^{n_1}a_1 + p^{n_2}a_2 = \gamma_1$, $p^{n_1}b_1 + p^{n_2}b_2 = \gamma_2$ (notice that $\mathcal{F}_\gamma \neq 0$ implies that $n_2 = v_p(\gamma_2) < 0$), and $s_1 = s(a_1)$, $s_2 = s(a_2)$.

If $\gamma_2 \in \mathbb{Z}$, then $\gamma_2 = bp^m$, where $m \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{Z}$ and $(b,p)=1$. In this case we set

$$\mathcal{F}_\gamma = bp^m D_{(a,b),m} - \sum_{n_1,n_2,j_1,j_2} \eta(n_1-s_1,n_2-s_2)p^{n_1}b_1 \left[D_{j_1n_1}, D_{j_2n_2}\right],$$

where $a = \gamma_1 p^{-m}$, $s_1 = s(j_1), s_2 = s(j_2)$, the sum is taken under the same restrictions as in the case of non-integral γ_2 (notice that everywhere $D_{j,n} = 0$ if $j \notin \mathcal{B}(\omega)$).

One can easily verify that the above definition gives elements $\mathcal{F}_\gamma$ from L_k^{pf} but, in fact, one has the following more strong property.

Proposition 4.1. *For any $\gamma \in J$, $\mathcal{F}_\gamma \in L_k^f$.*

Proof. The only non-obvious case appears when $\gamma = (\gamma_1,\gamma_2) \in J_2$, $\gamma_1 > 0$. We must prove the finiteness of the set of collections of the form (j_1,n_1,j_2,n_2), where $j_1 = (a_1,b_1), j_2 = (a_2,b_2) \in \mathcal{B}(\omega)$, $n_1,n_2 \in \mathbb{Z}$, $0 \leq n_1 \leq M$, $n_1 - s_1 \geq n_2 - s_2$, $p^{n_1}a_1 + p^{n_2}a_2 = \gamma_1$ and $p^{n_1}b_1 + p^{n_2}b_2 = \gamma_2$.

Let $a_1^0 = a_1p^{s_1}, a_2^0 = a_2p^{s_2}$, then a_1^0 and a_2^0 are integers from the interval $[0, I(\omega)]$ and we can assume that they are fixed.

Assume that $a_1^0, a_2^0 \neq 0$. Then the equality $a_1p^{n_1} + a_2p^{n_2} = \gamma_1$ implies that

$$\delta_1(\omega,\gamma_1) \leq a_1^0p^{n_1-s_1}, a_2^0p^{n_2-s_2} < \gamma_1,$$

therefore, there are $m_1(\omega,\gamma_1), m_2(\omega,\gamma_1) \in \mathbb{Z}$ such that

$$m_1(\omega,\gamma_1) \leq n_1 - s_1, n_2 - s_2 \leq m_2(\omega,\gamma_1).$$

So, we can assume that the values $n_1 - s_1$ and $n_2 - s_2$ are fixed. The equality $b_1 p^{n_1} + b_2 p^{n_2} = \gamma_2$ implies the double inequality

$$\gamma_2 - J(\omega)p^{m_2(\omega,\gamma_1)} \leq b_1 p^{n_1} \leq J(\omega)p^{m_2(\omega,\gamma_1)}.$$

Together with the obvious inequality $n_1 \geq n_1 - s_1$ it implies the finiteness of the set of all different collections (b_1, n_1). All other components of the collection (j_1, n_1, j_2, n_2) can be recovered uniquely from (b_1, n_1) and the values $a_1^0, a_2^0, n_1 - s_1, n_2 - s_2$, which were fixed earlier. Therefore, our proposition is proved in the case $a_1^0, a_2^0 \neq 0$.

Suppose now that $a_1^0 = 0$. Then $s_1 = 0$, $b_1 \geq 1$ and the relation $p^{n_2 - s_2} a_2^0 = \gamma_1$ determines uniquely the value of $n_2 - s_2$. The inequalities

$$J(\omega)p^M \geq b_1 p^{n_1} \geq 1$$

imply the finiteness of the set of different collections (b_1, n_1). As earlier, this gives the finiteness of the set of all collections (j_1, n_1, j_2, n_2) such that $a_1^0 = 0$.

Suppose, finally, that $a_2^0 = 0$. Then $s_2 = 0$, $b_2 \geq 1$ and the value of $n_1 - s_1$ is determined uniquely. The finiteness of the set of all collections (b_2, n_2) follows from the inequalities

$$1 \leq b_2 p^{n_2} \leq J(\omega)p^{n_1 - s_1}, \quad n_2 \geq \min\{n_1, v_p(\gamma_2)\} \geq \min\{0, v_p(\gamma_2)\}.$$

This gives the finiteness of the set of all collections (j_1, n_1, j_2, n_2) such that $a_2^0 = 0$. The proposition is completely proved. $\square$

4.3. Ideals $L_k^f(j)$, $j \in J$. For any $j \in J$, define the ideal $L_k^f(j)$ of L_k^f as its minimal σ-invariant ideal containing the elements $\mathcal{F}_\gamma$ for all $\gamma \geq j$, $\gamma \in J$.

Clearly, $\left\{L_k^f(j)\right\}_{j \in J}$ is a decreasing filtration of ideals of L_k^f.

For $a \in \mathbb{Q}_{\geq 0}$, set

$$L_k^f(a+) = \bigcup \left\{ L_k^f(j) \mid j = (a', b') \in J_2, a' > a \right\}.$$

Notice that for a given a and all sufficiently large b, the ideals $L_k^f(j)$, where $j = (a, b)$, coincide.

Proposition 4.2. *Let $j_1 = (a_1, b_1) \in J_2$, $m \in \mathbb{Z}_{\geq 0}$ and $p^m a_1 = a$. Then for any $j_2 = (a_2, b_2) \in J_2$, where $a_2 > 0$, and any $n_1, n_2 \in \mathbb{Z}$, it holds*

$$p^m \left[D_{j_1 n_1}, D_{j_2 n_2}\right] \in L_k^f(a+).$$

Proof. We can set $n_1 = m$ because the statement of our proposition is invariant under action of σ. By induction we can assume, that our proposition holds for all $j' = (a', b') \in J_2$ and $m' \in \mathbb{Z}_{\geq 0}$, such that $p^{m'} a' = a$ and $p^{m'} b' > p^m b_1$ (notice that if $p^m b_1 > p^M J(\omega)$, then $D_{j_1 m} = 0$). Because

$D_{j_2n_2}$ depends only on the residue $n_2 \bmod N_0$ we can assume also that n_2 is a "sufficiently big" negative integer such that

$$p^{n_2}I(\omega) < \delta_1(\omega, a),\ b_2 + p^{-n_2} > J(\omega),\ m - s_1 \geq n_2 - s_2$$

(where, as usually, $s_1 = s(a_1)$ and $s_2 = s(a_2)$).

Let $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_1 = a + p^{n_2}a_2$ and $\gamma_2 = p^{n_1}b_1 + p^{n_2}b_2$. Consider the expression for $\mathcal{F}_\gamma \in L_k^f(\gamma)$. This expression is a linear combination of commutators of second order of the form

$$p^{m_1}d_1 \left[D_{(c_1,d_1),m_1}, D_{(c_2,d_2),m_2}\right], \tag{11}$$

where $m_1 \in \mathbb{Z}_{\geq 0}$, $m_2 \in \mathbb{Z}$, $(c_1, d_1), (c_2, d_2) \in \mathcal{B}(\omega)$, $m_1 - s_1' \geq m_2 - s_2'$ with $s_1' = s(c_1)$, $s_2' = s(c_2)$, and $c_1p^{m_1} + c_2p^{m_2} = \gamma_1$, $d_1p^{m_1} + d_2p^{m_2} = \gamma_2$.

First, notice that $m_2 = v_p(\gamma_2) = n_2$.

