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# S-expansions in dimension two

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Résumé. Nous généralisons en dimension deux la méthode de singularisation développée par C. Kraikamp au cours des années 90 dans ses travaux sur les systèmes dynamiques associées aux fractions continues, en relation avec certaines propriétés d'approximations diophantiennes. Nous appliquons la méthode à l'algorithme de Brun en dimension 2 et montrons comment utiliser cette technique et d'autres analogues pour transférer des propriétés métriques et diophantiennes d'un algorithme à l'autre. Une conséquence de cette étude est la construction d'un algorithme qui améliore les propriétés d'approximations par comparaisons avec celles de l'algorithme de Brun.

Abstract. The technique of singularization was developped by C. Kraaikamp during the nineties, in connection with his work on dynamical systems related to continued fraction algorithms and their diophantine approximation properties. We generalize this technique from one into two dimensions. We apply the method to the the two dimensional Brun's algorithm. We discuss, how this technique, and related ones, can be used to transfer certain metrical and diophantine properties from one algorithm to the others. In particular, we are interested in the transferability of the density of the invariant measure. Finally, we use this method to construct an algorithm which improves approximation properties, as opposed to Brun's algorithm.

## 1. Introduction

The technique of *singularization*, as described in details by M. Iosifescu and C. Kraaikamp [7] (see also [9]), was introduced to improve some diophantine approximation properties of the regular one-dimensional continued fraction algorithm in the following sense: Let $\{p^{(t)}/q^{(t)}\}_{t=1}^{\infty}$ be the sequence of convergents of an arbitrary real number $x$ in $(0,1)$, produced by the regular continued fraction algorithm. Singularization methods allow to transform the original (regular continued fraction) algorithm into new ones (depending on the actual setting of the applications), such that the

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sequences of convergents built from the new algorithms are subsequences of the previous one. This technique also allows to transfer the underlying ergodic properties of one algorithm to the other.

A large family of semi-regular continued fraction algorithms, called *S-expansions*, can be related to each other via singularizations (e.g. the nearest integer continued fraction [14], Hurwitz' singular continued fraction [6], Minkowski's diagonal expansion [13], Nakada's $\alpha$-expansions [15, 16] or Bosma's optimal continued fraction [2]).

In this paper, we show that similar techniques can be applied in dimension two. We describe singularization processes, based on the two-dimensional Brun's algorithm, and analyze how to use singularizations to transfer certain statistical and approximation properties towards the resulting algorithm. In particular, using natural extensions of the underlying ergodic dynamical systems, we are interested in how to deduce the corresponding invariant measure of the new algorithms from the density of the invariant measure of the original algorithm. This is of a special interest with respect to recent investigations by the author on similar relations between Brun's algorithm and the Jacobi-Perron algorithm in two dimensions [19] .

Finally, we present an algorithm $\overline{T}_q$ with improved approximation properties, as opposed to the underlying Brun's algorithm.

## 2. Definitions

We recall some basic definitions and results on fibered systems. For an extensive summary, we refer to a monograph of F. Schweiger [23].

**Definition.** Let $X$ be a set and $T : X \to X$. If there exists a partition $\{X(i) : i \in I\}$ of $X$, where $I$ is finite or countable, such that the restriction of $T$ to $X(i)$ is injective, then $(X, T)$ is called a *fibred system*.

$I$ is the *set of digits*, and the partition $\{X(i) : i \in I\}$ is called the *time-1-partition*.

**Definition.** A *cylinder of rank $t$* is the set

$$X(i^{(1)}, \dots, i^{(t)}) := \{x : i(x) = i^{(1)}, \dots, i(T^{t-1}(x)) = i^{(t)}\}\,.$$

A block of digits $(i^{(1)}, \dots, i^{(t)})$ is called *admissible*, if

$$X(i^{(1)}, \dots, i^{(t)}) \neq \emptyset\,.$$

Since $T : X(i) \to TX(i)$ is bijective, there exists an inverse map $V(i) : TX(i) \to X(i)$ which will be called a *local inverse branch* of $T$. Define $V(i^{(1)}, i^{(2)}, \dots, i^{(t)}) := V(i^{(1)}) \circ V(i^{(2)}, \dots, i^{(t)})$; then $V(i^{(1)}, i^{(2)}, \dots, i^{(t)})$ is a local inverse branch of $T^t$.

**Definition.** The fibred system $(X,T)$ is called a *multidimensional continued fraction algorithm* if

(1) $X$ is a subset of the Euclidean space $\mathbb{R}^n$

(2) For every digit $i\in I$, there is an invertible matrix $\alpha=\alpha(i)=((a_{kl}))$, $0\le k,l\le n$, such that $x^{(1)}=Tx^{(0)}$, $x^{(0)}\in X$, is given as

$$x_k^{(1)}=\frac{a_{k0}+\sum_{l=1}^n a_{kl}x_l^{(0)}}{a_{00}+\sum_{l=1}^n a_{0l}x_l^{(0)}}\,.$$

In this paper, the process of singularization will be applied to the following algorithm:

**Definition** (Brun 1957)**.** Let $M:=\{(b_0,b_1,b_2):b_0\ge b_1\ge b_2\ge 0\}$. Brun's Algorithm is generated by the map $\tau_S:M\to M$, where

$$\tau_S(b_0,b_1,b_2)=\begin{cases}(b_0-b_1,b_1,b_2), & b_0-b_1\ge b_1 \qquad (j=0)\,,\\ (b_1,b_0-b_1,b_2), & b_1\ge b_0-b_1\ge b_2 \quad (j=1)\,,\\ (b_1,b_2,b_0-b_1), & b_2\ge b_0-b_1 \qquad (j=2)\,.\end{cases}$$

Let $X_B:=\{(x_1,x_2):1\ge x_1\ge x_2\ge 0\}$; using the projection map $p:M\to X_B$, defined by

$$p(b_0,b_1,b_2)=\Big(\frac{b_1}{b_0},\frac{b_2}{b_0}\Big)\,,$$

we obtain the corresponding two-dimensional map $T_S:X_B\to X_B$,

$$T_S(x_1,x_2)=\begin{cases}(\frac{x_1}{1-x_1},\frac{x_2}{1-x_1}), & 1-x_1\ge x_1 \qquad (j=0)\,,\\ (\frac{1-x_1}{x_1},\frac{x_2}{x_1}), & x_1\ge 1-x_1\ge x_2 \quad (j=1)\,,\\ (\frac{x_2}{x_1},\frac{1-x_1}{x_1}), & x_2\ge 1-x_1 \qquad (j=2)\,.\end{cases}$$

We refer to $j$ as the *type* of the algorithm. Denote

$$\begin{aligned}X_B(0)&:=\{(x_1,x_2)\in X_B:j(x_1,x_2)=0\}\,,\\ X_B(1)&:=\{(x_1,x_2)\in X_B:j(x_1,x_2)=1\}\,,\\ X_B(2)&:=\{(x_1,x_2)\in X_B:j(x_1,x_2)=2\}\,.\end{aligned}$$

Further, for $t\ge 1$, define $j^{(t)}=j^{(t)}(x_1^{(0)},x_2^{(0)}):=j(T_S^{t-1}(x_1^{(0)},x_2^{(0)}))$. The cylinders $X_B(j^{(1)},\dots,j^{(t)})$ of the fibred system $(X_B,T_S)$ are *full*, *i.e.*, $T_S^tX_B(j^{(1)},\dots,j^{(t)})=X_B$. The algorithm is ergodic and conservative with respect to Lebesgue measure (see Theorem 21 in [23], p. 50).

Let $t\ge 1$; the matrices $\alpha_S^{(t)}:=\alpha_S(j^{(t)})$ of Brun's Algorithm are given as

$$\alpha_S(0)=\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix},\quad \alpha_S(1)=\begin{pmatrix}0&1&0\\1&-1&0\\0&0&1\end{pmatrix},\quad \alpha_S(2)=\begin{pmatrix}0&1&0\\0&0&1\\1&-1&0\end{pmatrix}.$$

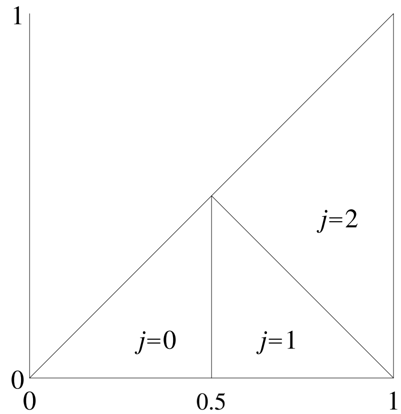


FIGURE 1. The time-1-partition of Brun's Algorithm $T_S$

The inverses $\beta_S^{(t)} := \beta_S(j^{(t)})$ of the matrices of the algorithm with

$$\beta_S(0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta_S(1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta_S(2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

produce a sequence of convergence matrices $\{\Omega_S^{(s)}\}_{s=0}^{\infty}$ as follows:

**Definition.** Let $E$ be the identity matrix. Then

$$\Omega_S^{(0)} = \begin{pmatrix} q^{(0)} & q^{(0')} & q^{(0'')} \\ p_1^{(0)} & p_1^{(0')} & p_1^{(0'')} \\ p_2^{(0)} & p_2^{(0')} & p_2^{(0'')} \end{pmatrix} := E$$

$$\Omega_S^{(t)} = \begin{pmatrix} q^{(t)} & q^{(t')} & q^{(t'')} \\ p_1^{(t)} & p_1^{(t')} & p_1^{(t'')} \\ p_2^{(t)} & p_2^{(t')} & p_2^{(t'')} \end{pmatrix} := \Omega_S^{(t-1)}\beta_S^{(t)}.$$

Hence, for $k = 1, 2$, $(x_1^{(t)}, x_2^{(t)}) = T_S^t(x_1^{(0)}, x_2^{(0)})$,

$$x_k^{(0)} = \frac{p_k^{(t)} + x_1^{(t)} p_k^{(t')} + x_2^{(t)} p_k^{(t'')}}{q^{(t)} + x_1^{(t)} q^{(t')} + x_2^{(t)} q^{(t'')}}.$$

The columns of the convergence matrices produce Diophantine approximations $(p_1^{(t)}/q^{(t)}, p_2^{(t)}/q^{(t)})$ to $(x_1^{(0)}, x_2^{(0)})$. Similar to the above, $j$ is referred to as the *type* of a matrix $\beta_S(j)$

## 3. The process of Singularization

The basic idea of singularization, as introduced by C. Kraaikamp [9], was to improve approximation properties of the (one-dimensional) regular continued fraction algorithm. In particular, C. Kraaikamp was interested in semi-regular continued fraction algorithms, whose sequences of convergents $\{p^{(t)}/q^{(t)}\}_{t=1}^{\infty}$ were subsequences of the sequence of regular convergents of $x$. To construct these algorithms, he introduced the *process of singularization*, which further led to the definition of a new class of semi-regular continued fraction algorithms, the *S-expansions*.

