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Weighted uniform densities

par Rita GIULIANO ANTONINI et Georges GREKOS

Résumé. Nous introduisons la notion de densité uniforme pondérée (supérieure et inférieure) d'une partie A de $\mathbf{N}^*$, par rapport à une suite de poids (a_n). Ce concept généralise la notion classique de la densité uniforme (pour laquelle les poids sont tous égaux à 1). Nous démontrons un théorème de comparaison de deux densités uniformes (ayant des suites de poids différentes) et un théorème de comparaison d'une densité pondérée uniforme et d'une densité pondérée classique (asymptotique; non uniforme). Comme conséquence, nous obtenons un nouveau majorant et un nouveau minorant pour l'ensemble des $\alpha-$densités (classiques) d'une partie A de $\mathbf{N}^*$.

Abstract. We introduce the concept of uniform weighted density (upper and lower) of a subset A of $\mathbb{N}^*$, with respect to a given sequence of weights (a_n). This concept generalizes the classical notion of uniform density (for which the weights are all equal to 1). We also prove a theorem of comparison between two weighted densities (having different sequences of weights) and a theorem of comparison between a weighted uniform density and a weighted density in the classical sense. As a consequence, new bounds for the set of (classical) α–densities of A are obtained.

1. Introduction

The uniform upper and lower densities of a subset A of $\mathbb{N}^*$ are defined respectively by

$$\lim_{h\to\infty} \limsup_{n\to\infty} \frac{\operatorname{card}\{k \in A, n+1 \le k \le n+h\}}{h},$$

$$\lim_{h\to\infty} \liminf_{n\to\infty} \frac{\operatorname{card}\{k \in A, n+1 \le k \le n+h\}}{h}.$$

The notion of uniform density is currently used and the first aim is generally to prove that the limits of limsup and liminf exist: see for instance [2], [3],

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[6], [8] as references on this topic. Notice that, setting

$$\operatorname{card}\{k \in A, n+1 \leq k \leq n+h\} = \sum_{\substack{n+1 \leq k \leq n+h \\ k \in A}} 1; \qquad h = \sum_{n+1 \leq k \leq n+h} 1,$$

the above ratio takes the form

$$\frac{\sum_{\substack{n+1 \leq k \leq n+h \\ k \in A}} a_k}{\sum_{k=n+1}^{n+h} a_k},$$

where $a_k = 1$ for every integer k, and the corresponding upper and lower densities reveal now their nature of "weighted" uniform densities with constant "weights" equal to 1.

Hence the problem of looking for a "general" definition of weighted uniform density (*i.e.,* with a not necessarily constant sequence of weights (a_n)) becomes quite natural. Such a definition is given in section 2 of the present paper; moreover, in Theorem (2.13) we prove that, under some very general conditions on the defining sequence (a_n), *every* $A \subseteq \mathbb{N}^*$ has upper and lower weighted uniform a–densities.

In section 3 the problem of comparing two such weighted uniform densities is studied. A general theorem of comparison is proved (Theorem (3.5)). Such a result is natural if one looks at the analogous well known result on weighted but non uniform densities (also studied for instance in [1] and [7]) proved in [5] (see also [4] for a recent extension).

Section 4 is devoted to the comparison between a weighted uniform and a weighted but not uniform density. The main result of this section is Theorem (4.5), which reveals the astonishing fact that it is possible to compare *any* weighted density with *any* uniform weighted density (provided some very general assumptions are satisfied, of course).

By specializing Theorem (4.5) in the case of α–densities, we obtain a result that provides new bounds (*i.e.*, different in principle from the trivial ones 0 and 1) for the set of α–densities (upper and lower) of a given set A.

