Journal de Théorie des Nombres
de Bordeaux **18** (2006), 677–682

On extreme forms in dimension 8

par Cordian RIENER

Résumé. Par un théoreme de Voronoi, un réseau est extrême si et seulement s'il est parfait et eutactique. La classification des réseaux parfaits a été récemment obtenue en dimension 8 ([5]). Il y a 10916 réseaux parfaits. En utilisant des méthodes de programmation linéaire, nous obtenons la liste de ceux de ces réseaux qui sont eutactiques. En petite dimension, presque tous les réseaux parfaits sont également eutactiques. Ce n'est plus le cas à partir de la dimension 8 : il n'y a que 2408 réseaux extrêmes de dimension 8.

Abstract. A theorem of Voronoi asserts that a lattice is extreme if and only if it is perfect and eutactic. Very recently the classification of the perfect forms in dimension 8 has been completed [5]. There are 10916 perfect lattices. Using methods of linear programming, we are able to identify those that are additionally eutactic. In lower dimensions almost all perfect lattices are also eutactic (for example 30 out of the 33 in dimension 7). This is no longer the case in dimension 8: up to similarity, there are only 2408 extreme 8-dimensional lattices.

1. Introduction

A lattice Λ is a discrete subgroup of rank n of the Euclidean vector space $\mathbb{R}^n$ endowed with a scalar product $(\ \cdot\ )$. For details on the theory of lattices we refer to Martinet's book [7]. We note $\mathrm{N}(x)$ the squared norm of an element x of $\mathbb{R}^n$. The norm or minimum of Λ is defined as $\mathrm{N}(\Lambda) = \min_{x\in\Lambda, x\neq 0} x\cdot x$ and the set $S(\Lambda) := \{x \in \Lambda : x\cdot x = \mathrm{N}(\Lambda)\}$ is the set of minimal vectors of Λ.

Given a family $\mathcal{F}$ of vectors $x_1, ..., x_n \in \mathbb{R}^n$, the Gram matrix of $\mathcal{F}$ is $\mathrm{Gram}(\mathcal{F}) := ((x_i \cdot x_j))$ and we call a *Gram matrix of* Λ the Gram matrix of any $\mathbb{Z}$-base $\mathcal{B} = (b_1, ..., b_n)$ of Λ. The determinant of such a matrix is the determinant $\det(\Lambda)$ of Λ. Given a pair $(\Lambda, \mathcal{B})$, we attach to it the quadratic from $q(x) = \mathrm{N}(\sum_i x_i b_i)$, which depends only on the isometry class of $(\Lambda, \mathcal{B})$. This dictionary gives a one-to-one correspondence between

Manuscrit reçu le 14 novembre 2005.

similarity classes of lattices and equivalence classes over $\mathbb{Z}$ of quadratic forms having the same minimum.

One important invariant of these similarity classes was introduced by Hermite in 1845. It is defined as:

$$\gamma(\Lambda) = \frac{\mathrm{N}(\Lambda)}{\det(\Lambda)^{\frac{1}{n}}}.$$

Geometrically $\gamma(\Lambda)$ is a measure for the density of the sphere packing attached to Λ. Its maximum $\gamma_n := \max_{\Lambda \subset E} \gamma(\Lambda)$, the so called *Hermite constant*, leads therefore to the densest lattice sphere packing in dimension n. Following the approach of Korkine and Zolotareff, we call a lattice *extreme* if $\gamma(\Lambda)$ attains a local maximum in Λ, i.e., one has $\gamma(\Lambda') \leq \gamma(\Lambda)$ for any Λ' close enough to Λ. In order to find the local maxima of the Hermite function one uses the properties of perfection and eutaxy, which are defined as follows: with $0 \neq x \in E$ we associate the rank one symmetric matrix $P_x := xx^t$. A lattice is called *perfect* if the $P_x, x \in S(\Lambda)$ span the space of all symmetric matrices Sym_n.

Let Λ be a lattice with Gram-Matrix A. A *eutaxy relation* for Λ is a set of coefficients $\lambda_x \in \mathbb{R}$, $x \in S(\Lambda)$ such that $A^{-1} = \sum_{x \in S(\Lambda)} \lambda_x P_x$. A lattice is called *weakly eutactic* if there exists a eutaxy relation. Furthermore it is called *semi-eutactic* if the λ_x are ≥ 0 and *eutactic* if they may be chosen strictly positive.
Note that perfection already implies weak eutaxy. Using the notions of perfection and eutaxy, Voronoi ([8]) has shown:

Theorem 1. (Voronoi) *A lattice Λ is extreme if and only if it is both perfect and eutactic.*

Using the algorithm developed by Voronoi himself, one can obtain a complete classification of all perfect lattices in a given dimension. This work has been carried out up to dimension 8. In the next section we will describe how to use methods of linear programming in order to decide whether a lattice is eutactic.