The process is defined by a *law of singularization* which, to a given continued fraction algorithm (or a class of such algorithms), determines in an unambiguous way the convergents to be singularized by using some specific *matrix identities*.

We give an example for Brun's Algorithm in two dimensions. The following matrix identities are easily checked (for an arbitrary $t$, $\phi_t$ and $\psi_t$ are defined such that either $\phi_t = 1$ and $\psi_t = 0$, or $\phi_t = 0$ and $\psi_t = 1$):

*type* $M_1$*:*

$$\begin{pmatrix} 1 & A_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & A_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & A_1 + A_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

*type* $M_2$*:*

$$\begin{pmatrix} 1 & A_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & \phi_2 & \psi_2 \\ 1 & 0 & 0 \\ 0 & \psi_2 & \phi_2 \end{pmatrix} = \begin{pmatrix} A_1 + A_2 & \phi_2 & \psi_2 \\ 1 & 0 & 0 \\ 0 & \psi_2 & \phi_2 \end{pmatrix}.$$

Based on these identities, we remove any matrix $\beta_S^{(t)}$ from the sequence of inverse matrices if $j^{(t)} = 0$. The matrix $\beta^{(t+1)}$ should then be replaced according to the above rule. Thus a new sequence of convergence matrices $\{\Omega_S^{*(s)}\}_{s=0}^{\infty}$ is obtained by removing $\Omega_S^{(t)}$ from $\{\Omega_S^{(s)}\}_{s=0}^{\infty}$. Clearly, the sequence of Diophantine approximations $\{(p_1^{*(s)}/q^{*(s)}, p_2^{*(s)}/q^{*(s)})\}_{s=0}^{\infty}$ obtained from the new convergence matrices is a subsequence of the original one.

Now we apply the same procedure to any remaining matrix of type 0, and continue until all such matrices have been removed. That way, a new algorithm is defined. This transformation of the original algorithm into a new one is called a *singularization*. We put this into a more general form:

**Definition.** A transformation $\sigma_t$, defined by a matrix identity that removes the matrix $\beta^{(t)}$ from the sequence of inverse matrices (which changes an algorithm into a new form such that the sequence of Diophantine approximations $\{(p_1^{*(s)}/q^{*(s)}, p_2^{*(s)}/q^{*(s)})\}_{s=0}^{\infty}$ obtained from the new algorithm is a subsequence of the original one) is called a *singularization*. We say we have *singularized* the matrix $\beta^{(t)}$.

By the definition, the sequence of convergents of the singularized algorithm is a subsequence of the sequence of convergents of the original one. Therefore, if the original algorithm converges to $(x_1, x_2)$ so does the new one. Now we define the *exponent of convergence* as the supremum of real numbers $d$ such that for almost all $(x_1, x_2)$ and all $t$ large enough, the inequalities

$$\left| x_k - \frac{p_k^{(t)}}{q^{(t)}} \right| \leq \frac{1}{(q^{(t)})^{1+d}} \quad (k = 1, 2)$$

hold. Notice that the exponent of convergence of the singularized algorithm is always larger or equal to the one of the original algorithm.

We generalize the definition of matrices $\beta_S(j)$, $j = 0, 1, 2$, to

$$\beta_M(0, A) = \begin{pmatrix} 1 & A & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta_M(1, A) = \begin{pmatrix} A & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\beta_M(2, A) = \begin{pmatrix} A & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We may thus define a law of singularization LM* for Brun's Algorithm, to obtain its multiplicative acceleration $T_M$ (a different, but equivalent rule LM will be introduced Section 4).

**Law of singularization LM*:** *Singularize every matrix $\beta_M(0, A)$, using identities $M_1$ and $M_2$.*

Consider a block of successive matrices of type 0. Note that matrix identities $M_1$ and $M_2$ allow singularizations of these matrices in an arbitrary order, yielding the same algorithm, as long as we remove every such matrix.

The resulting algorithm is the well-known multiplicative acceleration of Brun's Algorithm $T_M : X_B \to X_B$,

$$T_M(x_1, x_2) = \begin{cases} (\frac{1}{x_1} - A, \frac{x_2}{x_1}), & \frac{1}{x_1} - A \geq \frac{x_2}{x_1} \quad (j = 1) \\ (\frac{x_2}{x_1}, \frac{1}{x_1} - A), & \frac{x_2}{x_1} \geq \frac{1}{x_1} - A \quad (j = 2) \end{cases}, \quad A := [\frac{1}{x_1}]$$

(compare [23], p. 48ff). All matrices of type 0 have been removed, and the new partition is defined by the types $j = 1, 2$ and the *partial quotients* $A \in \mathbb{N}$ of the algorithm.

In particular, we denote

$$X_M(1) := \{(x_1, x_2) \in X_B : j(x_1, x_2) = 1\},$$
$$X_M(2) := \{(x_1, x_2) \in X_B : j(x_1, x_2) = 2\}.$$

Similarly to the above, for $t \geq 1$, $j^{(t)} := j(T_M^{t-1}(x_1^{(0)}, x_2^{(0)}))$ and $A^{(t)} := A(T_M^{t-1}(x_1^{(0)}, x_2^{(0)}))$. The cylinders are defined by the pairs $(j^{(t)}, A^{(t)})$, while the inverses $\beta_M^{(t)} := \beta_M(j^{(t)}, A^{(t)})$, as well as the convergence matrices $\Omega_M^{(t)}$, are defined as above.

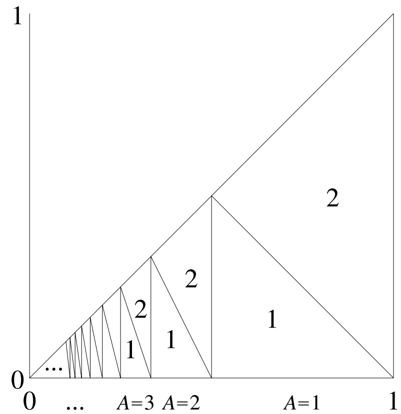


FIGURE 2. The time-1-partition of the multiplicative acceleration of Brun's Algorithm

## 4. The natural extension $(\overline{X}, \overline{T})$

Following [7], we may describe the law of singularization in defining a *singularization area i.e.*, a set $S$ such that every $x \in S$ specifies a matrix $\beta$ to be singularized. To describe $S$ we use the *natural extension* of a fibred system, introduced by Nakada, Ito and Tanaka [16] (see also [15] and [3]) but we follow F. Schweiger ([23], p. 22f).

**Definition.** Let $(X, T)$ be a multidimensional continued fraction algorithm. A fibred system $(X^{\#}, T^{\#})$ is called a *dual algorithm* if

(1) $(i^{(1)}, \ldots, i^{(t)})$ is an admissible block of digits for $T$ if and only if $(i^{(t)}, \ldots, i^{(1)})$ is an admissible block of digits for $T^{\#}$;
(2) there is a partition $X^{\#}(i)$ such that the matrices $\alpha^{\#}(i)$ of $T^{\#}$ restricted to $X^{\#}(i)$ are the transposed matrices of $\alpha(i)$.

Similar to the above, let $V^{\#}(i) : T^{\#}X^{\#}(i) \to X^{\#}(i)$ denote the local inverse branches of $T^{\#}$.

**Definition.** For any $x \in X$, the *polar set* $D(x)$ is defined as follows:

$$D(x) := \{y \in X^{\#} : x \in \bigcap_{t=1}^{\infty} T^t X(i^{(t)}(y), \ldots, i^{(1)}(y))\}\,.$$

Let $y \in X^{\#}(i^{(1)}, \ldots, i^{(t)})$. By the definition, $y \in D(x)$ if and only if $V(i^{(t)}, \ldots, i^{(1)})(x)$ is well defined for all $t$. In particular, if all cylinders are full, then $D(x) = X^{\#}$.

**Definition.** The dynamical system $(\overline{X}, \overline{T})$, where $\overline{X} := \{(x, y) : x \in X, y \in D(x)\}$ and

$$\overline{T} : \overline{X} \to \overline{X}, \quad \overline{T}(x, y) = (T(x), V^{\#}(i(x))(y))$$

is called a *natural extension* of $(X, T)$.

**Definition.** Let $t \geq 1$. A *singularization area* is a set $S \subset \overline{X}$ such that, for some fixed $k$, $\beta^{(t+k)}$ should be singularized if and only if $(x^{(t)}, y^{(t)}) \in S$.

**Remark.** In theory, the singularization area could be chosen arbitrarily. However, since the process is based on some matrix identities which have an effect on the remaining matrices, there are some restrictions similar to the ones described in [7] (section 4.2.3). Since we are more 'liberal' in the sense that, throughout this paper, several matrix identities will be used, there is no such general description of these restraints.