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2. Weighted uniform density

Let $a = (a_n)_{n \geq 1}$ be a sequence of non-negative real numbers. For any $n \in \mathbb{N}$, $h \in \mathbb{N}^*$, put

$$S_{n,h} \doteq \sum_{k=n+1}^{n+h} a_k.$$

For simplicity we set $S_n \doteq S_{0,n}$, and we assume that

$$\lim_{n \to \infty} S_n = +\infty.$$

For every subset A of $\mathbb{N}^*$, put now

$$1_A(k) = \begin{cases} 1 & \text{if } k \in A, \\ 0 & \text{if } k \in A^c, \end{cases}$$

and for every pair of integers n, h,

$$s_{n,h}(A) \doteq \sum_{\substack{n+1 \le k \le n+h \\ k \in A}} a_k = \sum_{k=n+1}^{n+h} a_k 1_A(k).$$

(2.1) **Definition.** Let $A \subseteq \mathbb{N}^*$ and assume that the two limits

$$\overline{u}_a(A) \doteq \lim_{h\to\infty} \limsup_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}, \qquad \underline{u}_a(A) \doteq \lim_{h\to\infty} \liminf_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}$$

exist; then $\overline{u}_a(A)$ and $\underline{u}_a(A)$ are called respectively the *upper* and the *lower uniform density* of A with respect to the weight sequence (a_n) (or, more briefly, *uniform a–densities*).

It is immediate that A has a lower (resp. upper) uniform a–density if and only if A^c has an upper (resp. lower) uniform a–density and that

$$\underline{u}_a(A) = 1 - \overline{u}_a(A^c). \tag{2.2}$$

The following assumption will be an essential tool in the sequel:

$$\textit{for every fixed integer } q \ge 1, \qquad \lim_{r\to\infty} \sup_{n\in\mathbb{N}} \frac{S_{n+rq,q}}{S_{n,rq}} = 0. \tag{2.3}$$

Note that (2.3) holds if (a_n) is not increasing, since in this case

$$S_{n+rq,q} = \sum_{k=n+rq+1}^{n+(r+1)q} a_k \le q\, a_{n+rq+1} = \frac{rq}{r}\, a_{n+rq+1} \le \frac{1}{r} \sum_{k=n+1}^{n+rq} a_k = \frac{1}{r} S_{n,rq}.$$

We now investigate another relevant situation in which (2.3) holds.

(2.4) **Proposition.** *Assume that*

(i) there exist two strictly positive constants $C_1 \le C_2$ such that, for every pair of integers $n \ge 0$ and $h \ge 1$, the following bounds hold

$$C_1\, h\, a_{n+h} \le S_{n,h} \le C_2\, h\, a_{n+h}; \tag{2.5}$$

(ii)

$$\sup_{n,r,q} \frac{a_{n+(r+1)q}}{a_{n+rq}} = M < +\infty, \tag{2.6}$$

where the sup *is taken over $n \in \mathbb{N}$, $r \in \mathbb{N}^*$, $q \in \mathbb{N}^*$.*
Then (2.3) holds (uniformly in q).

Proof. It is straightforward, since, from (2.5) and (2.6) we get, for every q, r, n,

$$\frac{S_{n+rq,q}}{S_{n,rq}} \leq \frac{C_2}{C_1}\frac{1}{r}\frac{a_{n+(r+1)q}}{a_{n+rq}} \leq M\frac{C_2}{C_1}\frac{1}{r}.$$

$\square$

A case in which (2.6) holds is exhibited in the following result.

(2.7) **Proposition.** *Assumption (2.6) holds if* (a_n) *is increasing and*

$$\sup_{n\in\mathbb{N}^*}\frac{a_{2n}}{a_n} = H < +\infty.$$

In such a case, we have $M \leq H$.

Proof. For every q, r, n,

$$a_{n+(r+1)q} \leq a_{2n+2rq} \leq Ha_{n+rq}.$$

$\square$

The following proposition concerns a case in which (2.5) holds. Let α be a real number, $\alpha > -1$, and consider the sequence $a_n = n^\alpha$.

(2.8) **Proposition.** *(i) Let* $\alpha \geq 0$. *Then, given a pair of integers* $n \geq 0$ *and* $h \geq 1$, *we have the inequalities*

$$\frac{1}{\alpha+1}h(n+h)^\alpha \leq \sum_{k=n+1}^{n+h} k^\alpha \leq h(n+h)^\alpha. \tag{2.9}$$

(ii) Let $-1 < \alpha < 0$. *Then, given a pair of integers* $n \geq 0$ *and* $h \geq 1$, *we have the inequalities*

$$h(n+h)^\alpha \leq \sum_{k=n+1}^{n+h} k^\alpha \leq \frac{1}{\alpha+1}h(n+h)^\alpha. \tag{2.10}$$

