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PAC fields over number fields

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Résumé. Soient K un corps de nombres et N une extension galoisienne de $\mathbb{Q}$ qui n'est pas algébriquement close. Alors N n'est pas PAC sur K.

Abstract. We prove that if K is a number field and N is a Galois extension of $\mathbb{Q}$ which is not algebraically closed, then N is not PAC over K.

1. Introduction

A central concept in Field Arithmetic is "pseudo algebraically closed (abbreviated **PAC**) field". Since our major result in this note concerns number fields, we focus our attention on fields of characteristic 0. If K is a countable Hilbertian field, then $\tilde{K}(\boldsymbol{\sigma})$ is PAC for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ [1, Thm. 18.6.1]. Aharon Razon observed that the proof of that theorem yields that the fields $\tilde{K}(\boldsymbol{\sigma})$ are even "PAC over K". Moreover, if K is the quotient field of a countable Hilbertian ring R (e.g. $R = \mathbb{Z}$ and $K = \mathbb{Q}$), then for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $\tilde{K}(\boldsymbol{\sigma})$ is PAC over R [4, Prop. 3.1].

Here $\tilde{K}$ denotes the algebraic closure of K and $\mathrm{Gal}(K) = \mathrm{Gal}(\tilde{K}/K)$ is its absolute Galois group. This group is equipped with a Haar measure and the close "almost all" means "for all but a set of measure zero". If $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_e) \in \mathrm{Gal}(K)^e$, then $\tilde{K}(\boldsymbol{\sigma})$ is the fixed field in $\tilde{K}$ of $\sigma_1, \dots, \sigma_e$.

Recall that a field M is said to be **PAC** if every nonempty absolutely irreducible variety V over M has an M-rational point. One says that M is **PAC over** a subring R if for every absolutely irreducible variety V over M of dimension $r \geq 0$ and every dominating separable rational map $\varphi\colon V \to \mathbb{A}^r_M$ there exists $\mathbf{a} \in V(M)$ with $\varphi(\mathbf{a}) \in R^r$.

When K is a number field, the stronger property of the fields $\tilde{K}(\boldsymbol{\sigma})$ (namely, being PAC over the ring of integers O of K) has far reaching arithmetical consequences. For example, $\tilde{O}(\boldsymbol{\sigma})$ (= the integral closure of O in $\tilde{K}(\boldsymbol{\sigma})$) satisfies Rumely's local-global principle [5, special case of Cor. 1.9]: If V is an absolutely irreducible variety over $\tilde{K}(\boldsymbol{\sigma})$ with $V(\tilde{O}) \neq \emptyset$, then V has an $\tilde{O}(\boldsymbol{\sigma})$-rational point. Here $\tilde{O}$ is the integral closure of O in $\tilde{K}$.

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For an arbitrary countable Hilbertian field K of characteristic 0 we further denote the maximal Galois extension of K in $\tilde{K}(\boldsymbol{\sigma})$ by $\tilde{K}[\boldsymbol{\sigma}]$. We know that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $\tilde{K}[\boldsymbol{\sigma}]$ is PAC [1, Thm. 18.9.3]. However, at the time we wrote [4], we did not know if $\tilde{K}[\boldsymbol{\sigma}]$ is PAC over K, so much the more we did not know if $\tilde{K}[\boldsymbol{\sigma}]$ is PAC over O when K is a number field. Thus, we did not know if $\tilde{O}[\boldsymbol{\sigma}]$ (= the integral closure of O in $\tilde{K}[\boldsymbol{\sigma}]$) satisfies Rumely's local global principle. We did not even know of any Galois extension of $\mathbb{Q}$ other than $\tilde{\mathbb{Q}}$ which is PAC over $\mathbb{Q}$. We could only give a few examples of distinguished Galois extensions of $\mathbb{Q}$ which are not PAC over $\mathbb{Q}$: The maximal solvable extension $\mathbb{Q}_{\mathrm{solv}}$ of $\mathbb{Q}$, the compositum $\mathbb{Q}_{\mathrm{symm}}$ of all symmetric extensions of $\mathbb{Q}$, and $\mathbb{Q}_{\mathrm{tr}}(\sqrt{-1})$ ($\mathbb{Q}_{\mathrm{tr}}$ is the maximal totally real extension of $\mathbb{Q}$). The proof of the second statement relies, among others, on Faltings' theorem about the finiteness of K-rational points of curves of genus at least 2. Note that $\mathbb{Q}_{\mathrm{symm}}$ is PAC [1, Thm. 18.10.3 combined with Cor. 11.2.5] and $\mathbb{Q}_{\mathrm{tr}}[\sqrt{-1}]$ is PAC [2, Remark 7.10(b)]. However, it is a major problem of Field Arithmetic whether $\mathbb{Q}_{\mathrm{solv}}$ is PAC [1, Prob. 11.5.8]. Thus, it is not known whether every absolutely irreducible equation $f(x, y) = 0$ with coefficients in $\mathbb{Q}$ can be solved by radicals.