In case of Brun's Algorithm, we consider the fibred system $(X_B, T_S)$ from above. A dual system can be described as follows. Let

$$X_S^{\#} := \{(y_1, y_2) : 0 \leq y_1; 0 \leq y_2 \leq 1\}$$

and set in particular

$$\begin{aligned} X_S^{\#}(0) &:= \{(y_1, y_2) \in X_S^{\#} : 1 \leq y_1\}\,, \\ X_S^{\#}(1) &:= \{(y_1, y_2) : 1 \geq y_1 \geq y_2 \geq 0\}\,, \\ X_S^{\#}(2) &:= \{(y_1, y_2) : 1 \geq y_2 \geq y_1 \geq 0\}\,. \end{aligned}$$

Define $V_S^{\#} : X_S^{\#} \to X_S^{\#}$,

$$V_S^{\#}(j)(y_1, y_2) = \begin{cases} (1 + y_1, y_2) & (j = 0)\,, \\ (\frac{1}{1+y_1}, \frac{y_2}{1+y_1}) & (j = 1)\,, \\ (\frac{y_2}{1+y_1}, \frac{1}{1+y_1}) & (j = 2)\,, \end{cases}$$

$\overline{X}_S := X_B \times X_S^{\#}$, and finally $\overline{T}_S : \overline{X}_S \to \overline{X}_S$,

$$\overline{T}_S(x_1, x_2, y_1, y_2) = (T_S(x_1, x_2), V_S^{\#}(j(x_1, x_2))(y_1, y_2)).$$

Then $(\overline{X}_S, \overline{T}_S)$ is a natural extension of $(X_B, T_S)$. We now define the singularization area

$$S_M := X_B \times ([1, \infty) \times [0, 1]),$$

and thus restate the law of singularization LM*:

**Law of singularization LM:** *Singularize* $\beta_S^{(t)}$ *if and only if*

$$(x_1^{(t)}, x_2^{(t)}, y_1^{(t)}, y_2^{(t)}) \in S_M,$$

*using matrix identities* $M_1$ *and* $M_2$.

By the definition of $S_M$, the laws of singularization LM and LM* are equivalent.

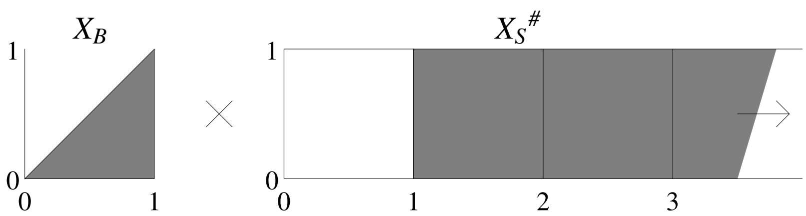


FIGURE 3. The singularization area $S_M$

**Remark.** The matrix identities $M_1$ and $M_2$, and thus the law of singularization LM, slightly differ from the process of singularization described in [7] and [9] (compare matrix identity $1_1$ given in Section 5). Nevertheless, there exists a relation similar to LM between the one-dimensional regular continued fraction algorithm and the *Lehner expansions* [11]. Lehner expansions are generated by a map isomorphic to Brun's Algorithm $T_B : [0,1) \to [0,1)$,

$$T_B(x) = \begin{cases} \frac{x}{1-x}, & 0 \leq x < \frac{1}{2}\,; \\ \frac{1-x}{x}, & \frac{1}{2} \leq x < 1\,. \end{cases}$$

This relation, again based on ideas similar to singularization, is described in Dajani and Kraaikamp [5] (see also Ito [8]).

## 5. Eliminating partial quotients $A^{(t)} = 1$, where $j^{(t)} = 1$

From now on, we assume that the law of singularization LM has already been applied to Brun's Algorithm $T_S$ *i.e.,* in the following, we consider the resulting multiplicative acceleration of Brun's Algorithm $T_M$.

We are now going to define another singularization process: The Singularization of matrices $\beta_M(1,1)$, which will lead to a new algorithm $T_1$ with better approximation properties (as opposed to both $T_S$ and $T_M$). We use the following identities:

*type* $1_1$*:*

$$\begin{pmatrix} A_1 & 1 & 0 \\ \epsilon_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_3 & \phi_3 & \psi_3 \\ 1 & 0 & 0 \\ 0 & \psi_3 & \phi_3 \end{pmatrix}$$
$$= \begin{pmatrix} A_1+1 & 1 & 0 \\ \epsilon_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_3+1 & \phi_3 & \psi_3 \\ -1 & 0 & 0 \\ 0 & \psi_3 & \phi_3 \end{pmatrix}.$$

*type* $1_2$*:*

$$\begin{pmatrix} A_1 & 0 & 1 \\ \epsilon_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_3 & \phi_3 & \psi_3 \\ 1 & 0 & 0 \\ 0 & \psi_3 & \phi_3 \end{pmatrix}$$
$$= \begin{pmatrix} A_1 & 0 & 1 \\ \epsilon_1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_3+1 & \phi_3 & \psi_3 \\ -1 & 0 & 0 \\ 0 & \psi_3 & \phi_3 \end{pmatrix}.$$

**Remark.** The matrix identity $1_1$ corresponds to the identity used in [7] and [9] to define the singularization process for the one-dimensional (regular) continued fraction algorithm.

Define, for $\epsilon \in \{-1, 1\}$, the matrices $\beta_1(j, A, \epsilon, C)$ as

$$\beta_1(1, A, \epsilon, C) = \begin{pmatrix} A & 1 & 0 \\ \epsilon & 0 & 0 \\ C & 0 & 1 \end{pmatrix}, \quad \beta_1(2, A, \epsilon, C) = \begin{pmatrix} A & 0 & 1 \\ \epsilon & 0 & 0 \\ C & 1 & 0 \end{pmatrix}.$$

Consider a block of pairs of digits $((j^{(t)}, A^{(t)}), (1, 1), (j^{(t+2)}, A^{(t+2)}))$. If we singularize $\beta_M^{(t+1)}$, then $\beta_M^{(t)}$ will be replaced by $\beta_1(1, A^{(t)} + 1, 1, 0)$ (if $j^{(t)} = 1$), or by $\beta_1(2, A^{(t)}, 1, 1)$ (if $j^{(t)} = 2$). The matrix $\beta_M^{(t+2)}$ will be replaced by $\beta_1(j^{(t+2)}, A^{*(t+2)} + 1, -1, 0)$. Hence, even if $A^{(t)} = 1$ and $j^{(t)} = 1$, or $A^{(t+2)} = 1$ and $j^{(t+2)} = 1$, we cannot singularize one of the resulting matrices using the above identities. In other words, from every block of consecutive matrices $\beta_M(1, 1)$ we can only singularize every other matrix.

We thus have to specify which of the matrices of such blocks should be singularized, and this choice determines the outcome, as it can be seen easily with the following example: Let $\{\beta_M^{(s)}\}_{s=0}^{\infty}$ be a sequence of matrices specified by the pairs of digits $((j^{(1)}, A^{(1)}), \ldots, (j^{(t)}, A^{(t)}), (1, 1), (1, 1), (1, 1), (j^{(t+4)}, A^{(t+4)}))$, where both $A^{(t)} \neq 1$ and $A^{(t+4)} \neq 1$. Singularizing $\beta_M^{(t+2)}$ obviously yields a different subsequence of $\{\beta_M^{(s)}\}_{s=0}^{\infty}$, and thus a different algorithm, than singularizing both $\beta_M^{(t+1)}$ and $\beta_M^{(t+3)}$. The class of S-expansions for the regular one-dimensional continued fraction algorithm was obtained in giving different laws of singularization *i.e.*, different choices of matrices to be singularized, for one single matrix identity. For example, the following law of singularization can be considered:

**Law of singularization L1*:** *From every block of $n$ consecutive matrices $\beta_M(1, 1)$, singularize the first, the third,... matrices, using identities $1_1$ and $1_2$.*

**Remark.** Due to the nature of matrix identities $1_1$ and $1_2$, we may not apply L1* until the first index $s$ with $j^{(s)} \neq 1$ or $A^{(s)} \neq 1$. Strictly speaking, L1* is only valid for matrices $\beta_M^{(t)}(1, 1)$ where $t > t_0$, and $t_0 := \min \{s : j^{(s)} = 2 \text{ or } A^{(s)} > 1\}$. Similar restrictions will be true for all laws of singularization proposed from now on.

Using the natural extension $(\overline{X}_M, \overline{T}_M)$ of $(X_B, T_M)$, where

$$X_M^{\#} := \{(y_1, y_2) : 0 \leq y_1 \leq 1; 0 \leq y_2 \leq 1\},$$

$$\begin{aligned} X_M^{\#}(1) &:= \{(y_1,y_2)\in X_M^{\#} : y_1 \geq y_2\}\,,\\ X_M^{\#}(2) &:= \{(y_1,y_2)\in X_M^{\#} : y_2 \geq y_1\}\,,\\ \overline{X}_M &:= X_B \times X_M^{\#}\,, \end{aligned}$$

and $V_M^{\#} : X_M^{\#} \to X_M^{\#}$,

$$V_M^{\#}(j,A)(y_1,y_2) := \begin{cases} (\frac{1}{A+y_1}, \frac{y_2}{A+y_1}) & (j=1)\,,\\ (\frac{y_2}{A+y_1}, \frac{1}{A+y_1}) & (j=2)\,, \end{cases}$$

such that $\overline{T}_M : \overline{X}_M \to \overline{X}_M$ is defined by

$$\overline{T}_M(x_1,x_2,y_1,y_2) = (T_M(x_1,x_2), V_M^{\#}(j(x_1,x_2),A(x_1,x_2))(y_1,y_2))\,.$$

Again, we may restate L1* in terms of a singularization area $S_1 \subset \overline{X}_M$. We use $V_M^{\#}$ to control the preceding pairs of digits $(j^{(t-1)}, A^{(t-1)})$, $(j^{(t-2)}, A^{(t-2)}), \ldots$ Denote $f_i$ the $i^{th}$ term of *Fibonacci's sequence* with initial terms $f_0 = 0$, $f_1 = 1$, and let $\Delta(P_1,P_2,P_3)$ be the triangle defined by the vertices $P_1$, $P_2$ and $P_3$. Then

$$\begin{aligned} S_1 := X_B(1) \times \Big( &\bigcup_{i=0}^{\infty} \Delta((\frac{f_{2i}}{f_{2i+1}},0),(\frac{f_{2i}}{f_{2i+1}},\frac{1}{f_{2i+1}}),(\frac{f_{2i+1}}{f_{2i+2}},\frac{1}{f_{2i+2}}))\\ &\cup \bigcup_{i=0}^{\infty} \Delta((\frac{f_{2i}}{f_{2i+1}},0),(\frac{f_{2i+2}}{f_{2i+3}},0),(\frac{f_{2i+2}}{f_{2i+3}},\frac{1}{f_{2i+3}}))\Big)\,. \end{aligned}$$