If $c_1p^{m_1} > a$, then the term (11) belongs to $[L_k^f(a+), L_k^f] \subset L_k^f(a+)$. Otherwise, the inequality $c_2p^{m_2} \leq I(\omega)p^{n_2} < \delta_1(\omega, a)$ implies that $c_1p^{m_1} = a$. If $d_1p^{m_1} > p^m b_1$, then the term (11) belongs to $L_k^f(a+)$ by the inductive assumption. If $d_1p^{m_1} = p^m b_1$, then the term (11) coincides with the term from our proposition multiplied by $b_1 \in \mathbb{Z}_p^*$. If $d_1p^{m_1} < p^m b_1$, then the equality $d_1p^{m_1} + d_2p^{m_2} = p^{n_1}b_1 + p^{n_2}b_2$ implies that

$$d_2 = b_2 + p^{-n_2}(p^m b_1 - p^{m_1}d_1) \geq b_2 + p^{-n_2} > J(\omega).$$

This gives $p^{s_2'}d_2 > J(\omega)$, i.e. $D_{(c_2,d_2),m_2} = 0$ and the term (11) is equal to 0.

It remains only to note that $\mathcal{F}_\gamma \in L_k^f(a+)$. The proposition is proved. □

4.4. Elements $\widetilde{D}_{j,0}(\omega)$, $j \in \mathcal{B}$, and their properties. For any $j = (a, b) \in J_2$, define the elements $\widetilde{D}_{j,0}(\omega) = \widetilde{D}_{j,0}$

$$= D_{j,0} - \sum_{m_1,m_2,j_1,j_2} \eta(m_2, m_1)\eta(m_1 - s_1, m_2 - s_2)\left[D_{j_1,m_1}, D_{j_2,m_2}\right],$$

where the sum is taken for all $m_1, m_2 \in \mathbb{Z}$ and $j_1 = (a_1, b_1), j_2 = (a_2, b_2) \in \mathcal{B}(\omega)$ such that $b_1p^{m_1} + b_2p^{m_2} = b$ and $a_1p^{m_1} + a_2p^{m_2} = a$. One can easily verify that the above expression gives the element from L_k^{pf}.

For any $n \in \mathbb{Z}$, set $\widetilde{D}_{j,n} = \sigma^n D_{j,0}$. Clearly, the family

$$\left\{\widetilde{D}_{j,n} \mid j \in \mathcal{B}(\omega), n \in \mathbb{Z} \bmod N_0\right\} \cup \{D_{(0,0)}\}$$

generates the algebra L_k^{pf}. Notice that if $j \in J_2 \setminus \mathcal{B}(\omega)$, then $\widetilde{D}_{j,n} \in C_2(L_k^{pf})$.

Proposition 4.3. *For any $j = (a, b) \in J$, $\widetilde{D}_{j,0} \in L_k^f$.*

Proof. We must prove the finiteness of the set of all collections of the form (j_1, m_1, j_2, m_2), where $j_1 = (a_1, b_1), j_2 = (a_2, b_2) \in \mathcal{B}(\omega)$ and $m_1, m_2 \in \mathbb{Z}$

are such that $m_2 \geq m_1$, $m_1 - s_1 \geq m_2 - s_2$, $a_1 p^{m_1} + a_2 p^{m_2} = a$ and $b_1 p^{m_1} + b_2 p^{m_2} = b$.

Let $a_1 = a_1^0 p^{-s_1}$, $a_2 = a_2^0 p^{-s_2}$, $b_1 = b_1^0 p^{-s_1}$ and $b_2 = b_2^0 p^{-s_2}$. Then a_1^0 and a_2^0 are integers from the interval $[0, I(\omega)]$ and we can assume that they are fixed.

Suppose $a_1^0, a_2^0 \neq 0$. Then the relation $a_1 p^{m_1} + a_2 p^{m_2} = a$ gives that

$$a > p^{m_1 - s_1} a_1^0 \geq \delta_1(\omega, a).$$

This implies the existence of $m_1(\omega, a), m_2(\omega, a) \in \mathbb{Z}$ such that

$$m_1(\omega, a) \leq m_1 - s_1 \leq m_2(\omega, a).$$

Therefore, we can fix the value of $m_1 - s_1$. We have also

$$p^{m_2(\omega,a)} J(\omega) \geq p^{m_1} b_1 = p^{m_1 - s_1} b_1^0 = b - p^{m_2 - s_2} b_2^0 \geq b - p^{m_1 - s_1} J(\omega).$$

Because $m_1 \geq m_1 - s_1 \geq m_1(\omega, a)$, this implies the finiteness of the set of all different collections (b_1, m_1). As in the proof of Prop. 4.1 any such collection determines uniquely the collection (j_1, m_1, j_2, m_2). This proves our proposition in the case $a_1^0, a_2^0 \neq 0$.

Let $a_1^0 = 0$, then $s_1 = 0$, $1 \leq b_1 \leq J(\omega)$ and $m_2 - s_2$ is uniquely determined. If $m_1 = m_2$ then $m_1 \leq v_p(b)$. If $m_1 < m_2$ then $m_2 = v_p(b)$. In the both cases $m_1 \leq v_p(b)$ and

$$p^{m_1} b_1 \leq J(\omega) p^{v_p(b)}.$$

Besides, we have $m_1 \geq m_2 - s_2$ and $b_1 p^{m_1} \geq p^{m_2 - s_2}$. This implies the finiteness of all collections of the form (b_1, m_1). This proves our proposition under the assumption $a_1^0 = 0$.

If $a_2^0 = 0$, then $s_2 = 0$, $1 \leq b_2 \leq J(\omega)$ and the value of $m_1 - s_1$ is determined uniquely. Here we have the inequalities

$$m_1 - s_1 \leq m_1 \leq v_p(b).$$

Apart from the trivial boundary $p^{m_1} b_1 < b$, we have also a lower boundary

$$p^{m_1} b_1 = b - b_2^0 p^{m_2 - s_2} \geq b - J(\omega) p^{m_1} \geq b - J(\omega) p^{v_p(b)}.$$

This gives the finiteness of the set of all collections (b_1, m_1) and we can finish the proof as earlier. The proposition is proved. $\square$

For $\gamma \in J_2$, define the elements $\widetilde{\mathcal{F}}_\gamma(\omega) = \widetilde{\mathcal{F}}_\gamma \in L_k^{pf}$ as follows.

Let $\gamma = (\gamma_1, \gamma_2)$.

If either $\gamma_2 = 0$ or $\gamma_2 \notin \mathbb{Z}$, set $\widetilde{\mathcal{F}}_\gamma = \mathcal{F}_\gamma$.

If $\gamma_2 \in \mathbb{Z}$, then $\gamma_2 = bp^m$ for $m \in \mathbb{Z}_{\geq 0}$ and $(b, p) = 1$. In this case we set

$$\widetilde{\mathcal{F}}_\gamma = p^m b \widetilde{D}_{(a,b),m} - \sum_{m_1, m_2, j_1, j_2} \eta(m_1, m_2) p^{m_1} b_1 \left[\widetilde{D}_{j_1, m_1}, \widetilde{D}_{j_2, m_2} \right],$$

where $a = p^{-m}\gamma_1$ and the sum is taken for all $m_1, m_2 \in \mathbb{Z}$ and $j_1 = (a_1, b_1), j_2 = (a_2, b_2) \in \mathcal{B}(\omega)$ such that $a_1 p^{m_1} + a_2 p^{m_2} = \gamma_1$ and $b_1 p^{m_1} + b_2 p^{m_2} = \gamma_2$ (one can verify that this expression gives an element from L_k^{pf}).

Proposition 4.4. *For any $\gamma \in J$, $\widetilde{\mathcal{F}}_\gamma \in L_k^f$.*

This proposition can be proved in the same way as Prop. 4.1 and Prop. 4.3.