Proof. The second (resp. first) inequality in (2.9) (resp. (2.10)) is immediate since $k \mapsto k^\alpha$ is increasing (resp. decreasing). For the first (resp. second) one, notice that

$$\sum_{k=n+1}^{n+h} k^\alpha \geq \int_n^{n+h} t^\alpha dt \qquad \text{(resp. } \sum_{k=n+1}^{n+h} k^\alpha \leq \int_n^{n+h} t^\alpha dt\text{)},$$

hence it is enough to prove that the same inequality holds for the integral above. The statement in the proposition can be reformulated as follows (immediate proof):

(2.11) **Lemma.** *(i) Let $\alpha \geq 0$. Then for every pair of positive real numbers x, y with $x \leq y$, we have*

$$(y-x)y^{\alpha} \leq y^{\alpha+1} - x^{\alpha+1}.$$

(ii) Let $-1 < \alpha < 0$. Then for every pair of positive real numbers x, y with $x \leq y$, we have

$$(y-x)y^{\alpha} \geq y^{\alpha+1} - x^{\alpha+1}.$$

$\square$

(2.12) **Remark.** For $-1 < \alpha \leq 0$, $a_n = n^{\alpha}$ is not increasing; thus, part (ii) of the above proposition is of no utility in this section. We have included it for future reference (see Remark (3.4)).

We now state our main result.

(2.13) **Theorem.** *Assume that the sequence (a_n) is such that (2.3) holds. Then, given a subset A of $\mathbb{N}^*$,*
(i) the two limits

$$\lim_{h\to\infty} \liminf_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}, \qquad \lim_{h\to\infty} \limsup_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}$$

exist;
(ii) the following relations hold

$$\lim_{h\to\infty} \liminf_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}} = \sup_{h} \liminf_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}, \tag{2.14}$$

$$\lim_{h\to\infty} \limsup_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}} = \inf_{h} \limsup_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}. \tag{2.15}$$

Proof. We put for simplicity

$$l_h \doteq \liminf_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}, \qquad L_h \doteq \limsup_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}}.$$

With these notations, formulas (2.14) and (2.15) take the forms

$$\lim_{h\to\infty} l_h = \sup_{h\in\mathbb{N}^*} l_h, \qquad \lim_{h\to\infty} L_h = \inf_{h\in\mathbb{N}^*} L_h$$

and are an immediate consequence of Lemma (2.16) below.

(2.16) **Lemma.** *Let $\delta > 0$ be fixed. Then there exist two integers n_0 and h_0 such that, for every $n > n_0$ and every $h > h_0$, we have*

$$l_q - 2\delta \leq \frac{s_{n,h}(A)}{S_{n,h}} \leq L_q + 2\delta.$$

Proof of Lemma (2.16). Since A is fixed, we shall adopt the simplified notation $s_{n,h}$ in place of $s_{n,h}(A)$. It will be enough to prove the right inequality above. By the definition of L_q, there exists an integer n_0 (depending on q of course) such that

$$\forall n > n_0, \ \frac{s_{n,q}}{S_{n,q}} < L_q + \delta\,. \tag{2.17}$$

Assumption (2.3) implies that there exists an r_0 such that

$$\forall r > r_0\,, \ \forall n \in \mathbb{N}^*, \ \frac{S_{n+rq,q}}{S_{n,rq}} < \delta\,. \tag{2.18}$$

Put now $h_0 = (r_0 + 1)q$ and let h be any fixed integer, with $h > h_0$. Then $h = rq + p$, where r and p are two integers satisfying the relations $r > r_0$ and $0 \le p < q$. After these preliminaries, we are ready to give the required estimation of

$$\frac{s_{n,h}}{S_{n,h}}, \quad n > n_0, \quad h > h_0.$$

We can write

$$s_{n,h} = \sum_{j=0}^{r-1} s_{n+jq,q} + s_{n+rq,p}.$$

Since $n + jq \ge n > n_0$ ($j = 0, 1, \ldots, r-1$), by (2.17) $s_{n,h}$ is not greater than

$$(L_q + \delta) \sum_{j=0}^{r-1} S_{n+jq,q} + S_{n+rq,q} = (L_q + \delta) S_{n,rq} + S_{n+rq,q},$$