**Law of singularization L1:** *Singularize $\beta_M^{(t)}$ if and only if*

$$(x_1^{(t-1)}, x_2^{(t-1)}, y_1^{(t-1)}, y_2^{(t-1)}) \in S_1\,,$$

*using identities $1_1$ or $1_2$, accordingly.*

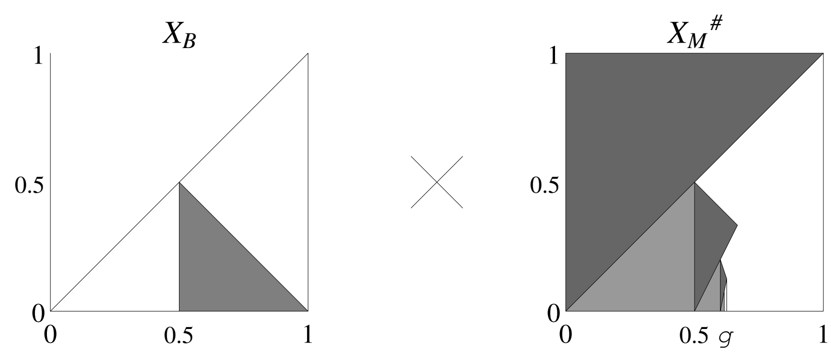


FIGURE 4. The singularization area $S_1$ ($g = \frac{\sqrt{5}-1}{2}$)

Thus L1 is equivalent to L1*. The singularization yields a new algorithm, which acts on the following set (compare Figure 5):

$$X_1 := \{(x_1,x_2) : 0 \le x_2 \le |x_1| \le 1/2\} \\ \cup\ \{(x_1,x_2) : 1/2 \le x_1 \le 1, 1-x_1 \le x_2 \le x_1\}$$

or, more precisely,

$$X_1(1) := \{(x_1,x_2) \in X_1 : \frac{x_2}{|x_1|} \le \frac{1}{|x_1|} - A \le \frac{1}{2}\} \\ \cup \left\{ (x_1,x_2) \in X_1 \ : \begin{array}{l} \max\{\frac{1}{2}, \frac{x_2}{|x_1|}\} \le \frac{1}{|x_1|} - A \\ A+1-\frac{1}{|x_1|} \le \frac{x_2}{|x_1|} \end{array} \right\},$$

$$X_1(2) := \{(x_1,x_2) \in X_1 : \frac{1}{|x_1|} - A \le \frac{x_2}{|x_1|} \le \frac{1}{2}\} \\ \cup\ \{(x_1,x_2) \in X_1 : \max\{\frac{1}{2}, \frac{1}{|x_1|} - A, (A+1) - \frac{1}{|x_1|}\} \le \frac{x_2}{|x_1|}\},$$

$$X_1(3) := \{(x_1,x_2) \in X_1 : \max\{\frac{1}{2}, \frac{x_2}{|x_1|}\} \le \frac{1}{|x_1|} - A \,\&\, \frac{x_2}{|x_1|} \le A+1-\frac{1}{|x_1|}\} \\ = \bigcup_{A=2}^{\infty} \Delta((\frac{1}{A+1},0),(\frac{2}{2A+1},0),(\frac{2}{2A+1},\frac{1}{2A+1})) \\ \cup \bigcup_{A=2}^{\infty} \Delta((-\frac{1}{A+1},0),(-\frac{2}{2A+1},0),(-\frac{2}{2A+1},\frac{1}{2A+1})),$$

$$X_1(4) := \{(x_1,x_2) \in X_1 : \max\{\frac{1}{2}, \frac{1}{|x_1|} - A\} \le \frac{x_2}{|x_1|} \le A+1-\frac{1}{|x_1|}\} \\ = \bigcup_{A=1}^{\infty} \Delta((\frac{1}{A},\frac{1}{A}),(\frac{1}{A},\frac{1}{2A}),(\frac{2}{2A+1},\frac{1}{2A+1})) \\ \cup \bigcup_{A=2}^{\infty} \Delta((-\frac{1}{A},\frac{1}{A}),(-\frac{1}{A},\frac{1}{2A}),(-\frac{2}{2A+1},\frac{1}{2A+1})).$$

The resulting algorithm $T_1 : X_1 \to X_1$ (the one defined by the new matrices $\beta_1^{(t)}$) can be described as follows:

$$T_1(x_1,x_2) = \begin{cases} (\frac{1}{|x_1|} - A, \frac{x_2}{|x_1|}) & (j=1), \\ (\frac{x_2}{|x_1|}, \frac{1}{|x_1|} - A) & (j=2), \\ (\frac{1}{|x_1|} - (A+1), \frac{x_2}{|x_1|}) & (j=3), \\ (\frac{x_2}{|x_1|} - 1, \frac{1}{|x_1|} - A) & (j=4), \end{cases}$$

$$(x_1,x_2) \in X_1(j), \quad A := [\frac{1}{|x_1|}].$$

We may further define an algorithm $V_1^{\#} : X_1^{\#} \to X_1^{\#}$, where

$$
\begin{aligned}
X_1^{\#}(1) &:= \{(y_1, y_2) : 0 \le y_2 \le y_1 \le \frac{1}{2}\}\,,\\
X_1^{\#}(2) &:= X_M^{\#}(2)\,,\\
X_1^{\#}(3) &:= \{(y_1, y_2) : 0 \le y_2 \le -y_1 \le \frac{1}{3}\}\\
&\qquad \cup \bigcup_{i=1}^{\infty} \Delta((-\frac{f_{2i+2}}{f_{2i+4}}, 0), (-\frac{f_{2i+2}}{f_{2i+4}}, \frac{1}{f_{2i+4}}), (-\frac{f_{2i}}{f_{2i+2}}, 0))\\
&\qquad \cup \bigcup_{i=1}^{\infty} \Delta((-\frac{f_{2i+1}}{f_{2i+3}}, \frac{1}{f_{2i+3}}), (-\frac{f_{2i}}{f_{2i+2}}, \frac{1}{f_{2i+2}}), (-\frac{f_{2i}}{f_{2i+2}}, 0))\,,\\
X_1^{\#}(4) &:= \{(y_1, y_2) : 0 \le -y_1 \le \min\{y_2, 1 - y_2\} \,\&\, y_2 \le 1\},
\end{aligned}
$$

$$
X_1^{\#} := \bigcup_{i=1}^{4} X_1^{\#}(i),
$$

$$
V_1^{\#}(j, A)(y_1, y_2) = \begin{cases} (\frac{1}{A+y_1}, \frac{y_2}{A+y_1}) & (j = 1)\,,\\ (\frac{y_2}{A+y_1}, \frac{1}{A+y_1}) & (j = 2)\,,\\ (-\frac{1}{A+1+y_1}, \frac{y_2}{A+1+y_1}) & (j = 3)\,,\\ (-\frac{y_2}{A+y_1+y_2}, \frac{1}{A+y_1+y_2}) & (j = 4)\,. \end{cases}
$$

Since the cylinders of the new algorithm are not full, we verify that

$$
D_1(x_1, x_2) = \begin{cases} X_1^{\#}(1) \cup X_1^{\#}(2) & \text{if } 0 \le x_1,\\ X_1^{\#}(3) \cup X_1^{\#}(4) & \text{if } x_1 \le 0 \end{cases}
$$

*i.e.*, whenever $x_1 \le 0$, so is $y_1$ (and conversely). Thus $D_1(x_1, x_2)$ is not empty. Finally, we put (see Figure 5):

$$
\begin{aligned}
\overline{X}_1 := &\{(x_1, x_2) \in X_1 : 0 \le x_1\} \times (X_1^{\#}(1) \cup X_1^{\#}(2))\\
&\cup\ \{(x_1, x_2) \in X_1 : x_1 \le 0\} \times (X_1^{\#}(3) \cup X_1^{\#}(4))
\end{aligned}
$$

and

$$
\overline{T}_1(x_1, x_2, y_1, y_2) = (T_1(x_1, x_2), V_1^{\#}(j(x_1, x_2), A(x_1, x_2))(y_1, y_2))
$$

to obtain the system $(\overline{X}_1, \overline{T}_1)$.

## 6. The ergodic system connected with the natural extension

Consider the fibred system $(X_B, T_M)$ and its natural extension $(\overline{X}_M, \overline{T}_M)$. Let $\Sigma_M$ be the $\sigma$-algebra generated by the cylinders of $\overline{X}_M$. The multiplicative acceleration of Brun's Algorithm is known to be ergodic and

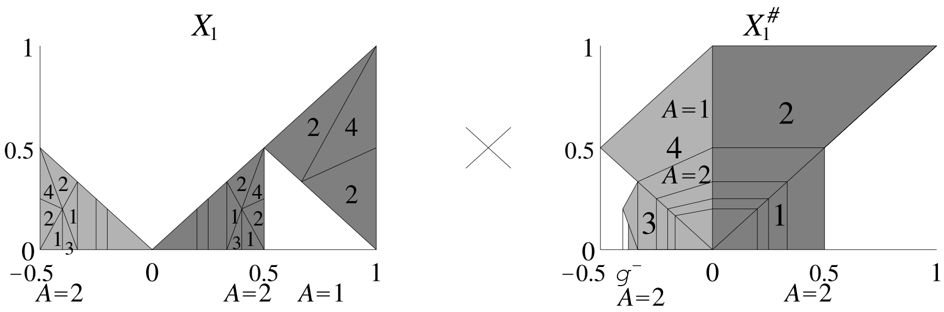


FIGURE 5. The set $\overline{X}_1$ ($g^- = g-1$)

conservative, and it admits an invariant probability measure $\mu_M$, whose density is given as

$$\frac{1}{C_M}\,\frac{1}{(1+x_1y_1+x_2y_2)^3}$$

(see e.g. Schweiger [20] or Arnoux and Nogueira [1]), where $C_M \approx 0,19$. Define

$$\begin{aligned} S_1^C &:= \overline{X}_M \setminus S_1\,, \\ S_1^+ &:= S_1^C \setminus \overline{T}_M S_1\,, \\ N_1 &:= \overline{T}_1\, \overline{T}_M^{-1} S_1\,. \end{aligned}$$

Note that, by the definitions $N_1 \cap \overline{X}_M = \emptyset$ and $\overline{X}_1 = S_1^+ \cup N_1$. Next we define a transformation $\varsigma_1$ that 'jumps' over the singularization area $S_1$.