Proposition 4.5. *For any $j \in J_2$, $L_k^f(j)$ is the minimal σ-invariant ideal of L_k^f such that for any $\gamma \geq j$, $\widetilde{\mathcal{F}}_\gamma \in L_k^f(j)$.*

Proof. It is sufficient to prove that for any $\gamma = (\gamma_1, \gamma_2) \in J$,

$$\widetilde{\mathcal{F}}_\gamma \equiv \mathcal{F}_\gamma \bmod \left[L_k^f(\gamma_1+), L_k^f\right]. \tag{12}$$

We can assume that $\gamma = p^m(a, b)$, where $m \in \mathbb{Z}_{\geq 0}$ and $(a, b) \in \mathcal{B}_2(\omega)$. Then

$$p^m b D_{(a,b),m} - p^m b \widetilde{D}_{(a,b),m}$$
$$= p^m b \sum_{m_1, m_2, j_1, j_2} \eta(m_2, m_1)\eta(m_1 - s_1, m_2 - s_2) \left[\widetilde{D}_{j_1, m_1}, \widetilde{D}_{j_2, m_2}\right],$$

where $m_1, m_2 \in \mathbb{Z}$, $j_1 = (a_1, b_1), j_2 = (a_2, b_2) \in \mathcal{B}(\omega)$, $p^{m_1}a_1 + p^{m_2}a_2 = p^m a$ and $p^{m_1}b_1 + p^{m_2}b_2 = p^m b$.

Notice that if $m_1 < 0$, then $m_1 = m_2$ and, therefore, either $a_1 p^{m_1}$ or $a_2 p^{m_2}$ is bigger than ap^m. Therefore, if we assume in addition that $m_1, m_2 \geq 0$, then the right-hand side of the above equality will not be changed modulo $\left[L_k^f(\gamma_1+), L_k^f\right]$ and can be rewritten in the form

$$\sum_{m_1, m_2, j_1, j_2} p^{m_1} b_1 \left[\widetilde{D}_{j_1, m_1}, \widetilde{D}_{j_2, m_2}\right] \eta(m_2, m_1)\eta(m_1 - s_1, m_2 - s_2) -$$
$$- \sum_{m_1, m_2, j_1, j_2} p^{m_1} b_1 \left[\widetilde{D}_{j_1, m_1}, \widetilde{D}_{j_2, m_2}\right] \eta(m_1, m_2)\eta(m_2 - s_2, m_1 - s_1)$$

(we substitute $p^{m_1}b_1 + p^{m_2}b_2$ instead of $p^m b$ and interchange indexes 1 and 2 in the second group of terms). The relation (12) can be obtained then by the use of the relations $\eta(m_2, m_1) = 1 - \eta(m_1, m_2)$ and $\eta(m_2 - s_2, m_1 - s_1) = 1 - \eta(m_1 - s_1, m_2 - s_2)$. □

Proposition 4.6. *Let $m \in \mathbb{Z}_{\geq 0}$ and $\tilde{\jmath} = (\tilde{a}, \tilde{b}) \in \mathcal{B}_2(\omega)$. If $a > 0$ is such that $p^m \tilde{a} > 2(a - \delta_1(\omega, a))$, then for any $n \in \mathbb{Z}$, one has $p^m \widetilde{D}_{\tilde{\jmath}, n} \in L_k^f(a+)$.*

Proof. The statement of the proposition is invariant under action of σ, therefore, we can assume that $m = n$.

Notice that $p^m \tilde{b} \widetilde{D}_{\tilde{\jmath}, m}$ is the only a first order term in the expression of $\widetilde{\mathcal{F}}_{p^m \tilde{\jmath}} \in L_k(p^m \tilde{\jmath}) \subset L_k(a+)$. Therefore, it is sufficient to verify that any commutator of second order from that expression belongs to $L_k(a+)$. Any such commutator is of the form

$$p^{m_1}b_1[D_{(a_1,b_1),m_1}, D_{(a_2,b_2),m_2}],$$

where $m_2 \leq m_1$, $m_1 \geq 0$, $(a_1,b_1),(a_2,b_2) \in \mathcal{B}(\omega)$, $a_1p^{m_1} + a_2p^{m_2} = p^m\tilde{a}$. Because $p^m\tilde{a} > 2(a - \delta_1(\omega,a))$, we have either $a_1p^{m_1} > a - \delta_1(\omega,a)$, or $a_2p^{m_2} > a - \delta_1(\omega,a)$.

In the first case $a_1p^{m_1} \geq a$. If $a_1p^{m_1} > a$, then our term belongs to $[L_k^f(a+), L_k^f] \subset L_k^f(a+)$. If $p^{m_1}a_1 = a$, then $a_2 > 0$ and our term belongs to $L_k^f(a+)$ by Prop. 4.2.

In the second case the inequality $m_2 \leq m_1$ implies $a_2p^{m_1} > a - \delta_1(\omega,a)$ and we finish the proof in the same way. The proposition is proved. $\square$

5. Ramification filtration modulo 3rd commutators

As usually, K is a complete discrete valuation field of dimension 2 given in the notation of n. 1.1. It has an additional structure given by a double valuation $v^{(0)}$ and a subfield of 1-dimensional constants satisfying the agreements from the beginning of n. 3. Consider the corresponding ramification filtration $\left\{\Gamma^{(j)}\right\}_{j\in J}$ of $\Gamma = \mathrm{Gal}(K_{\mathrm{sep}}/K)$. Fix $\omega \in \Omega$, $M \in \mathbb{Z}_{\geq 0}$, set $L = \mathcal{L}_{M+1}(\omega) \bmod C_3(\mathcal{L}_{M+1}(\omega))$, and consider the group epimorphism

$$\psi = \psi_{M+1}(\omega) \bmod C_3(\mathcal{L}_{M+1}(\omega)) : \Gamma \longrightarrow G(L),$$

cf. n. 1.5. This gives the decreasing filtration of ideals $\left\{L^{(j)}\right\}_{j\in J}$ of L such that $\psi(\Gamma^{(j)}) = L^{(j)}$. For any $j \in J$, denote by $L(j)$ the ideal of L generated by elements of the ideal $L^f(j)$ from n. 4. The following theorem gives an explicit description of the image of the ramification filtration of Γ in its maximal p-quotient of nilpotent class 2.

Theorem 5.1. *For any $j \in J$, $L(j) = L^{(j)}$.*

The rest of section deals with the proof of this theorem.

5.1. The cases $j = (v, \mathrm{c}) \in J_1$ and $j = (0, v) \in J_2$.

Set $L_k = L \otimes W_{M+1}(k)$, $L_K = L \otimes O_{M+1}(K)$ and $L_{\mathrm{sep}} = L \otimes O_{M+1}(K_{\mathrm{sep}})$. Let

$$e = e_{M+1}(\omega) \bmod C_3(\mathcal{L}_{M+1}(\omega)) = \sum_{\substack{(a,b)\in\mathcal{B}^0(\omega)\\ s=s(a)}} \tau^{-bp^s}t^{-ap^s}D_{(a,b),s} \in L_K,$$

$$f = f_{M+1}(\omega) \bmod C_3(\mathcal{L}_{M+1}(\omega)) \in L_{\mathrm{sep}}.$$

Then $\sigma f = f \circ e$ and for any $g \in \Gamma$, one has $\psi(g) = (gf) \circ (-f)$.

Let I_{c} be the minimal ideal of L such that $I_{\mathrm{c}} \otimes W_{M+1}(k)$ contains all $D_{(a,b),0}$ with indexes $(a,b) \in \mathcal{B}^0(\omega)$ with $b \neq 0$.

Consider the natural projection $\mathrm{pr}_{\mathrm{c}} : L \longrightarrow L/I_{\mathrm{c}} := L_{\mathrm{c}}$.