and dividing by $S_{n,h}$ ($h = rq + p \ge rq$), we get

$$\frac{s_{n,h}}{S_{n,h}} \le (L_q + \delta)\frac{S_{n,rq}}{S_{n,h}} + \frac{S_{n+rq,q}}{S_{n,h}} \le (L_q + \delta) + \frac{S_{n+rq,q}}{S_{n,rq}} = L_q + 2\delta,$$

where we have used relation (2.18) since $r > r_0$. □

3. Comparing two uniform weighted densities

In this section we suppose that (a_n) and (b_n) are two sequences of non-negative real numbers. For (a_n) the notations and assumptions will be as in Theorem (2.13). Analogous notations will be in force for (b_n); more precisely, for every pair of integers n and h, we put

$$T_{n,h} \doteq \sum_{k=n+1}^{n+h} b_k$$

and, for every subset A of $\mathbb{N}^*$,

$$t_{n,h}(A) \doteq \sum_{\substack{n+1 \le k \le n+h \\ k \in A}} b_k = \sum_{k=n+1}^{n+h} b_k 1_A(k).$$

For simplicity we set $T_n \doteq T_{0,n}$. Suppose that

$$\lim_{n\to\infty} T_n = +\infty.$$

We assume in addition that there exist two strictly positive constants $D_1 \leq D_2$ such that, for every pair of integers $n \geq 0$ and $h \geq 1$, the following bounds

$$D_1\, h\, b_{n+h} \leq T_{n,h} \leq D_2\, h\, b_{n+h} \tag{3.1}$$

hold; moreover

$$\sup_{n,r,q} \frac{b_{n+(r+1)q}}{b_{n+rq}} = N < +\infty, \tag{3.2}$$

(where the supremum is taken over $n \in \mathbb{N}$, $r \in \mathbb{N}^*$, $q \in \mathbb{N}^*$) and

$$\sup_h \limsup_{n\to\infty} \frac{b_n}{b_{n+h}} = R < +\infty. \tag{3.3}$$

(3.4) **Remark.** (i) We recall that assumptions (3.1) and (3.2) guarantee that (2.3) holds (see Proposition (2.4)), hence the definitions of $\underline{u}_b(A)$ and $\overline{u}_b(A)$ make sense.
(ii) As to the validity of assumption (3.1) in the case of weights $b_n = n^\alpha$ ($\alpha > -1$), we refer to Proposition (2.8) and Remark (2.12).
(iii) Assumption (3.3) holds trivially if

$$\limsup_n \frac{b_n}{b_{n+1}} = l \leq 1.$$

Since A will be fixed throughout, in the sequel we shall write more simply $s_{n,h}$, $\underline{u}_a$, $\overline{u}_a$ instead of $s_{n,h}(A)$, $\underline{u}_a(A)$, $\overline{u}_a(A)$ respectively. The analogous shortened notations $t_{n,h}$, $\underline{u}_b$, $\overline{u}_b$ will be used in place of $t_{n,h}(A)$, $\underline{u}_b(A)$, $\overline{u}_b(A)$ respectively.

Assume now that $a_n \neq 0$ for all n and define the sequence (c_n) by

$$\forall\, n \in \mathbb{N}^*, \quad c_n \doteq \frac{b_n}{a_n}\,.$$

We prove the following result.

(3.5) **Theorem.** *Assume that $(c_n)_n$ is not increasing. Then*

$$\underline{u}_a \leq \underline{u}_b \leq \overline{u}_b \leq \overline{u}_a.$$

Proof. By relation (2.2), it is enough to prove the part concerning upper densities, *i.e.,* the inequality

$$\overline{u}_b \leq \overline{u}_a. \tag{3.6}$$

The following auxiliary result can be easily deduced from Lemma (2.16).

