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Non-degenerate Hilbert cubes in random sets

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Résumé. Une légère modification de la démonstration du lemme des cubes de Szemerédi donne le résultat plus précis suivant: si une partie S de $\{1,\dots,n\}$ vérifie $|S| \geq \frac{n}{2}$, alors S contient un cube de Hilbert non dégénéré de dimension $\lfloor \log_2 \log_2 n - 3\rfloor$. Dans cet article nous montrons que dans un ensemble aléatoire avec les probabilités $\Pr\{s \in S\} = 1/2$ indépendantes pour $1 \leq s \leq n$, la plus grande dimension d'un cube de Hilbert non dégénéré est *proche* de $\log_2 \log_2 n + \log_2 \log_2 \log_2 n$ presque sûrement et nous déterminons la fonction seuil pour avoir un k-cube non dégénéré.

Abstract. A slight modification of the proof of Szemerédi's cube lemma gives that if a set $S \subset [1,n]$ satisfies $|S| \geq \frac{n}{2}$, then S must contain a non-degenerate Hilbert cube of dimension $\lfloor \log_2 \log_2 n - 3\rfloor$. In this paper we prove that in a random set S determined by $\Pr\{s \in S\} = \frac{1}{2}$ for $1 \leq s \leq n$, the maximal dimension of non-degenerate Hilbert cubes is a.e. nearly $\log_2 \log_2 n + \log_2 \log_2 \log_2 n$ and determine the threshold function for a non-degenerate k-cube.

1. Introduction

Throughout this paper we use the following notations: let $[1,n]$ denote the first n positive integers. The coordinates of the vector $\mathbf{A}^{(k,n)} = (a_0, a_1, \dots, a_k)$ are selected from the positive integers such that $\sum_{i=0}^{k} a_i \leq n$. The vectors $\mathbf{B}^{(k,n)}$, $\mathbf{A}_{\mathbf{i}}^{(k,n)}$ are interpreted similarly. The set S_n is a subset of $[1,n]$. The notations $f(n) = o(g(n))$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. An arithmetic progression of length k is denoted by AP_k. The rank of a matrix A over the field $\mathbb{F}$ is denoted by $r_{\mathbb{F}}(A)$. Let $\mathbb{R}$ denote the set of real numbers, and let $\mathbb{F}_2$ be the finite field of order 2.

Let n be a positive integer, $0 \leq p_n \leq 1$. The random set $S(n, p_n)$ is the random variable taking its values in the set of subsets of $[1,n]$ with the law determined by the independence of the events $\{k \in S(n,p_n)\}$, $1 \leq k \leq n$ with the probability $\Pr\{k \in S(n,p_n)\} = p_n$. This model is often used for

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proving the existence of certain sequences. Given any combinatorial number theoretic property P, there is a probability that $S(n,p_n)$ satisfies P, which we write $\Pr\{S(n,p_n) \models P)$. The function $r(n)$ is called a threshold function for a combinatorial number theoretic property P if

(i) When $p_n = o(r(n))$, $\lim_{n\to\infty} \Pr\{S(n,p_n) \models P\} = 0$,

(ii) When $r(n) = o(p(n))$, $\lim_{n\to\infty} \Pr\{S(n,p_n) \models P\} = 1$,

or visa versa. It is clear that threshold functions are not unique. However, threshold functions are unique within factors $m(n)$, $0 < \liminf_{n\to\infty} m(n) \le \limsup_{n\to\infty} m(n) < \infty$, that is if p_n is a threshold function for P then p'_n is also a threshold function iff $p_n = O(p'_n)$ and $p'_n = O(p_n)$. In this sense we can speak of the threshold function of a property.

We call $H \subset [1,n]$ a Hilbert cube of dimension k or, simply, a k-cube if there is a vector $\mathbf{A}^{(k,n)}$ such that

$$H = \mathbf{H}_{\mathbf{A}^{(k,n)}} = \{a_0 + \sum_{i=1}^{k} \epsilon_i a_i : \epsilon_i \in \{0,1\}\}.$$

The positive integers $a_1, \dots, a_k$ are called the generating elements of the Hilbert cube. The k-cube is non-degenerate if $|H| = 2^k$ i.e. the vertices of the cube are distinct, otherwise it is called degenerate. The maximal dimension of a non-degenerate Hilbert cube in S_n is denoted by $H_{max}(S_n)$, i.e. $H_{max}(S_n)$ is the largest integer l such that there exists a vector $\mathbf{A}^{(l,n)}$ for which the non-degenerate Hilbert cube $\mathbf{H}_{\mathbf{A}^{(l,n)}} \subset S_n$.