**Definition.** The transformation $\varsigma_1 : S_1^C \to S_1^C$ is defined by

$$\varsigma_1(x_1,x_2,y_1,y_2) = \begin{cases} \overline{T}_M(x_1,x_2,y_1,y_2), & (x_1,x_2,y_1,y_2) \in S_1^C \setminus \overline{T}_M^{-1} S_1\,, \\ \overline{T}_M^2(x_1,x_2,y_1,y_2), & (x_1,x_2,y_1,y_2) \in \overline{T}_M^{-1} S_1\,. \end{cases}$$

Using the theory of jump transformations (see e.g. [22]), this yields an ergodic system $(S_1^C, \Sigma_{S_1^C}, \mu_{S_1^C}, \varsigma_1)$, where $\Sigma_{S_1^C}$ is the restriction of $\Sigma_M$ to $S_1^C$ and $\mu_{S_1^C}$ is the probability induced by $\mu_M$ on $\Sigma_{S_1^C}$. Notice that $C_{S_1^C} := \mu_M(S_1^C) \approx 0,78$. Now we may identify the set $N_1$ with $\overline{T}_M S_1$ by a bijective map $M_1 : S_1^C \to \overline{X}_1$, where $M_1 \overline{T}_M S_1 = N_1$, while $M_1$ is the identity on $S_1^+$:

**Definition.** The map $M_1 : S_1^C \to \overline{X}_1$ is defined by,

$$M_1(x_1,x_2,y_1,y_2) = \begin{cases} (x_1,x_2,y_1,y_2), & (x_1,x_2,y_1,y_2) \in S_1^+\,, \\ (-\frac{x_1}{1+x_1}, \frac{x_2}{1+x_1}, y_1-1, y_2), & (x_1,x_2,y_1,y_2) \in \overline{T}_M S_1\,. \end{cases}$$

We may illustrate the relations between $S_1$, $\overline{T}_M S_1$ and $M_1\overline{T}_M S_1$ with two sets $E_1 \in S_1$, $E_2 \in S_1$, where $E_1$ is defined by the block of pairs of digits ((1, 2), (1, 1), (1, 2)) (*i.e.*, $E_1 = \{(x_1, x_2, y_1, y_2) : (j^{(0)}, A^{(0)}) = (1,2), (j^{(1)}, A^{(1)}) = (1,1), (j^{(2)}, A^{(2)}) = (1,2)\}$), while $E_2$ is defined by ((2, 2), (1, 1), (2, 2)).

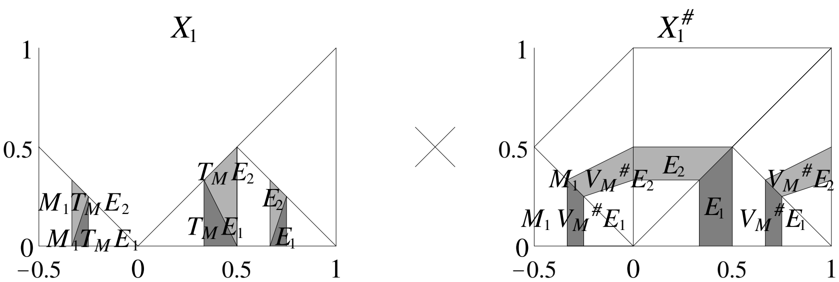


FIGURE 6. Evolution of the sets $E_1 \subset S_1$ and $E_2 \subset S_1$

We get the following

**Theorem 6.1.** *Consider* $\tau_1 : \overline{X}_1 \to \overline{X}_1$, $\tau_1(x_1, x_2, y_1, y_2) = M_1 \varsigma_1 M_1^{-1}$. *Then* $(\overline{X}_1, \Sigma_1, \mu_1, \tau_1)$ *is an ergodic dynamical system, where* $\Sigma_1$ *is the* $\sigma$-*algebra generated by the cylinders of* $\overline{X}_1$, $\mu_1$ *is the probability measure with density function*

$$\frac{1}{C_1}\,\frac{1}{(1 + |x_1|y_1 + x_2y_2)^3},$$

$C_1 = C_M C_{S_1^C} \approx 0,15$, *and for all* $(x_1, x_2, y_1, y_2) \in \overline{X}_1$,

$$\tau_1(x_1, x_2, y_1, y_2) = \overline{T}_1(x_1, x_2, y_1, y_2).$$

## 7. A cyclic version of the algorithm

Consider the multiplicative acceleration of Brun's Algorithm $T_M$ as described above and, for $t$ large enough, the convergence matrix

$$\Omega_M^{(t)} = \begin{pmatrix} q^{(t)} & q^{(t')} & q^{(t'')} \\ p_1^{(t)} & p_1^{(t')} & p_1^{(t'')} \\ p_2^{(t)} & p_2^{(t')} & p_2^{(t'')} \end{pmatrix}$$

*i.e.*, the approximations $(p_1^{(t)}/q^{(t)}, p_2^{(t)}/q^{(t)})$, $(p_1^{(t')}/q^{(t')}, p_2^{(t')}/q^{(t')})$ and $(p_1^{(t'')}/q^{(t'')}, p_2^{(t'')}/q^{(t'')})$ to some $(x_1^{(0)}, x_2^{(0)}) \in X_B$ generated by the algorithm. Define $P_M^{(s)} := (p_1^{(s)}/q^{(s)}, p_2^{(s)}/q^{(s)})$, then

$$(x_1^{(0)}, x_2^{(0)}) \in \Delta(P_M^{(t)}, P_M^{(t')}, P_M^{(t'')}).$$

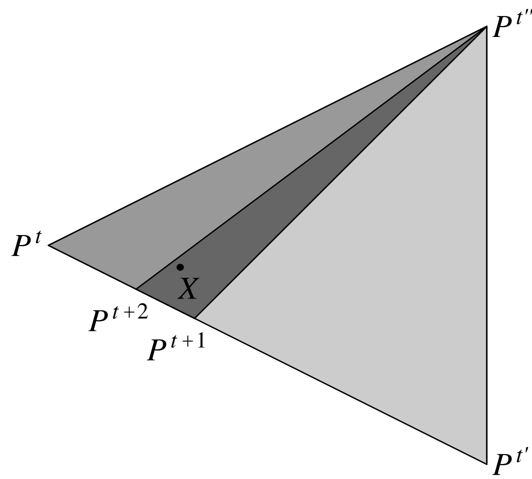


FIGURE 7. Example for an approximation where $j^{(t+1)} = 1$

Let $\Gamma(P_1, P_2)$ be defined as the line segment between the points $P_1$ and $P_2$. We observe that, by construction of the approximations, $P_M^{(t+1)} \in \Gamma(P_M^{(t)}, P_M^{(t')})$. Further, as long as $j^{(t+1)} = 1, \ldots, j^{(t+i)} = 1$ for some $i \geq 1$, then $P_M^{(t+2)}, \ldots, P_M^{(t+i+1)}$ lie on that same line segment. In particular, $P_M^{(t+i+1)} \in \Gamma(P_M^{(t+i)}, P_M^{(t+i-1)}) \subset \Gamma(P_M^{(t)}, P_M^{(t')})$. Thus the approximation triangles $\Delta(P_M^{(t+i+1)}, P_M^{((t+i+1)')}, P_M^{((t+i+1)'')})$ get very 'long' *i.e.*, the vertex $P_M^{(t'')}$ is not replaced until some $j^{(t+l)} = 2$, $l > i$ (Fig. 7).

On the other hand, if $j^{(t+1)} = 2$, then $P_M^{(t+2)} \in \Gamma(P_M^{(t+1)}, P_M^{(t'')})$, and both $P_M^{(t')}$ and $P_M^{(t'')}$ have been replaced with $P_M^{(t+1)}$ and $P_M^{(t+2)}$, respectively. We call this a *cyclic* approximation.

We are now going to construct an algorithm that 'jumps' over the 'bad' (in the above sense) types $j = 1$. The following matrix identity (and thus the corresponding law of singularization) somewhat is a generalization of the identity type $1_2$:

*type Q:*

$$\begin{pmatrix} A_1 & 0 & 1 \\ B_1 & 0 & 0 \\ C_1 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_2 & 1 & 0 \\ B_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_3 & \phi_3 & \psi_3 \\ 1 & 0 & 0 \\ 0 & \psi_3 & \phi_3 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 A_2 & 0 & 1 \\ B_1 A_2 & 0 & 0 \\ B_1 + C_1 A_2 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_3 + \frac{1}{A_2} & \phi_3 & \psi_3 \\ -\frac{B_2}{A_2} & 0 & 0 \\ 0 & \psi_3 & \phi_3 \end{pmatrix}.$$

Similar to the above, we further generalize the definition of the matrices $\beta_M(j,A)$ $(j \in \{1,2\})$ to

$$\beta(1,A,B,C) = \begin{pmatrix} A & 1 & 0 \\ B & 0 & 0 \\ C & 0 & 1 \end{pmatrix}, \quad \beta(2,A,B,C) = \begin{pmatrix} A & 0 & 1 \\ B & 0 & 0 \\ C & 1 & 0 \end{pmatrix},$$

where $\beta_M(j,A) = \beta(j,A,1,0)$, and define a law of singularization as follows:

**Law of singularization LQ*:** *From any block of matrices* $(\beta^{(t)},...,\beta^{(t+i)})$, *where* $j^{(t)} = 1,...,j^{(t+i)} = 1$, *and both* $j^{(t-1)} = 2$ *and* $j^{(t+i+1)} = 2$, *singularize the first, the second, ... the last matrix, using identity type Q.*
Or equivalently, in terms of the singularization area $S_Q := X_M(1) \times X_M^{\#}$,

**Law of singularization LQ:** *Singularize* $\beta^{(t)}$ *if and only if*

$$(x_1^{(t-1)}, x_2^{(t-1)}, y_1^{(t-1)}, y_2^{(t-1)}) \in S_Q,$$

*using matrix identity type Q.*

Similar to LM and LM*, the order of singularizing matrices in LQ* only is of a certain technical importance, and we could define matrix identities which would allow singularization independent of the order. The resulting algorithm, a 'cyclic' acceleration of Brun's Algorithm, would be the same. Therefore LQ, where the order is not determined, is equivalent to LQ*.