(3.7) **Lemma.** *For any fixed $\epsilon > 0$, there exist two integers n_0 and h_0 such that, for every $n > n_0$ and every $h > h_0$, we have*

$$\underline{u}_a - \epsilon \leq \frac{s_{n,h}}{S_{n,h}} \leq \overline{u}_a + \epsilon. \tag{3.8}$$

Let now ϵ be fixed, and choose n_0 and h_0 such that (3.8) holds ($n > n_0$, $h > h_0$). For every *fixed* $n > n_0 + h_0$ consider the two sequences

$$s'_k \doteq s_{n-h_0,k-n+h_0},$$

$$S'_k \doteq S_{n-h_0,k-n+h_0}$$

defined for $k \geq n - h_0 + 1$. Notice that, if $k > n$, we have $k - n + h_0 > h_0$. Since $n - h_0 > n_0$, Lemma (3.7) yields

$$\forall\, k > n\,,\ \ \underline{u}_a - \epsilon \leq \frac{s'_k}{S'_k} \leq \overline{u}_a + \epsilon\,; \tag{3.9}$$

moreover

$$\forall\, k \geq n - h_0 + 2\,,\ \ a_k 1_A(k) = s'_k - s'_{k-1}\,. \tag{3.10}$$

Then, for every $h \in \mathbb{N}^*$, (3.10) gives

$$\begin{aligned} t_{n,h} &= \sum_{k=n+1}^{n+h} c_k(a_k 1_A(k)) = \sum_{k=n+1}^{n+h} c_k\big(s'_k - s'_{k-1}\big) = \sum_{k=n+1}^{n+h} c_k s'_k - \sum_{k=n}^{n+h-1} c_{k+1} s'_k \\ &= \sum_{k=n+1}^{n+h-1} (c_k - c_{k+1}) s'_k + c_{n+h} s'_{n+h} - c_{n+1} s'_n, \end{aligned} \tag{3.11}$$

and analogously

$$T_{n,h} = \sum_{k=n+1}^{n+h-1} (c_k - c_{k+1}) S'_k + c_{n+h} S'_{n+h} - c_{n+1} S'_n. \tag{3.12}$$

Now, take $h > h_0$.

Since (c_n) is not increasing, by (3.9) the second member in (3.11) is not greater than

$$(\overline{u}_a+\epsilon)\Big(\sum_{k=n+1}^{n+h-1}(c_k-c_{k+1})S'_k+c_{n+h}S'_{n+h}\Big)-c_{n+1}S'_n(\underline{u}_a-\epsilon)$$
$$=(\overline{u}_a+\epsilon)\Big(\sum_{k=n+1}^{n+h-1}(c_k-c_{k+1})S'_k+c_{n+h}S'_{n+h}-c_{n+1}S'_n\Big)$$
$$+(\overline{u}_a-\underline{u}_a+2\epsilon)c_{n+1}S'_n=(\overline{u}_a+\epsilon)T_{n,h}+(\overline{u}_a-\underline{u}_a+2\epsilon)c_{n+1}S'_n\,,$$

where in the last equality we have used relation (3.12).

Dividing by $T_{n,h}$ gives, for $h>h_0$,

$$\frac{t_{n,h}}{T_{n,h}}\le(\overline{u}_a+\epsilon)+(\overline{u}_a-\underline{u}_a+2\epsilon)c_{n+1}\frac{S'_n}{T_{n,h}}\,. \tag{3.13}$$

We now prove that

$$\lim_{h\to\infty}\limsup_{n\to\infty}c_{n+1}\frac{S'_n}{T_{n,h}}=0\,. \tag{3.14}$$

Indeed, note that (c_n) is not increasing, so that $c_{n+1}\le c_k$ for $k\le n+1$ and, for every $h\in\mathbb{N}^*$,

$$c_{n+1}\frac{S'_n}{T_{n,h}}=c_{n+1}\frac{S_{n-h_0,h_0}}{T_{n,h}}=c_{n+1}\frac{\sum_{k=n-h_0+1}^{n+1}a_k}{T_{n,h}}$$
$$\le\frac{\sum_{k=n-h_0+1}^{n+1}b_k}{T_{n,h}}=\frac{T_{n-h_0,h_0}}{T_{n,h}}\le\frac{C_2}{C_1}\frac{h_0}{h}\frac{b_n}{b_{n+h}},$$

by assumption (3.1). We thus get (3.14) using assumption (3.3).