Hilbert originally proved that if the positive integers are colored with finitely many colors then one color class contains a k-cube. The density version of theorem was proved by Szemerédi and has since become known as "Szemerédi's cube lemma". The best known result is due to Gunderson and Rödl (see [3]):

Theorem 1.1 (Gunderson and Rödl)**.** *For every $d \ge 3$ there exists $n_0 \le (2^d - 2/\ln 2)^2$ so that, for every $n \ge n_0$, if $A \subset [1,n]$ satisfies $|A| \ge 2n^{1-\frac{1}{2^d-1}}$, then A contains a d-cube.*

A direct consequence is the following:

Corollary 1.2. *Every subset S_n such that $|S_n| \ge \frac{n}{2}$ contains a $\lfloor \log_2 \log_2 n \rfloor$-cube.*

A slight modification of the proof gives that the above set S_n must contain a non-degenerate $\lfloor \log_2 \log_2 n - 3 \rfloor$-cube.

Obviously, a sequence S has the Sidon property (that is the sums $s_i + s_j$, $s_i \le s_j$, $s_i, s_j \in S$ are distinct) iff S contains no 2-cube. Godbole, Janson, Locantore and Rapoport studied the threshold function for the Sidon property and gave the exact probability distribution in 1999 (see [2]):

Theorem 1.3 (Godbole, Janson, Locantore and Rapoport). *Let $c > 0$ be arbitrary. Let P be the Sidon property. Then with $p_n = cn^{-3/4}$,*

$$\lim_{n\to\infty} Pr\{S(n,p_n) \models P\} = e^{-\frac{c^4}{12}}.$$

Clearly, a subset $H \subset [1,n]$ is a degenerate 2-cube iff it is an AP_3. Moreover, an easy argument gives that the threshold function for the event "AP_3-free" is $p_n = n^{-2/3}$. Hence

Corollary 1.4. *Let $c > 0$ be arbitrary. Then with $p_n = cn^{-3/4}$,*

$$\lim_{n\to\infty} Pr\{S(n,p_n) \text{ contains no non-degenerate 2-cube}\} = e^{-\frac{c^4}{12}}.$$

In Theorem 1.5 we extend the previous Corollary.

Theorem 1.5. *For any real number $c > 0$ and any integer $k \geq 2$, if $p_n = cn^{-\frac{k+1}{2^k}}$,*

$$\lim_{n\to\infty} Pr\{S(n,p_n) \text{ contains no non-degenerate } k\text{-cube}\} = e^{-\frac{c^{2^k}}{(k+1)!k!}}.$$

In the following we shall find bounds on the maximal dimension of non-degenerate Hilbert cubes in the random set $S(n,\frac{1}{2})$. Let

$$D_n(\epsilon) = \lfloor \log_2\log_2 n + \log_2\log_2\log_2 n + \frac{(1-\epsilon)\log_2\log_2\log_2 n}{\log 2\log_2\log_2 n} \rfloor$$

and

$$E_n(\epsilon) = [\log_2\log_2 n + \log_2\log_2\log_2 n + \frac{(1+\epsilon)\log_2\log_2\log_2 n}{\log 2\log_2\log_2 n} \rfloor.$$

The next theorem implies that for almost all n, $H_{max}(S(n,\frac{1}{2}))$ concentrates on a single value because for every $\epsilon > 0$, $D_n(\epsilon) = E_n(\epsilon)$ except for a sequence of zero density.

Theorem 1.6. *For every $\epsilon > 0$*

$$\lim_{n\to\infty} Pr\{D_n(\epsilon) \leq H_{max}(S(n,\frac{1}{2})) \leq E_n(\epsilon)\} = 1.$$

2. Proofs

In order to prove the theorems we need some lemmas.