The goal of the present note is to prove that the above examples are only special cases of a general result:

Main Theorem. *No number field K has a Galois extension N which is PAC over K except $\tilde{\mathbb{Q}}$.*

The proof of this theorem relies on a result of Razon about fields which are PAC over subfields, on Frobenius density theorem, and on Neukirch's recognition of p-adically closed fields among all algebraic extensions of $\mathbb{Q}$. The latter theorem has no analog for finitely generated extensions over $\mathbb{F}_p$ but it has one for finitely generated extensions of $\mathbb{Q}$ (a theorem of Efrat-Koenigsmann-Pop). However, at one point of our proof we use the basic fact that $\mathbb{Q}$ has no proper subfields. That property totally fails if we replace $\mathbb{Q}$ say by $\mathbb{Q}(t)$ with t indeterminate. Thus, any generalization of the main theorem to finitely generated fields or, more generally, to countable Hilbertian fields, should use completely other means.

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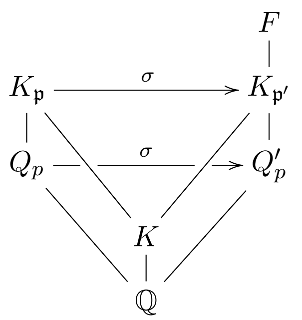
2. Galois extensions of number fields

Among all Hilbertian fields $\mathbb{Q}$ is the only one which is a prime field. This simple observation plays a crucial role in the proof of the main theorem (see Remark 2).

Lemma 1. *Let K be a finite Galois extension of $\mathbb{Q}$, $\mathfrak{p}$ an ultrametric prime of K, $K_\mathfrak{p}$ a Henselian closure of K at $\mathfrak{p}$, and F an algebraic extension of K such that $\mathrm{Gal}(K_\mathfrak{p}) \cong \mathrm{Gal}(F)$. Then $F = K_\mathfrak{p}^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q})$. Thus, $F = K_{\mathfrak{p}'}$ for some prime $\mathfrak{p}'$ of K conjugate to $\mathfrak{p}$ over $\mathbb{Q}$.*

Proof. Let p be the prime number lying under $\mathfrak{p}$. Denote the closure of $\mathbb{Q}$ in $K_\mathfrak{p}$ under the $\mathfrak{p}$-adic topology by Q_p. Then Q_p is isomorphic to the field of all algebraic elements in $\mathbb{Q}_p$ (= the field of p-adic integers). By [7, Satz 1], F is Henselian and it contains an isomorphic copy Q'_p of Q_p such that $[F : Q'_p] = [K_\mathfrak{p} : Q_p]$. In particular, the prime $\mathfrak{p}'$ which F induces on K lies over p. Hence, KQ'_p is a Henselian closure of K at $\mathfrak{p}'$ which we denote by $K_{\mathfrak{p}'}$. Since $K/\mathbb{Q}$ is Galois, there is a $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ with $\mathfrak{p}^\sigma = \mathfrak{p}'$. Moreover, σ extends to an element $\sigma \in \mathrm{Gal}(\mathbb{Q})$ with $K_\mathfrak{p}^\sigma = K_{\mathfrak{p}'}$.