Denote by the same symbol extensions of scalars of that projection. Then we obtain the elements

$$e_{\mathrm{c}} = \mathrm{pr}_{\mathrm{c}}(e) \in G(L_{\mathrm{c}} \otimes O_{M+1}(K_{\mathrm{c}})), \quad f_{\mathrm{c}} = \mathrm{pr}_{\mathrm{c}}(f) \in G(L_{\mathrm{c}} \otimes O_{M+1}(K_{\mathrm{sep}}))$$

Here $O_{M+1}(K_{\mathrm{c}})$ and $O_{M+1}(K_{\mathrm{c,sep}})$ are liftings modulo p^{M+1}, constructed via the p-basis of K determined by uniformising element $t_0 \in K_{\mathrm{c}}$. Notice that

$$e_{\mathrm{c}} = \sum_a t^{-a} \bar{D}_{a,0},$$

where a runs over the set $\{a \in \mathbb{N} \mid (a,p) = 1, a \leq I(\omega)\} \cup \{0\}$. There is also an equation $\sigma f_{\mathrm{c}} = f_{\mathrm{c}} \circ e_{\mathrm{c}}$ and a group epimorphism

$$\psi_{\mathrm{c}} : \Gamma_{\mathrm{c}} = \mathrm{Gal}(K_{\mathrm{c,sep}}/K_{\mathrm{c}}) \longrightarrow G(L_{\mathrm{c}})$$

such that for any $g \in \Gamma_{\mathrm{c}}$, $\psi_{\mathrm{c}}(g) = (gf_{\mathrm{c}}) \circ (-f_{\mathrm{c}})$.

Notice in addition, the composition

$$\Gamma \xrightarrow{\psi} \Gamma_{\mathrm{c}} \xrightarrow{\mathrm{pr}_{\mathrm{c}}} G(L_{\mathrm{c}})$$

coincides with the composition

$$\Gamma \longrightarrow \Gamma_{\mathrm{c}} \xrightarrow{\psi_{\mathrm{c}}} G(L_{\mathrm{c}})$$

(where the first arrow is the natural projection from n. 2.1).

One can easily see that for any $v \geq 0$,

$$\psi(\Gamma^{(v,\mathrm{c})}) = \mathrm{pr}_{\mathrm{c}}^{-1}\left(\psi_{\mathrm{c}}(\Gamma_{\mathrm{c}}^{(v)})\right),$$

where $\Gamma_{\mathrm{c}}^{(v)}$ is the ramification subgroup of the Galois group Γ_{c} of the 1-dimensional field K_{c}. The case $j = (v, \mathrm{c})$ of our theorem follows now from the description of ramification filtration for 1-dimensional local field from [1, 2].

The case $j = (0, v), v \geq 0$, can be considered similarly, because the ramification subgroup $\Gamma^{(0,v)}$ appears from the ramification subgroup $\Gamma^{(1)(v)}$, where $\Gamma^{(1)} = \mathrm{Gal}(K_{\mathrm{sep}}^{(1)}/K^{(1)})$ is the Galois group of the first residue field $K^{(1)}$ of K.

5.2. Abelian case. Let $j^0 = (a^0, b^0) \in J_2$. The ideal $L(j^0)_k \bmod C_2(L_k)$ is the minimal ideal of $L_k \bmod C_2(L_k)$ containing all elements of the form $p^s D_{(a,b),s}$, where $s \in \mathbb{Z}_{\geq 0}$ and the indexes $(a,b) \in J$ are such that $p^s(a,b) \geq j^0$. We can apply Prop. 3.1 to deduce that

$$L^{(j^0)} = L(j^0) \bmod C_2(L),$$

what gives the assertion of our theorem modulo 2nd commutators.

5.3. Application of the criterion. Until the end of the paper we assume that $j^0 = (a^0, b^0) \in J_2$ is such that $a^0 > 0$. Consider the rational number $\delta_1(\omega, a^0)$ defined in the beginning of n. 4. Define similarly $\delta_2(\omega, b^0)$ as the minimal value of all positive differences of the form $b^0 - p^n b_1$, where $(0, b_1) \in \mathcal{B}^0(\omega)$ and $n \in \mathbb{Z}$.

Choose $j^* = (a^*, b^*) \in J_2$ and $N^* \in \mathbb{N}$ satisfying the assumptions from the beginning of n. 3.2, and the following conditions: $a^* = a^0$, $b^* < b^0$, and

$$q > \max\left\{ \frac{a^0 + p^M I(\omega)}{\delta_1(\omega, a^0)};\quad \frac{p^M I(\omega)}{a^0} + 2;\quad \frac{p^M J_0(\omega)}{\delta_2(\omega, b^0)};\quad \frac{2p^M I(\omega)}{a^0} + 1 \right\}.$$

Consider the fields from n.3.3: $K_0 = K(N^*, j^*)$, $K' = K(t_{10})$ with $t_{10}^q = t_0$ and $K'_2 = K'(t_{20})$ with $t_{20}^{p^{s^*}} = t_{10}$, where $s^* = s(a^*)$, the liftings $O'_{M+1}(K')$, $O'_{M+1}(K'_{\rm sep})$ and $O'_{M+1}(K'_{2,\rm sep})$, the field isomorphisms $\eta : K \longrightarrow K'$ and $\eta_{\rm sep} : K_{\rm sep} \longrightarrow K'_{\rm sep}$ and the elements

$$\eta(e) = e_1 = \sum_{\substack{(a,b)\in\mathcal{B}^0(\omega)\\ s=s(a)}} \tau_1^{-bp^s} t_1^{-ap^s} D_{(a,b),s-N^*} \in L_{K'} := L \otimes O'_{M+1}(K'),$$

$$\eta_{\rm sep}(f) = f_1 \in L'_{\rm sep} := L \otimes O'_{M+1}(K'_{\rm sep}).$$

Then we can use the equations

$$\sigma f = f \circ e = f + e + \frac{1}{2}[f, e], \quad \sigma f_1 = f_1 \circ e_1 = f_1 + e_1 + \frac{1}{2}[f_1, e_1]$$

to obtain for the element

$$X = (\sigma^M f) \circ \left(-\sigma^{M+N^*} f_1\right) \in L_{2,\rm sep} := L \otimes O'_{M+1}(K_{2,\rm sep})$$

the following equation

$$X - \sigma X = \mathcal{A} - \frac{1}{2}\left[\mathcal{A}, \sigma^{M+N^*} e_1\right] - \left[\sigma X, \sigma^{M+N^*} e_1\right] + \frac{1}{2}\left[\sigma X, \mathcal{A}\right], \qquad (13)$$

where

$$\mathcal{A} = \sigma^{M+N^*} e_1 - \sigma^M e =$$

$$- \sum_{\substack{(a,b)\in\mathcal{B}^0(\omega)\\ s=s(a)}} \left(E(b, \Theta^*)^{p^{s+M}} - 1\right) \tau_1^{-bp^{s+M}q} t_1^{-ap^{s+M}q} D_{(a,b),s+M}$$

The criterion from n.3.3 implies that the ideal $L^{(j^0)}$ is the minimal ideal in L such that the element $X \bmod L^{(j^0)}_{2,\rm sep}$ is invariant with respect to the action of $\Gamma_2^{(j^0)}$, where $L^{(j^0)}_{2,\rm sep} := L^{(j^0)} \otimes O'_{M+1}(K_{2,\rm sep})$. By n. 5.2 one can assume that $L^{(j^0)}$ contains $\left[L(j^0),L\right]$. We are going to prove our theorem by decreasing induction on j^0. For this reason introduce the ideal

$$L(j^0+):=\bigcup_{j>j^0}L(j).$$

Then it will be natural to look for the ideal $L^{(j^0)}$ in the family of all ideals of L containing the ideal $L(j^0+)+[L(j^0),L]$. In order to realise this idea we shall simplify the relation (13) in nn. 5.4-5.5 below modulo the ideal $\{L(j^0+)+[L(j^0),L]\}_{\mathrm{sep}}$ generated by elements of $L(j^0+)+[L(j^0),L]$ in $L_{2,\mathrm{sep}}$.

5.4. Auxiliary statements. Let $O_2'=W_{M+1}((\tau_1))[[t_2]]\subset O_{M+1}'(K_2')$. For $b\in\mathbb{Z}_p$ and $1\le r\le M$, set

$$E_{-1}(b)=1,\quad E_0(b)=E(b,\Theta^*),$$

$$E_r(b)=\sigma^rE_0(b)\prod_{1\le u\le r}\sigma^{r-u}\exp\left(bp^u\Theta^*\right),$$

where E is an analogue of the Artin-Hasse exponential from the beginning of n. 3.