2. Eutactic perfect lattices

The perfect lattices of dimension 8 have been found by Bergé, Martinet and their students and very recently Dutour, Schürmann and Vallentin [5] have been able to show that this classification is complete. An interesting question still remains: Which are the eutactic forms within those 10916 perfect forms? In low dimensions almost all perfect lattices are also eutactic. Martinet conjectures in his book [7] that this no longer is the case from dimension 8 onwards. In order to transfer the question of eutaxy into the language of linear programming, we just present some simple facts about convex cones.

Let E be a Euclidean vector space of dimension m. A set $\mathcal{C} \subset E$ is called a *cone* if for $x, y \in \mathcal{C}$ we have $x + y \in \mathcal{C}$ and furthermore $\alpha x \in \mathcal{C}$ for $0 \leq \alpha$. Moreover $\mathcal{C}$ is called *pointed* if $\mathcal{C} \bigcap (-\mathcal{C}) = \{0\}$.
For $X \subset E$ the set

$$\mathit{cone}\ (X) := \left\{ \sum a_i x_i : 0 \leq a_i, x_i \in X \right\}$$

is called the cone induced by X.If X is finite cone (X) is called *polyhedral*. For every convex cone $\mathcal{C}$, we can form its *dual cone*, which is defined as

$$\mathcal{C}^* := \{u \in E \mid \forall\, x \in \mathcal{C} : \langle u, x \rangle \leq 0\}.$$

This is again a convex cone and if $\mathcal{C}$ is closed we have $\mathcal{C} = \mathcal{C}^{**}$.
If $\mathcal{C} := \mathit{cone}(x_1, .., x_n)$ is a polyhedral cone, the dual cone is just

$$\mathcal{C}^* := \{y \in E \mid \langle x_i, y \rangle \leq 0 \ \forall\ 1 \leq i \leq n\}.$$

On the space Sym_n we consider the scalar product $\langle A, B \rangle := Tr(AB)$. Now we want to express eutaxy and perfection in the language of cones. For this aim we give the following definition:

Definition 2. *Let $\Lambda \subset \mathbb{R}^n$ be a lattice. Then*

$$D_\Lambda := \mathrm{cone}\,(P_x)_{x \in S(\Lambda)}$$

is called the Voronoi domain of Λ.

For a lattice Λ we can now translate the notions of perfection and eutaxy:

(1) A lattice Λ is perfect if $\mathcal{D}_\Lambda^*$ is pointed.
(2) A lattice Λ is semi-eutactic if $A^{-1} \in \mathcal{D}_\Lambda$ and eutactic if $A^{-1} \in ri(\mathcal{D}_\Lambda)$ (ri stands for the *relative interior*).

Now to say that a perfect form is only weakly eutactic is equivalent to say that A^{-1} is not inside the Voronoi domain. This is the case when A^{-1} can be separated from the Voronoi domain by a hyperplane. We have the following:

Theorem 3. *Let $\mathcal{C}$ be a closed cone in E. Then $p \in V \backslash \mathcal{C}$ if and only if there exists $u \in \mathcal{C}^*$ such that $\langle p, u \rangle > 0$.*

Now the distinction between semi-eutaxy and eutaxy necessitates that we winnow the interior and the boundary of a convex cone:

For $u \in E$ we define the hyperplane $H(u) := \{x \in V | \langle u, x \rangle = 0\}$ and the half space $H^-(u) := \{x \in V | \langle u, x \rangle \leq 0\}$. We say that such a hyperplane is a supporting hyperplane if $H(u) \cap \mathcal{C} \neq \emptyset$ and $\mathcal{C} \subset H^-(u)$. Such a support is called non-trivial if $\mathcal{C} \cap H \neq \mathcal{C}$. If $\mathcal{C}$ is a polyhedral cone, only the boundary points can be in a hyperplane that defines a non-trivial support. Thus we have

Theorem 4. *Let $\mathcal{C}$ be a closed cone then $x \in int(\mathcal{C}^*)$ if and only if $H(u)$ is a supporting hyperplane and $x \in H(u)$ implies $u \in \mathcal{C}^{\perp}$.*

With these properties of polyhedral cones we want to decide the question whether a perfect lattice is eutactic or not using linear programming methods. Linear programming (abbreviated by (LP)) is a technique to find a maximizer or minimizer of a linear function subject to linear inequalities More precisely:

Definition 5. *Let $A \in \mathbb{R}^{m\times n}$ a $m \times n$ matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. A linear problem (LP) is of the form*

$$\begin{array}{rrcl} & c \cdot x & \rightarrow & \max \\ \textit{subject to} & Ax & \geq & b \end{array}$$

The convex polyhedron $\mathcal{P} := \{x \in \mathbb{R}^n | Ax \geq b\}$ is called the feasible region.

An efficient algorithm to solve such problems is the so called Simplex Algorithm introduced by Danzig [4]. It uses the fact that among the optimal solutions of an (LP), there is always one that is a vertex of $\mathcal{P}$.

Using the separation theorems for polyhedral cones, we are now able to investigate the list of perfect lattices in dimension 8. To this end we proceed in three steps:

(1) We try to find a separating hyperplane. If we find one, the lattice is only weakly eutactic and we are done
(2) When we do not find a separating hyperplane, we look for a supporting hyperplane which contains A^{-1}. If there is such a supporting hyperplane, the lattice is semi-eutactic.
(3) If we are not able to find neither a separation nor a supporting hyperplane, the lattice must be eutactic. To prove this, we look for a eutaxy relation.