Since Q_p (resp. Q'_p) is the $\mathfrak{p}$-adic (resp. $\mathfrak{p}'$-adic) closure of $\mathbb{Q}$ in $K_\mathfrak{p}$ (resp. $K_{\mathfrak{p}'}$), we have $Q_p^\sigma = Q'_p$. Hence, $[K_\mathfrak{p} : Q_p] = [K_{\mathfrak{p}'} : Q'_p]$. Therefore, $[F : K_{\mathfrak{p}'}] = 1$, so $F = K_{\mathfrak{p}'} = K_\mathfrak{p}^\sigma$. □

Remark 2. *The arguments of Lemma 1 can not be generalized to finitely generated extensions of $\mathbb{Q}$ which are transcendental over $\mathbb{Q}$. For example, suppose $K = \mathbb{Q}(t)$ with t indeterminate. If K is a Galois extension a field K_0, then, by Lüroth, $K_0 = \mathbb{Q}(u)$ with u transcendental over $\mathbb{Q}$. As such, K_0 has infinitely many automorphisms τ, each of which extends to $\tilde{K}$ and, in the notation of Lemma 1, $\mathrm{Gal}(K_\mathfrak{p}^\tau) \cong \mathrm{Gal}(K_\mathfrak{p})$. However, the prime of K induced by the Henselian valuation of $K_\mathfrak{p}^\tau$ is in general not conjugate to $\mathfrak{p}|_{K_0}$ over K_0.*

Observation 3. *Let V be a vector space of dimension d over $\mathbb{F}_p$ and $V_1, \ldots, V_n$ subspaces of dimensions $d-1$. Suppose $n < p$. Then, $\bigcup_{i=1}^n V_i$ is a proper subset of V. Indeed, $|\bigcup_{i=1}^n V_i| \le \sum_{i=1}^n |V_i| = np^{d-1} < p^d = |V|$, as required.*

Let N/K be an algebraic extension of fields. We say that N is **Hilbertian over** K if each separable Hilbertian subset of N contains elements of K.

Lemma 4. *Let N be an algebraic extension of a field K. Suppose N is Hilbertian over K. Then, K has for each finite abelian group A a Galois extension K' with Galois group A such that $N \cap K' = K$.*

Proof. Let t be a transcendental element over K. By [1, Prop. 16.3.5], $K(t)$ has a Galois extension F with Galois group A such that F/K is regular. In particular, $FN/N(t)$ is Galois with Galois group A. By [1, Lemma 13.1.1], N has a Hilbertian subset H such that for each $a \in H$, the specialization $t \to a$ extends to an N-place φ of FN with residue field N' which a Galois extension of N having Galois group A. Moreover, omitting finitely many elements from H, we have that if $a \in K$, then the residue field K' of F at φ is a Galois extension of K, $\mathrm{Gal}(K'/K)$ is isomorphic to a subgroup of A and $NK' = N'$.

Since N is Hilbertian over K, we may choose $a \in K \cap H$. Then,

$$|A| = [N' : N] \leq [K' : K] \leq [F : K(t)] = |A|.$$

Consequently, $\mathrm{Gal}(K'/K) \cong A$ and K' is linearly disjoint from N over K, as desired. □

Theorem 5. *Let N be a Galois extension of a number field K which is different from $\tilde{\mathbb{Q}}$. Then N is not PAC over K.*

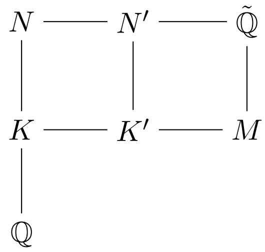
Proof. Assume N is PAC over K. First we replace K and N by fields satisfying additional conditions.