Lemma 2.1. *For $k_n = o(\frac{\log n}{\log\log n})$ the number of non-degenerate k_n-cubes in $[1,n]$ is $(1+o(1))\binom{n}{k_n+1}\frac{1}{k_n!}$, as $n \to \infty$.*

Proof. All vectors $\mathbf{A}^{(k_n,n)}$ are in 1-1 correspondence with all vectors $(v_0, v_1, \dots, v_{k_n})$ with $1 \le v_1 < v_2 < \cdots < v_{k_n} \le n$ in $\mathbb{R}^{k_n+1}$ according to the formulas $(a_0, a_1, \dots, a_{k_n}) \mapsto (v_0, v_1, \dots, v_{k_n}) = (a_0, a_0 + a_1, \dots, a_0 + a_1 + \cdots + a_{k_n})$; and $(v_0, v_1, \dots, v_{k_n}) \mapsto (a_0, a_1, \dots, a_{k_n}) = (v_0, v_1 - v_0, \dots, v_{k_n} - v_{k_n-1})$. Consequently,

$$\binom{n}{k_n+1} = |\{\mathbf{A}^{(k_n,n)} : \mathbf{H}_{\mathbf{A}^{(k_n,n)}} \text{ is non-degenerate}\}| + |\{\mathbf{A}^{(k_n,n)} : \mathbf{H}_{\mathbf{A}^{(k_n,n)}} \text{ is degenerate}\}|.$$

By the definition of a non-degenerate cube the cardinality of the set $\{\mathbf{A}^{(k_n,n)} : \mathbf{H}_{\mathbf{A}^{(k_n,n)}} \text{ is non-degenerate}\}$ is equal to

$$k_n!|\{\text{non-degenerate } k_n\text{-cubes in } [1,n]\}|,$$

because permutations of $a_1, \dots, a_k$ give the same k_n-cube. It remains to verify that the number of vectors $\mathbf{A}^{(k_n,n)}$ which generate degenerate k_n-cubes is $o(\binom{n}{k_n+1})$. Let $\mathbf{A}^{(k_n,n)}$ be a vector for which $\mathbf{H}_{\mathbf{A}^{(k_n,n)}}$ is a degenerate k_n-cube. Then there exist integers $1 \le u_1 < u_2 < \ldots < u_s \le k_n$, $1 \le v_1 < v_2 < \ldots < v_t \le k_n$ such that

$$a_0 + a_{u_1} + \ldots + a_{u_s} = a_0 + a_{v_1} + \ldots + a_{v_t},$$

where we may assume that the indices are distinct, therefore $s + t \le k_n$. Then the equation

$$x_1 + x_2 + \ldots + x_s - x_{s+1} - \ldots - x_{s+t} = 0$$

can be solved over the set $\{a_1, a_2 \dots, a_{k_n}\}$. The above equation has at most $n^{s+t-1} \le n^{k_n-1}$ solutions over $[1,n]$. Since we have at most k_n^2 possibilities for (s,t) and at most n possibilities for a_0, therefore the number of vectors $\mathbf{A}^{(k_n,n)}$ for which $\mathbf{H}_{\mathbf{A}^{(k_n,n)}}$ is degenerate is at most $k_n^2 n^{k_n} = o(\binom{n}{k_n+1})$. □

In the remaining part of this section the Hilbert cubes are non-degenerate.

The proofs of Theorem 1.5 and 1.6 will be based on the following definition. For two intersecting k-cubes $\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}$ let $\mathbf{H}_{\mathbf{A}^{(k,n)}} \cap \mathbf{H}_{\mathbf{B}^{(k,n)}} = \{c_1, \dots, c_m\}$ with $c_1 < \ldots < c_m$, where

$$c_d = a_0 + \sum_{l=1}^{k} \alpha_{d,l} a_l = b_0 + \sum_{l=1}^{k} \beta_{d,l} b_l, \quad \alpha_{d,l}, \beta_{d,l} \in \{0,1\}$$

for $1 \le d \le m$ and $1 \le l \le k$. The rank of the intersection of two k-cubes $\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}$ is defined as follows: we say that $r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}})$=(s,t) if for the matrices $A = (\alpha_{d,l})_{m \times k}$, $B = (\beta_{d,l})_{m \times k}$ we have $r_{\mathbb{R}}(A) = s$ and $r_{\mathbb{R}}(B) = t$. The matrices A and B are called matrices of the common vertices of $\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}$.