Relation (3.14), used in (3.13), easily yields (3.6) (by passing to the $\limsup_n$ and after to the $\lim_h$), since ϵ is arbitrary. □

In some particular cases, it is possible to compare the uniform a–densities of a set A with its classical uniform densities $\underline{u}_0(A)$ and $\overline{u}_0(A)$ (obtained for $b_n=1=n^0$, $n\in\mathbb{N}^*$). More precisely we prove the following result.

(3.15) **Theorem.** *For every $n\ge0$ and $h\ge1$ put*

$$m_{n,h}=\min_{n+1\le k\le n+h}a_k;\qquad M_{n,h}=\max_{n+1\le k\le n+h}a_k.$$

Assume that

$$\limsup_{h\to\infty}\limsup_{n\to\infty}\frac{M_{n,h}}{m_{n,h}}\doteq\sigma<+\infty.$$

Note that $\sigma \geq 1$. Then, for every $A \subseteq \mathbb{N}^*$ such that $\overline{u}_a(A)$ (resp. $\underline{u}_a(A)$) exists,

$$\overline{u}_a(A) \leq \sigma \overline{u}_0(A) \qquad (\textit{resp.}\, \underline{u}_a(A) \geq \sigma^{-1} \underline{u}_0(A)). \tag{3.16}$$

Proof. We prove only the first relation in (3.16). Let $A(x) = \operatorname{card} A \cap [1, x]$ be the counting function of the set A; then

$$\begin{aligned}\overline{u}_a(A) = \lim_{h\to\infty} \limsup_{n\to\infty} \frac{s_{n,h}(A)}{S_{n,h}} &\leq \limsup_{h\to\infty} \limsup_{n\to\infty} \frac{M_{n,h}}{m_{n,h}} \frac{A(n+h) - A(n)}{h} \\ &\leq \sigma \overline{u}_0(A).\end{aligned}$$

□

(3.17) **Remark.** The particular case $\sigma = 1$ is worth being pointed out, since in this situation there is equality between $\overline{u}_a(A)$ and $\overline{u}_0(A)$ (resp. $\underline{u}_a(A)$ and $\underline{u}_0(A)$). The relation $\sigma = 1$ holds for instance in the case of weights $a_n = n^\alpha$ (and, more generally, for a sequence of weights which is the restriction to $\mathbb{N}^*$ of a regularly varying monotone function defined on $[1, +\infty)$).

(3.18) **Remark.** The following examples show that a uniform weighted density does not always coincide with the classical uniform one. Denote by E the set of even integers; as it is well known, $\underline{u}_0(E) = \overline{u}_0(E)\,(= u_0(E)) = 1/2$.
(i) Consider the sequence $a_k = 1 + 1_E(k)$. It is easy to see that the uniform a–density of E (upper and lower) is equal to the number $2/3$.
(ii) Let $b > 1$ be fixed, and put $a_k = b^k$. It is an easy exercise to see that the uniform upper (resp. lower) a–density of the set E is equal to $b/(b+1)$ (resp. $1/(b+1)$). Observe that the sequence of weights considered in this example does not verify condition (2.3).

4. Comparing a weighted density and a uniform weighted density

Let (a_n) and (b_n) be two sequences of strictly positive real numbers. In particular the sequence (c_n), with $c_n \doteq b_n/a_n$ is defined. With the notations of the previous sections, assume that, for every fixed integer N, the following relations hold

$$\text{(i)} \quad \lim_{n\to\infty} \frac{S_{n,N}}{N\, a_n} = 1; \qquad \text{(ii)} \quad \lim_{n\to\infty} \frac{T_{n,N}}{N\, b_n} = 1; \tag{4.1}$$

$$\forall\, h \in \mathbb{N}^*, \quad \lim_{n\to\infty} \frac{S_{n,h}}{S_n} = 0; \tag{4.2}$$

put

$$m_n^{(N)} \doteq \min_{n+1 \le k \le n+N} c_k;$$

then

$$\lim_{n\to\infty} \frac{m_n^{(N)}}{c_n} = 1. \tag{4.3}$$

(4.4) **Remark.** (i) Assumption (4.3) holds if (c_n) is monotone and

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1.$$

(ii) The whole set of assumptions is verified for $a_n = n^\alpha$ and $b_n = n^\beta$, with $\alpha \ge -1$ and $\beta \ge -1$.