To simplify the notation of the upcoming LPs, we will use $a \in \mathbb{R}^d$ for the point that represents A^{-1} and $p_1, ..., p_s$ for the points that represent the P_x. In order to deal with the problem of the first step, we consider the following program:

$$\begin{array}{rrcl} & x \cdot a & \rightarrow & \max \\ \text{subject to} & x \cdot p_i & \leq & 0 \ \forall\, i = 1, ..., s \\ & x \cdot a & \leq & 100 \end{array}$$

The last inequality is artificially added in order that the LP should have a bounded solution. If we find any solution vector x^*, for which the optimal solution of the LP is strictly positive, we get a separating hyperplane, thus the lattice is not semi-eutactic and therefore also not eutactic.

For the lattices that passed the first test we now want to sort out those that are only semi-eutactic. For this aim, we investigate the following (LP):

$$\begin{array}{rrcl} & \sum x \cdot p_i & \to & \min \\ \text{subject to} & x \cdot p_i & \leq & 0 \ \forall i = 1, ..., s \\ & x \cdot a & = & 0 \\ & \sum x \cdot p_i & \geq & -100 \end{array}$$

Here again the last inequality was added to ensure the existence of a bounded solution. As for perfect lattices $\mathcal{D}^*_\Lambda$ is pointed, the existence of an optimal solution which is strictly negative provides a supporting hyperplane that contains A^{-1}. The lattice thus is only semi-eutactic.

The lattices that passed these two tests are expected to be eutactic. In order to prove this, we look for a eutaxy relation. As in the last section the Matrix $B := (p_i)_{i=1}^s$ is the matrix whose columns are the points p_i. The eutaxy relation therefore has the form: $a = Bx$, where x should be a vector with strictly positive components. In order to find such a vector x, we substract from each of its components a slack variable λ. We then try to maximize the value of λ, in other words we look at the following LP:

$$\begin{array}{rrcl} & \lambda & \to & \max \\ \text{subject to} & Bx & = & a \\ & x_i - \lambda & \geq & 0 \ \forall \ i = 1, .., s \end{array}$$

If the lattice we investigate is eutactic, the optimal value of λ must be strictly positive and then the vector x provides a eutaxy relation.

3. Results

There are many implementations of the simplex algorithms available on the Internet. We decided to use the one that is implemented in the lrs-package of David Avis[1]. Since it works with exact arithmetics, we can be sure to avoid rounding errors.

In the following table we summarize the results for dimension 8. We order them by increasing half-kissing numbers $s := \frac{1}{2}|S(\Lambda)|$. More detailed tables are available on the *A2x-Web-Pages on Lattices*[2].

Aknowledgements. The author is very grateful to Christine Bachoc, Jacques Martinet and Gabriele Nebe for helpful discussions.

References

[1] D. Avis, *The lrs homepage*, http://cgm.cs.mcgill.ca/~avis/C/lrs.html.

[2] Ch. Batut, J. Martinet, *A2x-Web-Pages on Lattices*, http://math.u-bordeaux.fr/~martinet/.

[3] Ch. Batut, *Classification of quintic eutactic forms.* Math. Comp. **70** (2001), 395–417.

[4] G. Danzig, *Linear Programming and Extensions.* Princeton University Press (1963).

[5] M. Dutour, A. Schürmann, F. Vallentin, *Classification of eight-dimensional perfect forms.* Electron. Res. Announc. Amer. Math. Soc. **13** (2007).

[6] D.-O. Jaquet-Chiffelle, *Énumération complète des classes de formes parfaites en dimension 7.* Ann. Inst. Fourier **43** (1993), 21–55.

[7] J. Martinet, *Perfect Lattices in Euclidean Spaces.* Springer–Verlag, Heidelberg (2003).

[8] G. VORONOI, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques: 1. Sur quelques propriétés des formes quadratiques parfaites.* J. reine angew. Math. **133** (1908), 97–178.

s	Extreme	Semi-Eutactic	Only Perfect	Total
36	858	8	5388	6254
37	513	1	1519	2033
38	471	6	1021	1498
39	212	1	288	501
40	156	6	180	342
41	71	3	47	121
42	44	1	24	69
43	19	0	7	26
44	20	0	4	24
45	11	1	1	13
46	10	1	1	12
47	4	0	0	4
48	4	0	0	4
49	1	0	0	1
50	2	0	0	2
51	3	0	0	3
52	1	0	0	1
54	4	0	0	4
56	1	0	0	1
58	1	0	0	1
71	1	0	0	1
120	1	0	0	1
total	2408	28	8480	10916

TABLE 1. Results in dimension 8

Cordian RIENER
J.W. Goethe-Universitaet
Fb. Mathematik u. Informatik
60054 Frankfurt am Main
E-mail : riener@math.uni-frankfurt.de