Clearly, for all $0\le r\le M$ and $b,b'\in\mathbb{Z}_p$, one has

$$E_r(b+b')=E_r(b)E_r(b').$$

For $0\le r\le M$ and $b\in\mathbb{Z}_p$, set $\mathcal{E}_r(b)=E_r(b)-\sigma E_{r-1}(b)$. Then $\mathcal{E}_r(b)\in p^rt_1^{a^*(q-1)}O_2'$ and

$$\mathcal{E}_r(b+b')=\mathcal{E}_r(b)+\mathcal{E}_r(b')+\sum_{i,j}\sigma^i\mathcal{E}_{r-i}(b)\sigma^j\mathcal{E}_{r-j}(b'),$$

where the sum is taken for all $0\le i,j\le r$, such that either $i=0$ or $j=0$.

For $0\le r\le M$ and $(a,b)\in\mathcal{B}^0(\omega)$ such that $s(a)\le N^*+r$, set

$$\mathcal{A}_r(a,b)=\mathcal{E}_r(b)\tau_1^{-bp^rq}t_1^{-ap^rq}D_{(a,b),r}\in L_{K_2'}.$$

Lemma 5.2. *There is an $\varepsilon>0$ such that*

$$\mathcal{A}\equiv-\sum_{\substack{M-r+s<N^*\\ r,(a,b)}}\sigma^{M-r+s}\mathcal{A}_r(a,b)\bmod t_1^{qp^MI(\omega)+\varepsilon}L_{O_2'},$$

where in the right-hand sum $s=s(a)$, r runs from 0 to M and (a,b) runs over the set $\mathcal{B}^0(\omega)$.

Proof. The terms of the expression for $\mathcal{A}$ from the end of n. 4.4, which do not appear in the right-hand sum, can be written in the form

$$\sigma^{M-r+s}\mathcal{E}_r(b)\tau_1^{-bp^{s+M}q}t_1^{-ap^{s+M}q}D_{(a,b),s+M},$$

where $M-r+s\geq N^*$, $0\leq r\leq M$ and $(a,b)\in\mathcal{B}^0(\omega)$. All those terms belong to $t_1^{qp^MI(\omega)+\varepsilon}L_{O_2'}$ for some $\varepsilon>0$, because

$$-ap^{s+M}q+p^{M-r+s}a^*(q-1)=p^{M-r+s}(-ap^{r+N^*}+a^*(q-1))>$$

$$q(-ap^{s+M}+a^*(q-1))>q(-p^MI(\omega)+a^*(q-1))>qp^MI(\omega)$$

(we use that $a^*(q-1)>2p^MI(\omega)$, cf. n. 5.3). □

Denote by $\mathcal{M}_0(\omega,j^*)$ the subset of all $W(k)$-linear combinations of elements from $L_{K_2'}$ of the form

$$\tau_1^{-p^rbq+b^*(q-1)}t_1^{-a^*}p^rD_{(a^*/p^r,b),m},$$

where $0\leq r\leq M$, $(a^*/p^r,b)\in\mathcal{B}^0(\omega)$, $p^rb<b^0$.

Lemma 5.3. *If $M-r+s<N^*$, then*

$$\mathcal{A}_r(a,b)\in L(j^0)_{K_2'}+t_1^{p^MI(\omega)+\varepsilon}L(0+)_{O_2'}+t_1^{a^*(q-1)}L_{O_2'}+\mathcal{M}_0(\omega,j^*)+C_2(L_{K_2'}).$$

Proof. Consider the decomposition

$$\mathcal{E}_r(b)=p^rb\Theta^*+p^r\Theta^{*2}\mu_{r,b}(\Theta^*),$$

where $\mu_{r,b}(X)\in W(k)[[X]]$ and $\Theta^*=\tau_1^{b^*(q-1)}t_1^{a^*(q-1)}$ (it follows easily from the definition of the Artin-Hasse exponential). This decomposition induces the decomposition of $\mathcal{A}_r(a,b)$ into 2 summands. Consider the first summand

$$S_1=b\tau_1^{-bp^rq+b^*(q-1)}t_1^{-ap^rq+a^*(q-1)}p^rD_{(a,b),r}.$$

If $p^ra>a^*=a^0$ or $p^ra=a^*$ and $p^rb\geq b^0$, then $S_1\in L(j^0)_{K_2'}+C_2(L_{K_2'})$. If $p^ra=a^*$ and $p^rb<b^0$, then $S_1\in\mathcal{M}_0(\omega,j^*)$. If $0<p^ra<a^*$, then $S_1\in t_1^{p^MI(\omega)+\varepsilon}L(0+)_{O_2'}+C_2(L_{K_2'})$ because

$$-ap^rq+a^*(q-1)\geq q\delta_1(\omega,a^0)-a^0>p^MI(\omega),$$

cf. the beginning of n. 5.3. If $a=0$, then $S_1\in t_1^{a^*(q-1)}L_{O_2'}$.

Consider the second summand

$$S_2=\tau_1^{-bp^rq+2b^*(q-1)}t_1^{-ap^rq+2a^*(q-1)}p^rD_{(a,b),r}\mu_{r,b}(\Theta^*).$$

If $p^ra>a^*$, then $S_2\in L(a^0+)_{K_2'}+C_2(L_{K_2'})\subset L(j^0)_{K_2'}+C_2(L_{K_2'})$. If $0<p^ra\leq a^*$, then $S_2\in t_1^{a^*(q-2)}L(0+)_{O_2'}+C_2(L_{K_2'})$ and $a^*(q-2)>p^MI(\omega)$, cf. the beginning of n. 5.3. If $a=0$, then $S_2\in t_1^{2a^*(q-1)}L_{O_2'}\subset t_1^{a^*(q-1)}L_{O_2'}$. The lemma is proved. □

Denote by $\mathcal{M}_1(\omega, j^*)$ the set of all $W(k)$-linear combinations of elements of $L_{2,\mathrm{sep}}$ of the form

$$Y(\alpha, p^r bq - b^*(q-1), a^*)p^r D_{(a^*/p^r, b), m},$$

where $\alpha \in W(k)$, $m \in \mathbb{Z} \bmod N_0$, $(a^*/p^r, b) \in \mathcal{B}(\omega)$, $p^r b < b^0$ and for $c \in \mathbb{Z}$, one has

$$Y(\alpha, c, a^*) - \sigma Y(\alpha, c, a^*) = \alpha \tau_1^{-c} t_1^{-a^*}.$$

This $W(k)$-module coincides with the $W(k)$-submodule of $L_{2,\mathrm{sep}}$ generated by all Z such that $Z - \sigma Z \in \mathcal{M}_0(\omega, j^*)$. One can easily verify that

$$\mathcal{M}_1(\omega, j^*) \subset L_{K_2(j^0)} + L(j^0+)_{2,\mathrm{sep}} + C_2(L_{2,\mathrm{sep}}),$$

where $K_2(j^0) = K_{2,\mathrm{sep}}^{\Gamma_2^{(j^0)}}$, by using that

$$Y(\alpha, p^r bq - b^*(q-1), a^*) \in K_2(j^0) \bmod pO_{M+1}(K_{2,\mathrm{sep}})$$

if $p^r b < b^0$, and that $p^{r+1} D_{(a^*/p^r, b), m} \in L(j^0+)_k + C_2(L_k)$.

For $0 \le r \le M$ and $(a, b) \in \mathcal{B}^0(\omega)$, denote by $X_r(a, b)$ the element of $L_{2,\mathrm{sep}}$ such that

$$X_r(a, b) - \sigma X_r(a, b) = \mathcal{A}_r(a, b).$$

Directly from the preceding lemma we obtain the following property.