Recall that the *upper* and *lower a–densities* of the set $A \subseteq \mathbb{N}^*$ are defined respectively as

$$\overline{d}_a(A) \doteq \limsup_{n\to\infty} \frac{s_{0,n}(A)}{S_n}; \qquad \underline{d}_a(A) \doteq \liminf_{n\to\infty} \frac{s_{0,n}(A)}{S_n}.$$

We shall again omit the reference to the set A in each of the following notations and statements (A being fixed throughout).

The main result of this section is the following theorem.

(4.5) **Theorem.** *If the two sequences (a_n) and (b_n) verify the assumptions (4.1), (4.2) and (4.3), then*

$$\underline{u}_b \le \underline{d}_a \le \overline{d}_a \le \overline{u}_b.$$

Let $\alpha \ge -1$ be fixed. By applying the above result to the two sequences

$$a_k = k^\alpha, \qquad b_k = 1 = k^0,$$

we get the next corollary.

(4.6) **Corollary.** *For every $\alpha \ge -1$, we have the relations*

$$\underline{u}_0 \le \underline{d}_\alpha \le \overline{d}_\alpha \le \overline{u}_0.$$

(4.7) **Remark.** Recall (see [5]) that the function $\alpha \mapsto \overline{d}_\alpha(A)$ (resp. $\alpha \mapsto \underline{d}_\alpha(A)$) is increasing (resp. decreasing). Hence Corollary (4.6) yields

$$\lim_{\alpha\to\infty} \overline{d}_\alpha(A) = \sup_\alpha \overline{d}_\alpha(A) \le \overline{u}_0(A);$$

$$(\text{resp. } \lim_{\alpha\to\infty} \underline{d}_\alpha(A) = \inf_\alpha \underline{d}_\alpha(A) \ge \underline{u}_0(A)).$$

It is an open problem to establish in which cases the equality sign holds in the two relations above.

Proof of Theorem (4.5).

Let $0 < \epsilon < 1$ be fixed. There exist two integers h_1 and n_0 such that

$$\forall\, n > n_0, \quad \frac{t_{n,h_1}}{T_{n,h_1}} \le \overline{u}_b + \epsilon \tag{4.8}$$

(see Lemma (3.7)).

Put

$$r_1 = \left[\frac{n_0}{h_1}\right] + 1.$$

Note that $r_1 h_1 > n_0$.

For every $j \ge r_1$ we have $jh_1 \ge r_1 h_1 > n_0$ so that, by (4.8),

$$t_{jh_1,h_1} \le (\overline{u}_b + \epsilon) T_{jh_1,h_1}, \quad j \ge r_1. \tag{4.9}$$

For the same ϵ, by the assumptions (4.1), (4.2) and (4.3), there exists an integer $\tilde{r}$ such that, for every $j \ge \tilde{r}$, all the following inequalities hold

$$\begin{cases} T_{jh_1,h_1} < (1+\epsilon) h_1 b_{jh_1}; \\ (1-\epsilon) h_1 a_{jh_1} < S_{jh_1,h_1} < (1+\epsilon) h_1 a_{jh_1}; \\ m_{jh_1}^{(h_1)} > (1-\epsilon) c_{jh_1}. \end{cases} \tag{4.10}$$

We now put $r_0 \doteq \max\{\tilde{r}, r_1\}$. Let n be any integer. Then n can be written in the form $n = r_n h_1 + p_n$, where $0 \le p_n < h_1$. Obviously

$$\lim_{n\to\infty} r_n = +\infty; \tag{4.11}$$

moreover, if $n \ge (r_0 + 1)h_1$, then

$$r_n \ge r_0 + 1 \tag{4.12}$$

(in fact the inequality $n = r_n h_1 + p_n \ge (r_0 + 1)h_1$ is equivalent to

$$r_n \ge (r_0 + 1) - (p_n/h_1) > r_0,$$

i.e., $r_n \ge r_0 + 1$ since r_n and r_0 are integers).

