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An Arakelov theoretic proof of the equality of conductor and discriminant

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Résumé. Nous donnons une preuve utilisant la théorie d'Arakelov de l'égalité du conducteur et du discriminant.

Abstract. We give an Arakelov theoretic proof of the equality of conductor and discriminant.

1. Introduction

Let K be a number field, $\mathcal{O}_K$ be the ring of integers of K, and S be $\mathrm{Spec}(\mathcal{O}_K)$. Let $f : X \to S$ be an arithmetic surface. By this we mean a regular scheme, proper and flat over S, of relative dimension one. We also assume that the generic fiber of X has genus≥ 1, and that X/S has geometrically connected fibers.

Let ω_X be the dualizing sheaf of X/S. The Mumford isomorphism ([Mumf], Theorem 5.10)

$$\det \mathrm{R}f_*(\omega_X^{\otimes 2}) \otimes K \to (\det \mathrm{R}f_*\omega_X)^{\otimes 13} \otimes K,$$

which is unique up to sign, gives a rational section Δ of

$$(\det \mathrm{R}f_*\omega_X)^{\otimes 13} \otimes (\det \mathrm{R}f_*(\omega_X^{\otimes 2}))^{\otimes -1}.$$

The discriminant $\Delta(X)$ of X/S is defined as the divisor of this rational section ([Saito]). If $\mathfrak{p}$ is a closed point of S, we denote the coefficient of $\mathfrak{p}$ in $\Delta(X)$ by $\delta_{\mathfrak{p}}$.

On the other hand X/S has an Artin conductor $\mathrm{Art}(X)$ (cf. [Bloch]), which is similarly a divisor on S. We denote the coefficient of $\mathfrak{p}$ in $\mathrm{Art}(X)$ by $\mathrm{Art}_{\mathfrak{p}}$. Let S' be the strict henselization of the complete local ring of S at $\mathfrak{p}$, with field of fractions K'. Let s be its special point, η be its generic point, and $\overline{\eta}$ be a geometric generic point corresponding to an algebraic closure $\overline{K'}$ of K'. Let ℓ be a prime different from the residue characteristic at $\mathfrak{p}$. Then

$$\mathrm{Art}_{\mathfrak{p}}(X) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}_\ell} \mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Q}_\ell) - \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}_\ell} \mathrm{H}^i_{\acute{e}t}(X_s, \mathbb{Q}_\ell) + \sum_{i\geq 0}(-1)^i \mathrm{Sw}_{\overline{K'}/K'}(\mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Q}_\ell)),$$

where $\mathrm{Sw}_{\overline{K'}/K'}$ denotes the Swan conductor of the Galois representation of $\overline{K'}/K'$. Both of these divisors are supported on the primes of bad reduction of X. We give another proof of Saito's theorem ([Saito], Theorem 1) in the number field case.

Theorem 1. *For any closed point $\mathfrak{p} \in S$, we have $\delta_{\mathfrak{p}} = -\mathrm{Art}_{\mathfrak{p}}$.*

Fix a Kähler metric on $X(\mathbb{C})$ invariant under complex conjugation, this gives metrics on $\Omega^1_{X_\nu}$'s, for each $\nu \in S(\mathbb{C})$. For a hermitian coherent sheaf $\mathcal{E}$, we endow $\det \mathrm{R}f_*(\mathcal{E})$ with its Quillen metric. The proof of the theorem has the following corollaries.

Proposition 1. *We have*

$$\deg\det Rf_*\omega_X = \frac{1}{12}[\deg f_*(\widehat{c_1}(\omega_X)^2) + \log\mathrm{Norm}(-\mathrm{Art}(X))] \\ [K:\mathbb{Q}](g-1)[2\zeta'(-1)+\zeta(-1)],$$

with ζ the Riemann zeta function.

Proposition 1 is an arithmetic analogue of Noether's formula in which $\det \mathrm{R}f_*\omega_X$ is endowed with the Quillen metric. Faltings [Falt] and Moret-Bailly [M-B] proved a similar formula for the Faltings metrics.

Proposition 2. *We have*

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\nu\in S(\mathbb{C})} \log\|\Delta_\nu\| = 12(1-g)[2\zeta'(-1)+\zeta(-1)],$$

where Δ_ν is the section on X_ν obtained by pulling back Δ via the map from Spec$(\mathbb{C})$ to Spec$(\mathcal{O}_K)$ that corresponds to $\nu \in S(\mathbb{C})$. In particular, the norm of the Mumford isomorphism does not depend on the metric.

2. Proof

First we prove Proposition 1. By duality ([Deligne], Lemme 1.3), $\deg\det \mathrm{R}f_*\omega_X = \deg\det \mathrm{R}f_*\mathcal{O}_X$. By the arithmetic Riemann-Roch theorem of Gillet and Soulé ([G-S], Theorem 7), we get

$$\deg\det \mathrm{R}f_*\mathcal{O}_X = \deg f_*(\widehat{Td}(\Omega^1_X)^{(2)}) - \frac{1}{2}\sum_{\nu\in S(\mathbb{C})}\int_{X_\nu} Td(T_{X_\nu})R(T_{X_\nu}).$$

Here Td and R are the Todd and Gillet-Soulé genera respectively, and the superscript (2) denotes the degree 2 component. Applying the definitions of these characteristic classes we obtain

$$\deg\det \mathrm{R}f_*\mathcal{O}_X = \frac{1}{12}\deg f_*(\widehat{c_1}(\Omega^1_X)^2 + \widehat{c_2}(\Omega^1_X)) + [K:\mathbb{Q}](g-1)[2\zeta'(-1)+\zeta(-1)].$$