Lemma 5.4. *If $M - r + s < N^*$, then*

$$X_r(a, b) \in L(j^0)_{K_{2,\mathrm{sep}}} + t_1^{p^M I(\omega)+\varepsilon} L(0+)_{O_2'} + t_1^{a^*(q-1)} L_{O_2'} + \mathcal{M}_1(\omega, j^*) + C_2(L_{2,\mathrm{sep}}).$$

Lemma 5.5. *For $0 \le n_1, n_2 < N^*$, $0 \le r \le M$ and $(a_1, b_1), (a_2, b_2) \in \mathcal{B}^0(\omega)$, there exists $\varepsilon > 0$ such that*

$$[\sigma^{n_1} \mathcal{A}_{r_1}(a_1, b_1), \sigma^{n_2} \mathcal{A}_{r_2}(a_2, b_2)] \in t_1^{\varepsilon} L_{O_2'} + L(j^0+)_{K_2'} + \left[L(j^0), L\right]_{K_2'}.$$

Proof. This lemma follows from estimates of Lemma 5.3. We only notice that

$$[L(0+)_{K_{2,\mathrm{sep}}}, \mathcal{M}_0(\omega, j^*)] \subset L(a^0+)_{K_{2,\mathrm{sep}}}$$

by Prop. 4.2, and for $0 \le n < N^*$,

$$t_1^{a^*(q-1)} \sigma^n \mathcal{M}_0(\omega, j^*) \in t_1^{\varepsilon'} L_{O_2'},$$

where $\varepsilon' = a^*(q - q/p - 1) > 0$. ∎

Remark. We note that if $Z\in L_{K_{2,\mathrm{sep}}}$ is such that with the notation of the above lemma one has

$$Z-\sigma Z=[\sigma^{n_1}\mathcal{A}_{r_1}(a_1,b_1),\sigma^{n_2}\mathcal{A}_{r_2}(a_2,b_2)]\,,$$

then $Z\in t_1^{\varepsilon}L_{O_2'}+\left\{L(j^0+)+\left[L(j^0),L\right]\right\}_{2,\mathrm{sep}}$.

Lemma 5.6. *If for any $0\le r_1,r_2\le M$, $0\le n<N^*$, $(a_1,b_1),(a_2,b_2)\in\mathcal{B}^0(\omega)$, an element $Z\in L_{2,\mathrm{sep}}$ satisfies the relation*

$$Z-\sigma Z=\left[\sigma^{n}\mathcal{A}_{r_1}(a_1,b_1),\sigma^{N^*}X_{r_2}(a_2,b_2)\right],$$

then $Z\in L_{K_2(j^0)}+\left\{L(j^0+)+[L(j^0),L]\right\}_{2,\mathrm{sep}}$ (where $K_2(j^0)=K_{2,\mathrm{sep}}^{\Gamma_2^{(j^0)}}$).

Proof. It is sufficient to use the estimates from Lemmas 5.3 and 5.4 and that σ is nilpotent on $t_1^{a^*(q-1)}\sigma^{N^*-1}\mathcal{M}_1(\omega,j^*)\subset L_{K_2(j^0)}$, what follows from the embedding $\sigma^{N^*+1}\mathcal{M}_1(\omega,j^*)\subset\sigma^{N^*}\mathcal{M}_1(\omega,j^*)+t_1^{-a^*q}L_{O_2'}$. □

Lemma 5.7. *In the notation of nn. 5.3-5.4 for some $\varepsilon>0$, it holds*

$$\left[\mathcal{A},\sigma^{M+N^*}e_1\right]\equiv-\sum_{\substack{r,(a,b)\\ M-r+s<N^*}}\left[\sigma^{M-r+s}\mathcal{A}_r(a,b),\sigma^{M+N^*}e_1\right]\bmod t_1^{\varepsilon}L_{O_2'}.$$

Proof. This follows from Lemma 5.2 because $e_1\in t_1^{-I(\omega)}L_{O_2'}$. □

Lemma 5.8. *In the notation of nn.5.3-5.4 if $Z\in L_{K_{2,\mathrm{sep}}}$ is such that*

$$\left[\sigma X,\sigma^{M+N^*}e_1\right]=-\sum_{\substack{r,(a,b)\\ M-r+s<n<N^*}}\left[\sigma^{n}\mathcal{A}_r(a,b),\sigma^{M+N^*}e_1\right]+Z-\sigma Z,$$

then $Z\in L_{K_2(j^0)}+\left\{L(j^0+)+\left[L(j^0),L\right]\right\}_{2,\mathrm{sep}}$.

Proof. We notice first that for some $\varepsilon>0$, it holds

$$\sigma X\equiv-\sum_{\substack{r,(a,b)\\ M-r+s<u<N^*}}\sigma^{u}\mathcal{A}_r(a,b)-\sum_{r,(a,b)}\sigma^{N^*}X_r(a,b)$$

$$\bmod t_1^{qp^MI(\omega)+\varepsilon}L_{O_2'}+C_2(L_{2,\mathrm{sep}}).$$

This is implied by the relation

$$X=-\sum_{\substack{r,(a,b)\\ M-r+s<N^*}}\sigma^{M-r+s}X_r(a,b)\bmod t_1^{qp^MI(\omega)+\varepsilon}L_{O_2'},$$

which follows from Lemma 5.2.

So, it is sufficient to prove that if $Z_r'(a,b)\in L_{2,\mathrm{sep}}$ is such that

$$Z_r'(a,b)-\sigma Z_r'(a,b)=\left[X_r(a,b),\sigma^{M}e_1\right],$$

then $Z'_r(a,b) \in L_{K_2(j^0)} + \left\{L(j^0+) + [L(j^0),L]\right\}_{2,\mathrm{sep}}$. By Lemma 5.4 this can be reduced to the following property: if $U \in L_{2,\mathrm{sep}}$ is such that

$$U - \sigma U \in \left[\mathcal{M}_1(\omega, j^*), \sigma^M e_1\right],$$

then $U \in L_{K_2(j^0)} + \left\{L(j^0+) + [L(j^0),L]\right\}_{2,\mathrm{sep}}$. By Prop. 4.2 we have

$$[\mathcal{M}_1(\omega, j^*), L(0+)_{K'_2}] \subset L(j^0+)_{2,\mathrm{sep}}.$$

We have also

$$p\left[\mathcal{M}_1(\omega, j^*), L_{K'_2}\right] \subset [L(a^0+)_{2,\mathrm{sep}}, L_{2,\mathrm{sep}}] \subset L(j^0+)_{2,\mathrm{sep}}.$$

So, the proof of lemma is reduced to the following statement.

— *let* $V, W \in W_{M+1}(K_{2,\mathrm{sep}})$ *be such that* $V - \sigma V = \beta\tau_1^{-b_1p^M} W$ *and* $W - \sigma W = \alpha\tau_1^{-bp^rq+b^*(q-1)}t_1^{-a}$, *where* $(0,b_1),(a^*/p^r,b) \in \mathcal{B}(\omega)$ *and* $bp^r < b^0$*; then* $V \in pW_{M+1}(K_{2,\mathrm{sep}}) + W_{M+1}(K_2(j^0))$

In other words, cf. n. 3.4, $j_2(V \bmod p/K_2) < j^0$ or, equivalently,

$$j'(V \bmod p/K') < qj^0 - (q-1)j^* = (a^0, qb^0 - (q-1)b^*).$$

The case $M = 0$ of Prop. 3.6 gives

$$j'(V \bmod p/K') = (a^0, b_1p^M + bp^rq - b^*(q-1)).$$

It remains only to notice that

$$b_1p^M + bp^rq < p^MJ_0(\omega) + q(b^0 - \delta_2(\omega, b^0)) < qb^0,$$

because $q\delta_2(\omega, b^0) > p^MJ_0(\omega)$, cf. n.5.3. The lemma is proved. $\square$

5.5. Simplification of relation (13). Consider $S = S_1 + S_2 \in L_{2,\mathrm{sep}}$, with

$$S_1 = -\sum_{n,(a,b)} \mathcal{A}_n(a,b),$$

$$S_2 = \sum_{n_1,n_2,j_1,j_2} \eta(n_1 - s_1, n_2 - s_2)\left[\mathcal{A}_{n_1}(a_1,b_1), \tau_1^{-b_2p^{n_2}q}t_1^{-a_2p^{n_2}q}D_{j_2,n_2}\right],$$

where the first sum is taken for $0 \le n \le M$ and $(a,b) \in \mathcal{B}^0(\omega)$ such that $M - n + s < N^*$ with $s = s(a)$; and the second sum is taken for all $0 \le n_1 \le M$, $n_2 > -N^* + M - s_2$, $j_1 = (a_1,b_1), j_2 = (a_2,b_2) \in \mathcal{B}^0(\omega)$ with $s_1 = s(a_1)$ and $s_2 = s(a_2)$ (cf. the definition of $\eta(n_1,n_2)$ in n. 4.2).