Since N is PAC, N is not real closed [1, Thm. 11.5.1]. Hence, as $N \neq \tilde{\mathbb{Q}}$, $[\tilde{\mathbb{Q}} : N] = \infty$ [6, p. 299, Cor. 3 and p. 452, Prop. 2.4], so $\mathbb{Q}$ has a finite Galois extension E containing K which is not contained in N. By Weissauer, NE is Hilbertian [1, Thm. 13.9.1]. Moreover, NE is Galois ever E, and by [1, Prop. 13.9.3], NE is Hilbertian over E. In addition, NE is PAC over E [4, Lemma 2.1]. Replacing N by NE and K by E, we may assume that, in addition to N being Galois and PAC over K, the extension $K/\mathbb{Q}$ is Galois and N is Hilbertian over K.

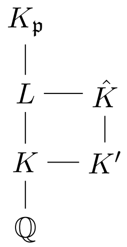
Let $n = [K : \mathbb{Q}]$ and choose a prime number $p > n$. Lemma 4 gives a cyclic extension K' of K of degree p which is linearly disjoint from N. Let $\hat{K}$ be the Galois closure of $K'/\mathbb{Q}$. Choose elements $\sigma_1, \ldots, \sigma_n$ of $\mathrm{Gal}(\hat{K}/\mathbb{Q})$ which lift the elements of $\mathrm{Gal}(K/\mathbb{Q})$. Finally let $K_i = (K')^{\sigma_i}$, $i = 1, \ldots, n$. Since $K/\mathbb{Q}$ is Galois, $K_1, \ldots, K_n$ are all of the conjugates of K' over $\mathbb{Q}$, so $\hat{K} = K_1 \cdots K_n$. Thus, $V = \mathrm{Gal}(\hat{K}/K)$ is a vector space over $\mathbb{F}_p$ of dimension d (which does not exceed n) and $V_i = \mathrm{Gal}(\hat{K}/K_i)$ is a subspace of V of dimension $d - 1$. Observation 3 gives a $\sigma \in V \smallsetminus \bigcup_{i=1}^n V_i$. Denote the fixed field of σ in $\hat{K}$ by L. Then $K_i \not\subseteq L$, $i = 1, \ldots, n$.

Now choose a primitive element x for the extension K'/K. By the preceding paragraph, for each $\sigma \in \mathrm{Gal}(\hat{K}/\mathbb{Q})$, there is an i such that x^σ is a primitive element of K_i/K, so $x^\sigma \notin L$.

Again, by [5, Lemma 2.1], $N' = NK'$ is PAC over K'. Hence, there exists a field M such that $N' \cap M = K'$ and $N'M = \tilde{\mathbb{Q}}$ [8, Thm. 5], so $N \cap M = K$ and $NM = \tilde{\mathbb{Q}}$. In particular, the restriction map $\mathrm{res}\colon \mathrm{Gal}(M) \to \mathrm{Gal}(N/K)$ is an isomorphism.



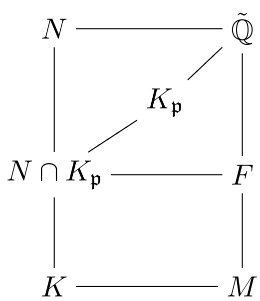
By the Frobenius density theorem, K has an ultrametric prime $\mathfrak{p}$ unramified in $\hat{K}$ such that each element of $\left(\frac{\hat{K}/K}{\mathfrak{p}}\right)$ generates $\mathrm{Gal}(\hat{K}/L)$ [3, p. 134, Thm. 5.2]. Hence, K has a Henselian closure $K_\mathfrak{p}$ at $\mathfrak{p}$ with $K_\mathfrak{p} \cap \hat{K} = L$. Therefore, no conjugate of x over $\mathbb{Q}$ belongs to $K_\mathfrak{p}$. Consequently, x belongs to no conjugate of $K_\mathfrak{p}$ over $\mathbb{Q}$.