Lemma 2.2. *The condition* $r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}) = (s,t)$ *implies that*

$$|\mathbf{H}_{\mathbf{A}^{(k,n)}} \cap \mathbf{H}_{\mathbf{B}^{(k,n)}}| \le 2^{min\{s,t\}}.$$

Proof. We may assume that $s \le t$. The inequality $|\mathbf{H}_{\mathbf{A}^{(k,n)}} \cap \mathbf{H}_{\mathbf{B}^{(k,n)}}| \le 2^s$ is obviously true for $s = k$. Let us suppose that $s < k$ and the number of common vertices is greater than 2^s. Then the corresponding $(0-1)$-matrices A and B have more than 2^s different rows, therefore $r_{\mathbb{F}_2}(A) > s$, but we know from elementary linear algebra that for an arbitrary $(0-1)$-matrix M we have $r_{\mathbb{F}_2}(M) \ge r_{\mathbb{R}}(M)$, which is a contradiction. □

Lemma 2.3. *Suppose that the sequences* $\mathbf{A}^{(k,n)}$ *and* $\mathbf{B}^{(k,n)}$ *generate non-degenerate* k*-cubes. Then*

(1) $|\{(\mathbf{A}^{(k,n)}, \mathbf{B}^{(k,n)}) : r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}) = (s,t)\}| \le 2^{2k^2}\binom{n}{k+1} n^{k+1-max\{s,t\}}$ *for all* $0 \le s,t \le k$*;*

(2) $|\{(\mathbf{A}^{(k,n)}, \mathbf{B}^{(k,n)}) : r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}) = (r,r), |\mathbf{H}_{\mathbf{A}^{(k,n)}} \cap \mathbf{H}_{\mathbf{B}^{(k,n)}}| = 2^r\}| \le 2^{2k^2}\binom{n}{k+1} n^{k-r}$ *for all* $0 \le r < k$*;*

(3) $|\{(\mathbf{A}^{(k,n)}, \mathbf{B}^{(k,n)}) : r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}) = (k,k), |\mathbf{H}_{\mathbf{A}^{(k,n)}} \cap \mathbf{H}_{\mathbf{B}^{(k,n)}}| > 2^{k-1}\}| \le 2^{2k^2+2k}\binom{n}{k+1}$.

Proof. (1) We may assume that $s \le t$. In this case we have to prove that the number of corresponding pairs $(\mathbf{A}^{(k,n)}, \mathbf{B}^{(k,n)})$ is at most $\binom{n}{k+1} 2^{2k^2} n^{k+1-t}$. We have already seen in the proof of Lemma 2.1 that the number of vectors $\mathbf{A}^{(k,n)}$ is at most $\binom{n}{k+1}$. Fix a vector $\mathbf{A}^{(k,n)}$ and count the suitable vectors $\mathbf{B}^{(k,n)}$. Then the matrix B has t linearly independent rows, namely $r_{\mathbb{R}}((\beta_{d_i,l})_{t\times k}) = t$, for some $1 \le d_1 < \dots < d_t \le m$, where

$$a_0 + \sum_{l=1}^{k} \alpha_{d_i,l} a_l = b_0 + \sum_{l=1}^{k} \beta_{d_i,l} b_l, \quad \alpha_{d_i,l}, \beta_{d_i,l} \in \{0,1\} \quad \text{for } 1 \le i \le t.$$

The number of possible b_0s is at most n. For fixed $b_0, \alpha_{d_i,l}, \beta_{d_i,l}$ let us study the system of equations

$$a_0 + \sum_{l=1}^{k} \alpha_{d_i,l} a_l = b_0 + \sum_{l=1}^{k} \beta_{d_i,l} x_l, \quad \alpha_{d_i,l}, \beta_{d_i,l} \in \{0,1\} \quad \text{for } 1 \le i \le t.$$

The assumption $r_{\mathbb{R}}(\beta_{d_i,l})_{t\times k} = t$ implies that the number of solutions over $[1,n]$ is at most n^{k-t}. Finally, we have at most 2^{kt} possibilities on the left-hand side for $\alpha_{d_i,l}$s and, similarly, we have at most 2^{kt} possibilities on the right-hand side for $\beta_{d_i,l}$s, therefore the number of possible systems of equations is at most 2^{2k^2}.