Proposition 5.9. *Suppose* $X \in L_{2,\mathrm{sep}}$ *satisfies relation (13). Then there is* $X' \in L_{2,\mathrm{sep}}$ *such that*

$$X' \equiv X \bmod L_{K_2(j^0)} + \left\{L(j^0+) + \left[L(j^0),L\right]\right\}_{2,\mathrm{sep}}$$

and

$$X' - \sigma X' = S. \tag{14}$$

Proof. From Lemmas 5.2, 5.7 and 5.8, we conclude that the right-hand side of the relation (13) is equivalent modulo

$$(\sigma - \mathrm{id})L_{K_2(j^0)} + \left\{L(j^0+) + [L(j^0), L]\right\}_{2,\mathrm{sep}}$$

to the expression

$$-\sum_{r,(a,b)} \sigma^{M-r+s}\mathcal{A}_r(a,b) + \sum_{r_1,(a_1,b_1),u} \eta(u,0)\left[\sigma^{M-r_1+s_1+u}\mathcal{A}_{r_1}(a_1,b_1), \sigma^{M+N^*}e_1\right],$$

where the summation indexes satisfy the conditions $0 \le r, r_1 \le M$, $(a,b),(a_1,b_1) \in \mathcal{B}^0(\omega)$, $u \in \mathbb{Z}_{\ge 0}$, $M - r + s < N^*$ and $0 \le u < N^* - (M - r_1 + s_1)$ (with $s = s(a)$ and $s_1 = s(a_2)$).

By changing the above expression modulo $(\sigma - \mathrm{id})L_{K_2'}$ and setting $r_1 = n_1$ and $n_2 = (s_2 - u) - (s_1 - r_1)$ we transform it to S. The proposition is proved. $\square$

By the use of the identity $1 = \eta(n_1, n_2) + \eta(n_2, n_1)$ we obtain the decomposition $S_2 = S_{21} + S_{22}$, where $S_{21} =$

$$\sum_{n_1,n_2,j_1,j_2} \eta(n_1 - s_1, n_2 - s_2)\eta(n_1, n_2)\left[\mathcal{A}_{n_1}(a_1,b_1), \tau_1^{-b_2p^{n_2}q}t_1^{-a_2p^{n_2}q}D_{j_2,n_2}\right],$$

and S_{22} is given by the same expression with $\eta(n_1,n_2)$ replaced by $\eta(n_2,n_1)$. By the use of the decomposition, cf. the proof of Lemma 5.3, $\mathcal{E}_{n_1}(b_1) = b_1p^{n_1}\Theta^* + p^{n_1}\Theta^{*2}\mu_{b_1,n_1}(\Theta^*)$, set $S_{21} = S'_{21} + S''_{21}$, where S'_{21} is given by the expression

$$\sum_{n_1,n_2,j_1,j_2} \eta(n_1 - s_1, n_2 - s_2)\eta(n_1,n_2)p^{n_1}b_1 \times \tau_1^{-(p^{n_1}b_1+p^{n_2}b_2)q}t_1^{-(p^{n_1}a_1+p^{n_2}a_2)q}\Theta^*[D_{j_1n_1}, D_{j_2n_2}].$$

Prove that

$$S''_{21} \in t_1^{\varepsilon}L_{O_2'} + \left\{L(j^0+) + [L(j^0), L]\right\}_{K_2'}.$$

It is sufficient to verify that the element of the form

$$t_1^{-(a_1p^{n_1}+a_2p^{n_2})q}\Theta^{*2}p^{n_1}[D_{j_1n_1}, D_{j_2,n_2}]$$

belongs to $t_1^{\varepsilon}L_{O_2'} + \left\{L(j^0+) + [L(j^0), L]\right\}_{K_2'}$ if $n_1 \ge n_2$.

If $p^{n_1}a_1 > a^0$, then our element belongs to $[L(j^0), L]_{K_2'}$. Let $p^{n_1}a_1 = a^0$. If $a_2 > 0$, then our element belongs to $L(j^0+)_{K_2'}$ by Prop. 4.2; if $a_2 = 0$, then it belongs to $t_1^{a^0(q-2)}L_{O_2'}$.

The case $p^{n_1}a_2 \ge a^0$ can be considered similarly.

If $p^{n_1}a_1, p^{n_1}a_2 < a^0$, then it remains to note that

$$(p^{n_1}a_1 + p^{n_2}a_2)q \leq (p^{n_1}a_1 + p^{n_1}a_2)q < 2a^0q - 2\delta_1(\omega, a^0)q < 2a^0(q-1),$$

because $q\delta_1(\omega, a^0) > a^0$, cf. n. 5.3.

Thus we have obtained

$$S_{21} \equiv S'_{21} \operatorname{mod} t_1^{\varepsilon} L_{O'_2} + \left\{L(j^0+) + [L(j^0), L]\right\}_{K'_2}.$$

For $n_2 \geq n_1$, $b = b_1$ and $b' = p^{n_2-n_1}b_2$, consider the identity from n. 5.4

$$\mathcal{E}_{n_1}(b_1) = \mathcal{E}_{n_1}(b_1 + p^{n_2-n_1}b_2) - \mathcal{E}_{n_1}(p^{n_2-n_1}b_2) - \sum_{i,j} \sigma^i \mathcal{E}_{n_1-i}(b_1)\sigma^j \mathcal{E}_{n_1-j}(p^{n_2-n_1}b_2).$$

With respect to three summands of the right-hand side of the above identity decompose S_{22} in the form $S_{221} + S_{222} + S_{223}$. Then

$$S_{223} = -\sum_{n_1,n_2,j_1,j_2} \eta(n_1 - s_1, n_2 - s_2)\eta(n_2, n_1) \times \left[\sigma^i \mathcal{A}_{n_1-i}(a_1, b_1), \sigma^j \mathcal{A}_{n_1-j}(p^{n_2-n_1}b_2)\right]$$

belongs to $t_1^{\varepsilon} L_{O'_2} + \left\{L(j^0+) + [L(j^0), L]\right\}_{K'_2}$ by Lemma 5.5, because one can repeat the arguments of the proof of Lemma 5.3 to obtain its estimate for the element given by the expression

$$\mathcal{E}_{n_1-j}(p^{n_2-n_1}b_2)t_1^{-a_2p^{n_2-j}q}\tau_1^{-b_2p^{n_2-j}q}D_{(a_2,b_2),n_2-j}.$$

By the use of the identity

$$\mathcal{E}_{n_1}(p^{n_2-n_1}b_2) = p^{n_2}b_2\Theta^* + p^{n_2}\Theta^{*2}\mu_{n_1,p^{n_2-n_1}b_2}(\Theta^*)$$

and the arguments we have used above to estimate S''_{21}, we obtain

$$\begin{aligned} S_{222} &\equiv -\sum_{n_1,n_2,j_1,j_2} \eta(n_1 - s_1, n_2 - s_2)\eta(n_2, n_1) \times \\ &\qquad p^{n_2}b_2\tau_1^{-(p^{n_1}b_1+p^{n_2}b_2)q}t_1^{-(p^{n_1}a_1+p^{n_2}a_2)q}\Theta^*[D_{j_1n_1}, D_{j_2,n_2}] \\ &\equiv \sum_{n_1,n_2,j_1,j_2} \eta(n_2 - s_2, n_1 - s_1)\eta(n_1, n_2) \times \\ &\qquad p^{n_1}b_1\tau_1^{-(p^{n_1}b_1+p^{n_2}b_2)q}t_1^{-(p^{n_1}a_1+p^{n_2}a_2)q}\Theta^*[D_{j_1n_1}, D_{j_2,n_2}]. \end{aligned}$$