For every integer n we have

$$S_n = \sum_{j=0}^{r_n - 1} S_{jh_1,h_1} + S_{r_n h_1, p_n} \ge \sum_{j=0}^{r_n - 1} S_{jh_1,h_1}. \tag{4.13}$$

In a similar way, for $n \ge (r_0 + 1)h_1$ (recall relation (4.12))

$$s_{0,n} = \sum_{j=0}^{r_0 - 1} s_{jh_1,h_1} + \sum_{j=r_0}^{r_n - 1} s_{jh_1,h_1} + s_{r_n h_1, p_n}. \tag{4.14}$$

Now, for every $j = r_0, \ldots, r_n - 1$,

$$s_{jh_1,h_1} = \sum_{k=jh_1+1}^{(j+1)h_1} a_k 1_A(k) = \sum_{k=jh_1+1}^{(j+1)h_1} c_k^{-1} b_k 1_A(k) \le \frac{1}{m_{jh_1}^{(h_1)}} t_{jh_1,h_1}$$

$$\le (\overline{u}_b + \epsilon)\frac{T_{jh_1,h_1}}{m_{jh_1}^{(h_1)}}, \tag{4.15}$$

where, in the last inequality, relation (4.9) has been used (since $j \ge r_0 \ge r_1$).

By inserting (4.15) (summed from r_0 to $r_n - 1$) into (4.14), we get

$$s_{0,n} \le M + (\overline{u}_b + \epsilon)\Big(\sum_{j=r_0}^{r_n-1} \frac{T_{jh_1,h_1}}{m_{jh_1}^{(h_1)}}\Big) + S_{r_nh_1,h_1},$$

where we have put for simplicity $M = \sum_{j=0}^{r_0-1} s_{jh_1,h_1}$. Dividing now the above relation by S_n and recalling (4.13) gives (for $n \ge (r_0+1)h_1$)

$$\frac{s_{0,n}}{S_{0,n}} \le \frac{M}{\sum_{j=0}^{r_n-1} S_{jh_1,h_1}} + (\overline{u}_b + \epsilon)\frac{\sum_{j=r_0}^{r_n-1} \frac{T_{jh_1,h_1}}{m_{jh_1}^{(h_1)}}}{\sum_{j=0}^{r_n-1} S_{jh_1,h_1}} + \frac{S_{r_nh_1,h_1}}{\sum_{j=0}^{r_n-1} S_{jh_1,h_1}}$$

$$= A_n + (\overline{u}_b + \epsilon)B_n + C_n. \tag{4.16}$$

Now

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{M}{\sum_{j=0}^{r_n-1} S_{jh_1,h_1}} = \lim_{n\to\infty} \frac{M}{S_{r_nh_1}} = 0 \tag{4.17}$$

by relation (4.11) and the basic assumption $\lim_{n\to\infty} S_n = \infty$ (see section 2).

Recalling the inequalities (4.10) (which hold for every $j \ge \tilde{r}$ and hence for every $j \ge r_0 \ge \tilde{r}$) we deduce

$$B_n \le \frac{\sum_{j=r_0}^{r_n-1} \frac{(1+\epsilon)h_1 b_{h_1 j}}{(1-\epsilon)c_{h_1 j}}}{\sum_{j=r_0}^{r_n-1} (1-\epsilon)h_1 a_{h_1 j}} = \frac{1+\epsilon}{(1-\epsilon)^2}; \tag{4.18}$$

and last

$$C_n = \frac{S_{r_nh_1,h_1}}{\sum_{j=0}^{r_n-1} S_{jh_1,h_1}} = \frac{S_{r_nh_1,h_1}}{S_{r_nh_1}},$$

whence, by (4.2) and (4.11),

$$\lim_{n\to\infty} C_n = 0. \tag{4.19}$$

Using relations (4.17), (4.18) and (4.19) in (4.16), the statement follows by the arbitrariness of ϵ. $\square$

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Rita Giuliano Antonini
Università di Pisa
Dipartimento di Matematica "L. Tonelli"
Largo Bruno Pontecorvo 5
56127 Pisa, Italia
E-mail : giuliano@dm.unipi.it
URL: http://www.dm.unipi.it/~giuliano/

Georges Grekos
Université Jean Monnet
23, rue du Dr Paul Michelon
42023 St Etienne Cedex 2, France
E-mail : grekos@univ-st-etienne.fr
URL: http://webperso.univ-st-etienne.fr/~grekos/