Let $[A_1, A_2, \dots, A_s]$ denote the regular one-dimensional continued fraction expansion with partial quotients $A_1, A_2, \dots, A_s$ *i.e.*,

$$[A_1, A_2, \dots, A_s] = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots \quad + \cfrac{1}{A_s}}}}.$$

Consider some $t > 0$ with $j^{(t)} = 2$. Let $k \geq 0$, $i \geq 0$ be such that $j^{(t-1)} = 1, \dots, j^{(t-k)} = 1$, $j^{(t-k-1)} = 2$, and $j^{(t+1)} = 1, \dots, j^{(t+i)} = 1$, $j^{(t+i+1)} = 2$. The integers $A^{(t-k-1)}, \dots, A^{(t)}, \dots, A^{(t+i+1)}$ are the corresponding partial quotients obtained by the multiplicative acceleration of Brun's Algorithm (the original algorithm).

Further, let the corresponding $t^* \leq t$ be such that $P_Q^{(t^*)}$ is the $t^{*\text{th}}$ convergent obtained by the new algorithm, where $P_Q^{(t^*)} = P_M^{(t)}$. By induction, we get the inverse matrices of the resulting algorithm

$$\beta_Q^{(t^*)}(x_1^{(0)}, x_2^{(0)}) = \begin{pmatrix} A_L^{(t^*)} A_R^{(t^*)} & 0 & 1 \\ \frac{(-1)^i}{A_R^{(t^*-1)}} A_R^{(t^*)} & 0 & 0 \\ C^{(t^*)} & 1 & 0 \end{pmatrix},$$

where

$$A_L^{(t^*)} = A_L^{(t^*)}(x_1^{(0)}, x_2^{(0)}) = [A^{(t)}, \dots, A^{(t-k)}]\,,$$

$$A_R^{(t^*)} = A_R^{(t^*)}(x_1^{(0)}, x_2^{(0)}) = \begin{cases} 1 & \text{if } i = 0\,, \\ [A^{(t+i)}, \dots, A^{(t+1)}] \cdots [A^{(t+1)}] & \text{if } i > 0\,, \end{cases}$$

$$C^{(t^*)} = C^{(t^*)}(x_1^{(0)}, x_2^{(0)}) = \begin{cases} 0 & \text{if } i = 0\,, \\ 1 & \text{if } i = 1\,, \\ [A^{(t+i)}, \dots, A^{(t+2)}] \cdots [A^{(t+2)}] & \text{if } i > 1\,. \end{cases}$$

Now let $i \geq 0$, $A_1 \geq 1, \dots, A_{i+1} \geq 1$ and define

$$E_1 = E_1(i; A_1, \dots, A_i, A_{i+1}) := \Big(\frac{(-1)^i}{[A_{i+1}, \dots, A_1] \cdots [A_1]}, 0\Big)\,,$$

$$E_2 = E_2(i; A_1, \dots, A_i, A_{i+1}) := \Big(\frac{(-1)^i}{[A_{i+1}, \dots, A_1] \cdots [A_1]}, \frac{1}{[A_{i+1}, \dots, A_1]}\Big)\,,$$

$$E_3 = E_3(i; A_1, \dots, A_i, A_{i+1}) :=$$
$$\Big(\frac{(-1)^i}{[A_{i+1}+1, A_i, \dots, A_1] \cdot [A_i, \dots, A_1] \cdots [A_1]}, \frac{1}{[A_{i+1}+1, A_i, \dots, A_1]}\Big)\,.$$

Then

$$X_Q = \bigcup_{i=1}^{\infty} \bigcup_{A_1=1}^{\infty} \cdots \bigcup_{A_{i+1}=1}^{\infty} \Delta(E_1, E_2, E_3) \subset [-1, 1] \times [0, 1].$$

**Remark.** As above, $i$ is the length of a block of consecutive matrices of type $j = 1$, and $A_1, \dots, A_i$ are the corresponding partial quotients (resulting from the original algorithm). $A_{i+1}$ is the partial quotient corresponding to the first type $j_{i+1} = 2$. If $i = 0$, then $\Delta(E_1, E_2, E_3)$ reduces to $\Delta((\frac{1}{A_1}, 0), (\frac{1}{A_1}, \frac{1}{A_1}), (\frac{1}{A_1+1}, \frac{1}{A_1+1}))$, and

$$\bigcup_{A_1=1}^{\infty} \Delta((\frac{1}{A_1}, 0), (\frac{1}{A_1}, \frac{1}{A_1}), (\frac{1}{A_1+1}, \frac{1}{A_1+1})) = X_M(2).$$

In principle, the resulting algorithm is defined by the inverse matrices $\beta_Q^{(t)}$. However, the actual construction yields the difficulty that the definitions of $A_L^{(t^*)}$ and $A_R^{(t^*-1)}$ depend on the explicit knowledge of the partial quotients $A^{(t-1)}, ..., A^{(t-k)}$. We may overcome this problem in using the natural extension: For $i \geq 1$, define

$$
\begin{aligned}
E_1^{\#} &= E_1^{\#}(i; A_1, \dots, A_i) := \big(\frac{1}{[A_i, \dots, A_1]}, 0\big)\,, \\
E_2^{\#} &= E_2^{\#}(i; A_1, \dots, A_i) := \big(\frac{1}{[A_i, \dots, A_1]}, \frac{1}{[A_i, \dots, A_1] \cdots [A_1 + 1]}\big)\,, \\
E_3^{\#} &= E_3^{\#}(i; A_1, \dots, A_i) := \\
&\qquad \big(\frac{1}{[A_i, \dots, A_2, A_1 + 1]}, \frac{1}{[A_i, \dots, A_2, A_1 + 1] \cdots [A_1 + 1]}\big)\,.
\end{aligned}
$$

If $i = 0$, then $E_1^{\#} = (0, 0), E_2^{\#} = (0, 1)$ and $E_3^{\#} = (1, 1)$. Let

$$
\overline{X}_Q = \bigcup_{i=0}^{\infty} \bigcup_{A_1=1}^{\infty} \cdots \bigcup_{A_{i+1}=1}^{\infty} \Delta(E_1, E_2, E_3) \times \Delta(E_1^{\#}, E_2^{\#}, E_3^{\#}).
$$

We may define a cyclic acceleration of Brun's Algorithm $\overline{T}_Q : \overline{X}_Q \to \overline{X}_Q$,

$$
\overline{T}_Q(x_1, x_2, y_1, y_2) = \big(\frac{A_R^+}{A_R^-|x_1|} - A_L A_R^+, \frac{A_R^+ x_2}{A_R^-|x_1|} - C, {V_M^{\#}}^{k^+ + 1}(y_1, y_2)\big)
$$

where

$$
\begin{aligned}
k^- &:= \min\{t : V_M^{t+1} \in X_M(2)\}\,, \\
k^+ &:= \min\{t : T_M^{t+1}\big(\frac{A_L^* A_R^- |x_1|}{A_L^* - A_R^- |x_1|}, \frac{A_L^* x_2}{A_L^* - A_R^- |x_1|}\big) \in X_M(2)\}\,, \\
A_L &:= [A_1, \dots, A_{1-k^-}], \\
A_L^* &:= [A_L] - A_L, \\
A_R^+ &:= \begin{cases} 1 & \text{if } k^+ = 0, \\ [A_{k^+ + 1}, \dots, A_2] \cdots [A_2] & \text{if } k^+ > 0\,. \end{cases}
\end{aligned}
$$

$$
A_R^- := \begin{cases} 1 & \text{if } k^- = 0, \\ [A_1, \dots, A_{2-k^-}] \cdots [A_{2-k^-}] & \text{if } k^- > 0\,. \end{cases}
$$

$$
C := \begin{cases} 0 & \text{if } k^+ = 0, \\ 1 & \text{if } k^+ = 1\,. \\ [A_{k^+ + 1}, \dots, A_3] \cdots [A_3] & \text{if } k^+ > 1\,. \end{cases}
$$

$$
A_i := \begin{cases} A\big(T_M^{i-1}\big(\frac{A_L^* A_R^- |x_1|}{A_L^* - A_R^- |x_1|}, \frac{A_L^* x_2}{A_L^* - A_R^- |x_1|}\big)\big) & \text{if } k^+ \geq i \geq 1, \\ [\frac{1}{{V_M^{\#}}^i(y_1, y_2)}] & \text{if } 0 \geq i \geq -k^-\,. \end{cases}
$$

Finally, we define

$$\begin{aligned} k^{(t)} &:= k^{+}\big(T_Q^{t-1}(x_1^{(0)}, x_2^{(0)})\big) = k^{-}\big(T_Q^{t}(x_1^{(0)}, x_2^{(0)})\big)\,, \\ A_L^{(t)} &:= A_L\big(T_Q^{t-1}(x_1^{(0)}, x_2^{(0)})\big)\,, \\ A_R^{(t)} &:= A_R^{+}\big(T_Q^{t-1}(x_1^{(0)}, x_2^{(0)})\big) = A_R^{-}\big(T_Q^{t}(x_1^{(0)}, x_2^{(0)})\big), \end{aligned}$$

and

$$C^{(t)} := C\big(T_Q^{t-1}(x_1^{(0)}, x_2^{(0)})\big)\,.$$

## 8. Convergence properties

In constructing the cyclic acceleration of the algorithm, we avoid more than three partial convergents lying on a line. Alas, the method yields another problem: While for $k^{(t)}$ even, $(x_1^{(0)}, x_2^{(0)}) \in \Delta(P_Q^{(t)}, P_Q^{(t-1)}, P_Q^{(t-2)})$, this is not true if $k^{(t)}$ is odd (Fig. 8). To overcome this problem, we have