As an extension of N, the field $NK_\mathfrak{p}$ is PAC [1, Cor. 11.2.5]. On the other hand, as an extension of $K_\mathfrak{p}$, $NK_\mathfrak{p}$ is Henselian. Therefore, by Frey-Prestel, $NK_\mathfrak{p} = \tilde{\mathbb{Q}}$ [1, Cor. 11.5.5], so

$$\mathrm{Gal}(N/N \cap K_\mathfrak{p}) \cong \mathrm{Gal}(K_\mathfrak{p}).$$

Let $F = (N \cap K_\mathfrak{p})M$. Since $\mathrm{res}\colon \mathrm{Gal}(M) \to \mathrm{Gal}(N/K)$ is an isomorphism, $\mathrm{Gal}(F) \cong \mathrm{Gal}(N/N \cap K_\mathfrak{p}) \cong \mathrm{Gal}(K_\mathfrak{p})$.



It follows from Lemma 1 that there exists $\sigma \in \mathrm{Gal}(\mathbb{Q})$ with $F = K_{\mathfrak{p}}^{\sigma}$. In particular, $x \notin F$, contradicting that $x \in M$ and $M \subseteq F$. ☐

Remark 6. *As already mentioned in the introduction, for almost all* $\boldsymbol{\sigma} \in \mathrm{Gal}(\mathbb{Q})^e$ *the field* $\tilde{\mathbb{Q}}[\boldsymbol{\sigma}]$ *is PAC [1, Thm. 18.9.3]. But, since* $\tilde{\mathbb{Q}}[\boldsymbol{\sigma}]$ *is Galois over* $\mathbb{Q}$*, it is not PAC over* $\mathbb{Q}$ *(Theorem 5), so much the more not PAC over* $\mathbb{Z}$*. However, the latter theorem does not rule out that* $\tilde{\mathbb{Q}}[\boldsymbol{\sigma}]$ *is PAC over its ring of integers* $\tilde{\mathbb{Z}}[\boldsymbol{\sigma}]$*. According to Lemma 7 below, the latter statement is equivalent to "*$\tilde{\mathbb{Z}}[\boldsymbol{\sigma}]$ *satisfies Rumely's local global theorem". We don't know whether these statements are true.*

Lemma 7 (Razon)**.** *The following statements on an algebraic extension* M *of* $\mathbb{Q}$ *are equivalent.*

(a) M *is PAC over* O_M.

(b) O_M *satisfies Rumely's local-global principle.*

Proof. The implication "(a)$\Longrightarrow$(b)" is a special case of [5, Cor. 1.9]. To prove (a) assuming (b), we consider an absolutely irreducible polynomial $f \in M[T, X]$ with $\frac{\partial f}{\partial X} \neq 0$ and a nonzero polynomial $g \in M[T]$. By [4, Lemma 1.3], it suffices to find $a \in O_M$ and $b \in M$ such that $f(a, b) = 0$ and $g(a) \neq 0$. Choose $a' \in \mathbb{Z}$ such that $g(a') \neq 0$ and $\frac{\partial f}{\partial X}(a', X) \neq 0$. Then choose $b' \in \tilde{\mathbb{Q}}$ with $f(a', b') = 0$. Next choose $c \in \mathbb{Z}$ with $b'c \in \tilde{\mathbb{Z}}$. For example, if $\sum_{i=0}^{n} c_i (b')^i = 0$ with $c_0, \ldots, c_n \in \mathbb{Z}$, then we may choose $c = c_n$. Now note that $(a', b'c)$ is a zero of the absolutely irreducible polynomial $f(T, c^{-1}X)$ with coefficients in M. By (a), there are $a \in O_M$ and $b'' \in M$ with $f(a, c^{-1}b'') = 0$. Then $b = c^{-1}b'' \in M$ satisfies $f(a, b) = 0$, as needed. ☐

Problem 8. Prove or disprove the following statement: Let K be a finitely generated transcendental extension of $\mathbb{Q}$. Let N be a Galois extension of K different from $\tilde{K}$. Then N is not PAC over K.

Problem 9. The fact that $\mathbb{Q}_{\mathrm{solv}}$ is not PAC over $\mathbb{Q}$ implies the existence of an absolutely irreducible polynomial $f \in \mathbb{Q}_{\mathrm{solv}}[X, Y]$ such that for all $a \in \mathbb{Q}$ the equation $f(a, Y) = 0$ has no solvable root. Is it possible to choose f in $\mathbb{Q}[X, Y]$?

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