Now we notice that

$$S_1 + S_{221} = -\sum_{n,(a,b)\in B^0} \mathcal{A}_n(a, b)\widetilde{D}_{(a,b),n}.$$

By the use of the decomposition $\mathcal{E}_n(b) = p^n b\Theta^* + p^n\Theta^{*2}\mu_{n,b}(\Theta^*)$ and Prop. 4.6 we obtain

$$S_1 + S_{221} \equiv -\sum_{n,(a,b)} p^n b\tau_1^{-bp^n q} t_1^{-ap^n q}\Theta^*\widetilde{D}_{(a,b),n}.$$

We summarize the above relations by the use of the identity

$$\eta(n_1 - s_1, n_2 - s_2) + \eta(n_2 - s_2, n_1 - s_1) = 1$$

and expressions for elements $\widetilde{\mathcal{F}}_\gamma(\omega)$, $\gamma \in J_2$,

$$S \equiv -\sum_{\gamma=(\gamma_1,\gamma_2)\in J_2} \tau_1^{-\gamma_1 q} t_1^{-\gamma_2 q}\Theta^*\widetilde{\mathcal{F}}_\gamma(\omega).$$

In order to simplify this equivalence modulo

$$(\sigma - \mathrm{id})L_{K_2(j^0)} + \left\{L(j^0+) + [L(j^0), L]\right\}_{2,\mathrm{sep}}$$

consider the set

$$\mathcal{G}_2(\omega) = \left\{\gamma = a_1^0 p^{m_1} + a_2^0 p^{m_2} \mid a_1^0, a_2^0 \in \mathbb{Z}\cap[0, I(\omega)], m_1, m_2 \in \mathbb{Z}\right\}.$$

It is easy to see the existence of $\delta(\omega, a^0) \in \mathbb{Q}_{>0}$ such that if $\gamma \in \mathcal{G}_2(\omega)$ and $\gamma < a^0$, then $\gamma \le a^0 - \delta(\omega, a^0)$.

Now suppose in addition to conditions for q from n. 5.3 that q satisfies also the equality

$$q\delta(\omega, a^0) > a^0. \tag{15}$$

Notice that $\widetilde{\mathcal{F}}_\gamma(\omega) = 0$ if $\gamma \notin \mathcal{G}_2(\omega)$.

For $\gamma = (\gamma_1, \gamma_2) \in \mathcal{G}_2(\omega)$, let $\widetilde{X}_\gamma \in L_{2,\mathrm{sep}}$ be such that

$$\widetilde{X}_\gamma - \sigma\widetilde{X}_\gamma = -\tau_1^{-\gamma_1 q} t_1^{-\gamma_2 q}\Theta^*\widetilde{\mathcal{F}}_\gamma.$$

If $\gamma_1 < a^0$ then

$$-\gamma_1 q + a^*(q-1) \ge q\delta(\omega, a^0) - a^0 > 0$$

by inequality (15) and, therefore, $\widetilde{X}_\gamma \in L_{K_2'}$. If $\gamma_1 \ge a^0$, then (cf. the relation (12) in the proof of Prop. 4.5) $\widetilde{\mathcal{F}}_\gamma(\omega) \equiv \mathcal{F}_\gamma \bmod \left[L(a^0+)_k, L_k\right]$ and $p\widetilde{\mathcal{F}}_\gamma(\omega) \in L(a^0+)_k$ (cf. the proof of Prop. 4.6). This implies that

— if $\gamma_1 = a^0$, $\gamma_2 < b^0$, then

$$X_\gamma \in L_{K_2(j^0)} + \left[L(j^0+), L\right]_{2,\mathrm{sep}};$$

— if $\gamma > j^0$, then

$$X_\gamma \in \left\{L(j^0+) + [L(j^0), L]\right\}_{2,\mathrm{sep}};$$

— $\widetilde{X}_{j^0} \equiv X_{j^0} \bmod \left\{L(j^0+) + [L(j^0), L]\right\}_{2,\mathrm{sep}}$, where

$$X_{j^0} - \sigma X_{j^0} = -\tau^{-b^0 q + b^*(q-1)} t_1^{-a^0 q + a^*(q-1)}\mathcal{F}_{j^0}.$$

With the above notation we obtain the following proposition.

Proposition 5.10.

$$X \equiv X_{j^0} \bmod L_{K_2(j^0)} + \left\{L(j^0+) + [L(j^0), L]\right\}_{2,\mathrm{sep}}.$$

5.6. The end of the proof of theorem. It is sufficient to prove that if L^0 is a finite Lie algebra and ρ^0 is a projection of L to L^0, then for any $j \in J$, one has $\rho^0(L^{(j)}) = \rho^0(L(j))$.

It is easy to see that the both filtrations $\{\rho^0(L(j))\}_{j\in J}$ and $\{\rho^0(L^{(j)})\}_{j\in J}$ are left-continuous, have jumps only in "finite points" $j^0 \in J$ and have trivial terms for a sufficiently large j. Therefore, we can use in the proof a transfinite decreasing induction on j, i.e. we can assume the existence of $j^0 \in J$ such that for all $j > j^0$ it holds

$$\rho^0(L(j)) = \rho^0(L^{(j)})$$

and must prove under this assumption that $\rho^0(L(j^0)) = \rho^0(L^{(j^0)})$.

By arguments of n.5.1 we can assume that $j^0 = (a^0, b^0)$ with $a^0 > 0$. By Prop. 5.10 and inductive assumption, $\rho^0(L^{(j^0)})$ is the minimal ideal in the family of all ideals I of L^0 such that

$$I \supset \rho^0(L(j^0+) + [L(j^0), L])$$

and

$$j_2\left(\rho^0(X_{j^0}) \bmod I_{2,\mathrm{sep}}/K_2\right) < j^0.$$

It remains only to note that $\rho^0(\mathcal{F}_{j^0}) \notin I_k$ if and only if $j'\left(\rho^0(X_{j^0}) \bmod I_{2,\mathrm{sep}}/K'\right) = qj^0 - (q-1)j^*$, and this is equivalent to the equality $j_2\left(\rho^0(X_{j^0}) \bmod I_{2,\mathrm{sep}}/K_2\right) = j^0$.

References

[1] V.A. Abrashkin *Ramification filtration of the Galois group of a local field. II.* Proceeding of Steklov Math. Inst. **208** (1995).

[2] V.A.Abrashkin *Ramification filtration of the Galois group of a local field. III.* Izvestiya RAN, ser. math. **62**, no 5, 3–48.

[3] V.A. Abrashkin, *On a local analogue of the Grothendieck Conjecture.* Intern. Journal of Math. **11** no.1 (2000), 3–43.

[4] H. Epp, *Eliminating wild ramification.* Invent. Math. **19** (1973), 235–249.

[5] K. Kato, *A generalisation of local class field theory by using K-groups I.* J. Fac. Sci. Univ. Tokyo Sec. IA **26** no.2 (1979), 303–376.

[6] Sh. Mochizuki, *A version of the Grothendieck conjecture for p-adic local fields.* Int. J. Math. **8** no. 4 (1997), 499–506.

[7] A.N. Parshin, *Local class field theory.* Trudy Mat. Inst. Steklov **165** (1984) English transl. in Proc.Steklov Math. Inst. **165** (1985) 157–185.

[8] J.-P. Serre, *Corps locaux.* Hermann, Paris, 1968.

[9] I. Zhukov, *On ramification theory in the imperfect residue field case.* Preprint of Nottingham University **98-02** (1998).

Victor ABRASHKIN
Math. Dept. of Durham University
South Road
Durham DH7 7QR, United Kingdom
&
Steklov Math. Institute
Gubkina 8, 117966, Moscow, Russia
E-mail : `victor.abrashkin dur.ac.uk`