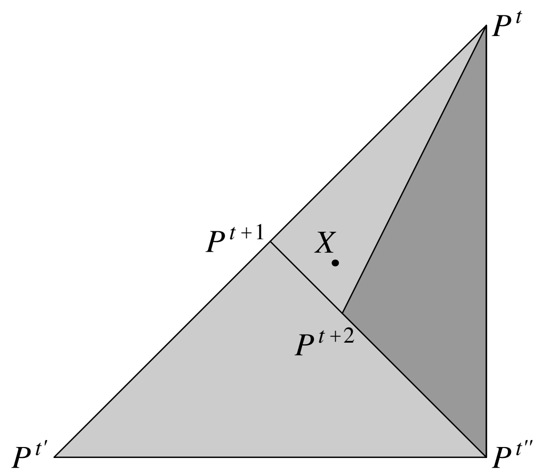


FIGURE 8. Example for an approximation after singularization, where $j^{(t+1)} = 2$ and $j^{(t+2)} = 1$

to accept single matrices of type $j = 1$. We propose the following law of singularization:

**Law of singularization Lq*:** *Let $(\beta^{(t)}, \dots, \beta^{(t+i)})$ be a block of matrices such that $j^{(t)} = 1, \dots, j^{(t+i)} = 1$, and both $j^{(t-1)} = 2$ and $j^{(t+i+1)} = 2$. If $i$ is even, then singularize the first, the second,… the last matrix, using identity type Q. If $i$ is odd, and $i \geq 3$, then singularize the first, the* *second,... matrix until (including) the first matrix before the last, using identity type Q.*

Denote

$$\begin{aligned}
F_1 = F_1(A_1, A_2) &:= \Big(\frac{1}{A_2 + \frac{1}{A_1}}, 0\Big)\,,\\
F_2 = F_2(A_1, A_2) &:= \Big(\frac{1}{A_2 + \frac{1}{A_1+1}}, 0\Big)\,,\\
F_3 = F_3(A_1, A_2) &:= \Big(\frac{1}{A_2 + \frac{1}{A_1}}, \frac{1}{A_1(A_2 + \frac{1}{A_1})}\Big)\\
F_4 = F_4(A_1, A_2) &:= \Big(\frac{1}{A_2 + \frac{1}{A_1+1}}, \frac{1}{(A_1+1)(A_2 + \frac{1}{A_1+1})}\Big)\,.
\end{aligned}$$

Setting

$$\begin{aligned}
S_q := &\bigcup_{A_1=1}^{\infty}\bigcup_{A_2=1}^{\infty} \Delta(F_1, F_2, F_3) \times X_M^{\#}\\
&\cup \bigcup_{A_1=1}^{\infty}\bigcup_{A_2=1}^{\infty} \Delta(F_1, F_3, F_4) \times \bigcup_{i=0}^{\infty}\bigcup_{A_1=1}^{\infty}\cdots\bigcup_{A_{2i+2}=1}^{\infty} \Delta(E_1^{\#}, E_2^{\#}, E_3^{\#})\,,
\end{aligned}$$

we find Lq as an equivalent law of singularization:

**Law of singularization Lq:** *Singularize $\beta^{(t)}$ if and only if*

$$(x_1^{(t-1)}, x_2^{(t-1)}, y_1^{(t-1)}, y_2^{(t-1)}) \in S_q,$$

*using matrix identity type Q.*

Define

$$\begin{aligned}
E_4 &= E_4(i; A_1, ..., A_i, A_{i+1})\\
&\quad := \Big(\frac{(-1)^i}{[A_{i+1}+1, A_i, \ldots, A_1]\cdot[A_i, \ldots, A_1]\cdots[A_1]}, 0\Big)\,,\\
\overline{X}_q(1) &:= \bigcup_{i=0}^{\infty}\bigcup_{A_1=1}^{\infty}\cdots\bigcup_{A_{2i+1}=1}^{\infty} \Delta(E_1, E_2, E_3) \times \Delta(E_1^{\#}, E_2^{\#}, E_3^{\#})\,,\\
\overline{X}_q(2) &:= \bigcup_{i=1}^{\infty}\bigcup_{A_1=1}^{\infty} (\Delta(E_1, E_2, E_3) \times \bigcup_{A_1=2}^{\infty}\cdots\bigcup_{A_{2i+2}=1}^{\infty} (1)\Delta(E_1^{\#}, E_2^{\#}, E_3^{\#}))\\
&\quad\ \cup \bigcup_{i=1}^{\infty}\bigcup_{A_1=0}^{\infty}\cdots\bigcup_{A_{2i+1}=1}^{\infty} \Delta(E_1, E_3, E_4) \times \Delta(E_1^{\#}, E_2^{\#}, E_3^{\#})\,,
\end{aligned}$$

and consequently,

$$\overline{X}_q := \overline{X}_q(1) \cup \overline{X}_q(2)\,.$$

The resulting algorithm $\overline{T}_q : \overline{X}_q \to \overline{X}_q$ can be defined similarly as above:

$$\overline{T}_q(x_1, x_2, y_1, y_2) = \begin{cases} (\frac{x_2}{A_R^-|x_1|}, \frac{1}{A_R^-|x_1|} - A_L, V_M^{\#}(y_1, y_2)) & \text{if } j = 1\,, \\ (\frac{A_R^+}{A_R^-|x_1|} - A_L A_R^+, \frac{A_R^+ x_2}{A_R^-|x_1|} - C, {V_M^{\#}}^{k^+ + 1}(y_1, y_2)) & \text{if } j = 2\,, \end{cases}$$

where

$$j := \begin{cases} 1 & \text{if } (x_1, x_2, y_1, y_2) \in \overline{X}_q(1)\,, \\ 2 & \text{if } (x_1, x_2, y_1, y_2) \in \overline{X}_q(2)\,. \end{cases}$$

The integers $k^-$ and $k^+$ are defined as above,

$$k_1 := \begin{cases} k^- & \text{if } k^- \text{ is even}\,, \\ 1 & \text{if } k^- \text{ is odd}\,, \end{cases}$$

$$k_2 := \begin{cases} k^+ & \text{if } k^+ \text{ is even}\,, \\ k^+ - 1 & \text{if } k^+ \text{ is odd}\,, \end{cases}$$

and $A_L$, $A_L^*$, $A_R^+$, $A_R^-$, $C$ and $A_i$ are defined as in Section 7, *in fine*, replacing $k^-$ by $k_1$ and $k^+$ by $k_2$, respectively.

Define $j^{(t)}$, $k^{(t)}$, $A_L^{(t)}$, $A_R^{(t)}$ and $C^{(t)}$ as above. Note that, by construction $A_R^{(t)}$ and $C^{(t)}$ are integers, and $A_R^{(t)} > C^{(t)}$. An invariant measure can be found, although requiring a certain technical effort, using the method described in Section 6. The inverse matrices of the algorithm are given by

$$\beta_q^{(t)}(1) = \begin{pmatrix} A_L^{(t)} & 1 & 0 \\ \frac{1}{A_R^{(t-1)}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta_q^{(t)}(2) = \begin{pmatrix} A_L^{(t)} A_R^{(t)} & 0 & 1 \\ \frac{1}{A_R^{(t-1)}} A_R^{(t)} & 0 & 0 \\ C^{(t)} & 1 & 0 \end{pmatrix},$$

where $A_R^{(t-1)} = 1$ if $j^{(t-1)} = 1$. The convergence matrices, and thus the sequences of Diophantine approximations, are defined as above. To estimate the exponent of convergence, we use the modified method of Paley and Ursell [17], as described in Schweiger [22]. It is based on the following quantities:

**Definition.** For $i = 1, 2$, set

$$[t, s] := q^{(t)} p_i^{(s)} - q^{(s)} p_i^{(t)}$$

and

$$\rho_{t+3} := \begin{cases} \max\left\{ \frac{[t+3,t+2]}{q^{(t+3)}}, \frac{[t+3,t]}{q^{(t+3)}} \right\} & \text{if } j^{(t+2)} = 1, \\ \\ \max\left\{ \frac{[t+3,t+2]}{q^{(t+3)}}, \frac{[t+3,t+1]}{q^{(t+3)}} \right\} & \text{if } j^{(t+2)} = 2. \end{cases}$$

It is known that

$$\left| x_i - \frac{p_i^{(t)}}{q^{(t)}} \right| \leq \frac{2\rho_t}{q^{(t)}}$$

(see e.g. [18]), thus exponential decay of $\rho_t$ yields exponential convergence to $(x_1, x_2)$. We have the following recursion relations (since the results hold for both $p_1^{(.)}$ and $p_2^{(.)}$, we write $p^{(.)}$ instead):

if $j^{(t+2)} = 1$ :

$$q^{(t+4)} = A_L^{(t+4)} A_R^{(t+4)} q^{(t+3)} + \frac{A_R^{(t+4)}}{A_R^{(t+3)}} q^{(t)} + C^{(t+4)} q^{(t+2)} \,,$$

if $j^{(t+3)} = 1$ :

$$q^{(t+4)} = A_L^{(t+4)} A_R^{(t+4)} q^{(t+3)} + A_R^{(t+4)} q^{(t+2)} + C^{(t+4)} q^{(t+1)} \,,$$

if $j^{(t+2)} = j^{(t+3)} = 2$ :

$$q^{(t+4)} = A_L^{(t+4)} A_R^{(t+4)} q^{(t+3)} + \frac{A_R^{(t+4)}}{A_R^{(t+3)}} q^{(t+1)} + C^{(t+4)} q^{(t+2)} \,,$$

if $j^{(t+2)} = 1$ :

$$[t+4, t+3] = -\frac{A_R^{(t+4)}}{A_R^{(t+3)}} [t+3, t] - C^{(t+4)} [t+3, t+2] \,,$$

$$[t+4, t+2] = A_L^{(t+4)} A_R^{(t+4)} [t+3, t+2] - \frac{A_R^{(t+4)}}{A_R^{(t+3)}} [t+2, t] \,,$$

if $j^{(t+3)} = 1$ :

$$[t+4, t+3] = -A_R^{(t+4)} [t+3, t+2] - C^{(t+4)} [t+3, t+1] \,,$$

$$[t+4, t+1] = A_L^{(t+4)} A_R^{(t+4)} [t+3, t+1] + A_R^{(t+4)} [t+2, t+1] \,,$$

if $j^{(t+2)} = j^{(t+3)} = 2$ :

$$[t+4, t+3] = -\frac{A_R^{(t+4)}}{A_R^{(t+3)}} [t+3, t+1] - C^{(t+4)} [t+3, t+2] \,,$$

$$[t+4, t+2] = A_L^{(t+4)} A_R^{(t+4)} [t+3, t+2] - \frac{A_R^{(t+4)}}{A_R^{(t+3)}} [t+2, t+1] \,.$$