Let Z denote the union of singular fibers of f, and let $c^Z_{2,X}(\Omega^1_X)$ be the localized Chern class of Ω^1_X with support in Z (cf. [Bloch], [Fulton]). Chinburg, Pappas, and Taylor ([CPT], Proposition 3.1) prove the formula

$$\deg f_*(\widehat{c_2}(\Omega^1_X)) = \log \mathrm{Norm}(c^Z_{2,X}(\Omega^1_X)).$$

Combining this with the fundamental formula of Bloch ([Bloch], Theorem 1)

$$-\mathrm{Art}_{\mathfrak{p}}(X) = \deg_{\mathfrak{p}} c^Z_{2,X}(\Omega^1_X),$$

we obtain the desired formula. Note that, since $\det\Omega^1_X = \omega_X$, $\widehat{c_1}(\Omega^1_X) = \widehat{c_1}(\omega_X)$. □

Taking degrees in the Mumford isomorphism gives

$$13\deg\det \mathrm{R}f_*\omega_X = \deg\det \mathrm{R}f_*(\omega_X^{\otimes 2}) + \log\mathrm{Norm}(\Delta(X)) - \sum_{\nu\in S(\mathbb{C})}\log\|\Delta_\nu\|.$$

The arithmetic Riemann-Roch theorem ([Falt], Theorem 3) gives

$$\deg\det \mathrm{R}f_*(\omega_X^{\otimes 2}) = \deg\det \mathrm{R}f_*\omega_X + \deg f_*(\widehat{c_1}(\omega_X)^2).$$

Therefore we get

(1)
$$\deg\det \mathrm{R}f_*\omega_X = \frac{1}{12}[\deg f_*(\widehat{c_1}(\omega_X)^2) + \log\mathrm{Norm}(\Delta(X)) - \sum_{\nu\in S(\mathbb{C})}\log\|\Delta_\nu\|.$$

Subtracting (1) from the expression in the statement of Proposition 1, we obtain

(2)
$$\log\Big(\frac{\mathrm{Norm}(\Delta(X/S))}{\mathrm{Norm}(-\mathrm{Art}(X/S))}\Big) = \sum_{\nu\in S(\mathbb{C})}\log\|\Delta_\nu\| + 12[K:\mathbb{Q}](g-1)[2\zeta'(-1)+\zeta(-1)].$$

Now X_K has semistable reduction after a finite base change K'/K. For semistable X'/S', both $-\mathrm{Art}_{\mathfrak{p}'}(X')$ ([Bloch]), and $\delta_{\mathfrak{p}'}(X')$ ([Falt], Theorem 6) are equal to the number of singular points in the geometric fiber over $\mathfrak{p}'$. Therefore $-\mathrm{Art}_{\mathfrak{p}'} = \delta_{\mathfrak{p}'}$, and hence

(3)
$$\mathrm{Norm}(\Delta(X'/S')) = \mathrm{Norm}(-\mathrm{Art}(X'/S')).$$

Applying this to a semistable model X' of $X \otimes_K K'$, and noting that the base change multiplies the right hand side of (2) by $[K' : K]$, we see that the right hand side of (2) is equal to zero, and hence that the equality

$$\operatorname{Norm}(\Delta(X/S)) = \operatorname{Norm}(-\operatorname{Art}(X/S)) \tag{4}$$

holds for X.

To prove the equality $\delta_{\mathfrak{p}} = -\operatorname{Art}_{\mathfrak{p}}$ for an arbitrary closed point $\mathfrak{p} \in S$, we will use the following lemma.

Lemma 1. *Fix distinct closed points $\beta_1, .., \beta_s \in S$. For each i such that $1 \leq i \leq s$, let L_i be an extension of the completion K_i of K at β_i such that $[L_i : K_i] = n$ is independent of i. Then there exists an extension L/K such that, for each $1 \leq i \leq s$, there is only one prime γ_i of L lying over β_i, and the completion of L at γ_i is isomorphic (over K_i) to L_i.*

Proof. The proof is an application of Krasner's lemma, and the approximation lemma. Details are omitted. □

Take $\mathfrak{p}=\beta_1$, a prime of bad reduction. Denote the remaining primes of bad reduction by β_i, $2 \leq i \leq s$. Choose extensions L_i of the local fields K_i, for all $1 \leq i \leq s$, such that L_1 is unramified over K_1, X has semistable reduction over L_i, for $2 \leq i \leq s$, and $[L_i : K_i] = n$, for some n. Applying the lemma to this data we obtain an extension L of K. Let $T = \operatorname{Spec}(\mathcal{O}_L)$. The curve $X \otimes_K L$ has a proper, regular model Y over T such that

(i) $Y \otimes_T T_{\gamma_1} \simeq X \otimes_S T_{\gamma_1}$, and
(ii) Y is semistable at γ_i, for $2 \leq i \leq s$.

Applying (4) to Y gives the equality

$$\sum_{1 \leq i \leq s} \delta_{\gamma_i} \log \operatorname{Norm}(\gamma_i) = \sum_{1 \leq i \leq s} -\operatorname{Art}_{\gamma_i} \log \operatorname{Norm}(\gamma_i).$$

On the other hand because of semistability, we have $\delta_{\gamma_i} = -\operatorname{Art}_{\gamma_i}$, for $2 \leq i \leq s$. Hence we get $\delta_{\gamma_1} = -\operatorname{Art}_{\gamma_1}$. Since T/S is étale at γ_1 , (i) implies

$$\delta_{\mathfrak{p}} = \delta_{\gamma_1} = -\operatorname{Art}_{\gamma_1} = -\operatorname{Art}_{\mathfrak{p}}.$$

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