From these relations, we deduce the following

**Lemma 8.1.**

$$\big|[t+4,t+3]\big| \le (q^{(t+4)} - q^{(t)}) \max\{\rho_{t+3}, \rho_{t+2}, \rho_{t+1}\}\,.$$

*Proof.* We only show the cyclic case $j^{(t+1)} = j^{(t+2)} = j^{(t+3)} = 2$. The other cases are similar. We use the above recursion relations. If $C^{(t+4)} = 0$, then $A_R^{(t+4)} = 1$. We get

$$\begin{aligned}
\big|[t+4,t+3]\big| &= \big| -\frac{1}{A_R^{(t+3)}}[t+3,t+1]\big| \\
&\le \frac{1}{A_R^{(t+3)}} q^{(t+3)} \rho_{t+3} \\
&\le (q^{(t+4)} - ((A_L^{(t+4)} - \frac{1}{A_R^{(t+3)}})q^{(t+3)} + \frac{1}{A_R^{(t+3)}} q^{(t+1)}))\rho_{t+3} \\
&\le (q^{(t+4)} - q^{(t+1)})\rho_{t+3}\,.
\end{aligned}$$

Now let $C^{(t+4)} \ge 1$, hence $A_R^{(t+4)} \ge 2$. If $[t+3,t+1][t+3,t+2] \le 0$, then

$$\begin{aligned}
\big|[t+4,t+3]\big| &= \big| -\frac{A_R^{(t+4)}}{A_R^{(t+3)}}[t+3,t+1] - C^{(t+4)}[t+3,t+2]\big| \\
&\le A_R^{(t+4)} q^{(t+3)} \rho_{t+3} \\
&\le (q^{(t+4)} - q^{(t+2)})\rho_{t+3}\,.
\end{aligned}$$

If $[t+3,t+1][t+3,t+2] \ge 0$, $C^{(t+3)} = 0$ and $A_R^{(t+3)} = 1$, then

$$\begin{aligned}
\big|[t+4,t+3]\big| &= \big| -\frac{A_R^{(t+4)}}{A_R^{(t+3)}}[t+3,t+1] - C^{(t+4)}[t+3,t+2]\big| \\
&= \big| - A_R^{(t+4)}[t+3,t+1] + \frac{C^{(t+4)}}{A_R^{(t+2)}}[t+2,t]\big| \\
&\le (A_R^{(t+4)} q^{(t+3)} + C^{(t+4)} q^{(t+2)}) \max\{\rho_{t+3}, \rho_{t+2}\} \\
&\le (q^{(t+4)} - q^{(t+1)}) \max\{\rho_{t+3}, \rho_{t+2}\}\,.
\end{aligned}$$

If $[t+3,t+1][t+3,t+2] \ge 0$, $C^{(t+3)} \ge 1$, and thus $A_R^{(t+3)} \ge 2$, we have two cases: $[t+2,t][t+2,t+1] \le 0$ yields

$$
\begin{aligned}
\big|[t+4,t+3]\big| &= \Big| -\frac{A_R^{(t+4)}}{A_R^{(t+3)}}[t+3,t+1] - C^{(t+4)}[t+3,t+2]\Big| \\
&= \Big| -A_L^{(t+3)}A_R^{(t+4)}[t+2,t+1] + \frac{A_R^{(t+4)}}{A_R^{(t+2)}}[t+1,t] \\
&\quad + \frac{A_R^{(t+3)}C^{(t+4)}}{A_R^{(t+2)}}[t+2,t] + C^{(t+4)}C^{(t+3)}[t+2,t+1]\Big| \\
&\le \big((A_L^{(t+3)}A_R^{(t+4)}A_R^{(t+3)} - 1)q^{(t+2)} + A_R^{(t+4)}q^{(t+1)} \\
&\quad + C^{(t+4)}q^{(t+2)}\big)\max\{\rho_{t+2},\rho_{t+1}\} \\
&\le (q^{(t+4)} - q^{(t+2)})\max\{\rho_{t+2},\rho_{t+1}\}\,,
\end{aligned}
$$

while if $[t+2,t][t+2,t+1] \ge 0$, then $[t+3,t+2][t+2,t+1] \le 0$ and

$$
\begin{aligned}
\big|[t+4,t+3]\big| &= \Big| -\frac{A_R^{(t+4)}}{A_R^{(t+3)}}[t+3,t+1] - C^{(t+4)}[t+3,t+2]\Big| \\
&= \Big| -A_L^{(t+3)}A_R^{(t+4)}[t+2,t+1] \\
&\quad + \frac{A_R^{(t+4)}}{A_R^{(t+2)}}[t+1,t] - C^{(t+4)}[t+3,t+2]\Big| \\
&\le \Big(A_R^{(t+4)}q^{(t+3)} + \frac{A_R^{(t+4)}}{A_R^{(t+3)}}q^{(t+1)}\Big)\max\{\rho_{t+3},\rho_{t+2},\rho_{t+1}\} \\
&\le (q^{(t+4)} - q^{(t+2)})\max\{\rho_{t+3},\rho_{t+2},\rho_{t+1}\}\,.
\end{aligned}
$$

$\square$

Similarly, we have

**Lemma 8.2.** *Let* $j^{(t+3)} = 2$*, then*

$$\big|[t+4,t+2]\big| \le (q^{(t+4)} - q^{(t)})\max\{\rho_{t+3},\rho_{t+2},\rho_{t+1},\rho_t\}\,.$$

**Lemma 8.3.** *Let* $j^{(t+4)} = 1$*, then*

$$\big|[t+5,t+2]\big| \le (q^{(t+5)} - q^{(t)})\max\{\rho_{t+3},\rho_{t+2},\rho_{t+1},\rho_t\}\,.$$

Now define $\tau_t :=\ \max\{\rho_{t+4},\rho_{t+3},\rho_{t+2},\rho_{t+1},\rho_t\}$. Using Lemmata 8.1 - 8.3 and the above definitions, we estimate

$$\tau_{t+5} \le \left(1 - \min\left\{\frac{q^{(t)}}{q^{(t+5)}}\,,\ \frac{q^{(t+1)}}{q^{(t+6)}}\,,\ \frac{q^{(t+2)}}{q^{(t+7)}}\,,\ \frac{q^{(t+3)}}{q^{(t+8)}}\,,\ \frac{q^{(t+4)}}{q^{(t+9)}}\right\}\right)\tau_t\,.$$

Since

$$\frac{q^{(t)}}{q^{(t+5)}} > 0$$

almost everywhere, we may define a function

$$g(x_1, x_2, y_1, y_2) := \log\left(1 - \min\left\{\frac{q^{(1)}}{q^{(6)}}, \frac{q^{(2)}}{q^{(7)}}, \frac{q^{(3)}}{q^{(8)}}, \frac{q^{(4)}}{q^{(9)}}, \frac{q^{(5)}}{q^{(10)}}\right\}\right)$$

to apply the ergodic theorem

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} g\big(\overline{T}_q^s(x_1, x_2, y_1, y_2)\big) = \int_{\overline{X}_q} g(x_1, x_2, y_1, y_2)d\mu =: \ \log K < 0$$

almost everywhere. Thus $\tau_t \leq cK^{\frac{t}{5}}$, and $\rho_t$ goes down exponentially. We state the following

**Theorem 8.4.** *For the algorithm $\overline{T_q}$, there exists a constant $d_q$ such that, for almost all $(x_1, x_2, y_1, y_2)$ in $\overline{X}_q$, there exist an integer $t(x_1, x_2, y_1, y_2)$, such that the inequality*

$$\left| x_i - \frac{p_i^{(t)}}{q^{(t)}} \right| \leq \frac{1}{(q^{(t)})^{1+d_q}} \ .$$

*hold for any $t \geq t(x_1, x_2, y_1, y_2)$.*

**Remark.** Exponential convergence of $\overline{T}_q$ follows directly from exponential convergence of the multiplicative acceleration of Brun's Algorithm. However, the proof of Theorem 8.4 is interesting for a different reason: Hitherto, proofs for exponential convergence of Brun's algorithm were based on considering a special subset of $X_B$ *i.e.*, the set where $j^{(1)} = \cdots = j^{(t)} = 2$ for some $t \geq 3$, and the induced transformation on this set (compare R. Meester [12] or [18]). We may now (with respect to the measure of the singularization area $S_q$) transfer the above result, especially the estimate of the decay using the function $g(x_1, x_2, y_1, y_2)$, to the multiplicative acceleration of Brun's algorithm using standard techniques, which essentially were described in the original work of Paley and Ursell [17]. We immediately see that not only these special sets, but all cylinders contribute to the exponential approximation. However, the estimate of the approximation speed depends on the size of the quantities $q^{(t+5)} - q^{(t)}$. The smaller this difference, the better the estimate. Thus the estimate gets worse if a large number of non-cyclic convergents with suitable partial quotients has been singularized, which essentially leads to the counterexample for cyclic algorithms in [17].

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