(2) The number of vectors $\mathbf{A}^{(k,n)}$ is $\binom{n}{k+1}$ as in (1). Fix a vector $\mathbf{A}^{(k,n)}$ and count the suitable vectors $\mathbf{B}^{(k,n)}$. It follows from the assumptions

$r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}) = (r,r)$, $|\mathbf{H}_{\mathbf{A}^{(\mathbf{k})}} \cap \mathbf{H}_{\mathbf{B}^{(\mathbf{k})}}| = 2^r$ that the vectors $(\alpha_{d,1}, \dots, \alpha_{d,k})$, $d = 1, \dots, 2^r$ and the vectors $(\beta_{d,1}, \dots, \beta_{d,k})$, $d = 1, \dots, 2^r$, respectively form r-dimensional subspaces of $\mathbb{F}_2^k$. Considering the zero vectors of these subspaces we get $a_0 = b_0$. The integers $b_1, \dots, b_k$ are solutions of the system of equations

$$a_0 + \sum_{l=1}^{k} \alpha_{d,l} a_l = b_0 + \sum_{l=1}^{k} \beta_{d,l} x_l \quad \alpha_{d,l}, \beta_{d,l} \in \{0,1\} \quad \text{for } 1 \le d \le 2^r.$$

Similarly to the previous part this system of equation has at most n^{k-r} solutions over $[1,n]$ and the number of choices for the r linearly independent rows is at most 2^{2k^2}.

(3) Fix a vector $\mathbf{A}^{(k,n)}$. Let us suppose that for a vector $\mathbf{B}^{(k,n)}$ we have $r(\mathbf{H}_{\mathbf{A}^{(k,n)}}, \mathbf{H}_{\mathbf{B}^{(k,n)}}) = (k,k)$ and $|\mathbf{H}_{\mathbf{A}^{(k,n)}} \cap \mathbf{H}_{\mathbf{B}^{(k,n)}}| > 2^{k-1}$. Let the common vertices be

$$a_0 + \sum_{l=1}^{k} \alpha_{d,l} a_l = b_0 + \sum_{l=1}^{k} \beta_{d,l} b_l, \quad \alpha_{d,l}, \beta_{d,l} \in \{0,1\} \quad \text{for } 1 \le d \le m,$$

where we may assume that the rows $d_1, \dots, d_k$ are linearly independent, i.e. the matrix $B_k = (\beta_{d_i,l})_{k\times k}$ is regular. Write the rows $d_1, \dots, d_k$ in matrix form as

$$\underline{a} = b_0 \underline{1} + B_k \underline{b}, \tag{1}$$

with vectors $\underline{a} = (a_0 + \sum_{l=1}^{k} \alpha_{d_i,l} a_l)_{k\times 1}$, $\underline{1} = (1)_{k\times 1}$ and $\underline{b} = (b_i)_{k\times 1}$. It follows from (1) that

$$\underline{b} = B_k^{-1}(\underline{a} - b_0\underline{1}) = B_k^{-1}\underline{a} - b_0 B_k^{-1}\underline{1}.$$

Let $B_k^{-1}\underline{1} = (d_i)_{k\times 1}$ and $B_k^{-1}\underline{a} = (c_i)_{k\times 1}$. Obviously, the number of subsets $\{i_1, \dots i_l\} \subset \{1, \dots, k\}$ for which $d_{i_1} + \dots + d_{i_l} \ne 1$ is at least 2^{k-1}, therefore there exist $1 \le u_1 < \dots < u_s \le k$ and $1 \le v_1 < \dots < v_t \le k$ such that $a_0 + a_{u_1} + \dots + a_{u_s} = b_0 + b_{v_1} + \dots + b_{v_t}$, and $d_{v_1} + \dots + d_{v_t} \ne 1$. Hence

$$a_0 + a_{u_1} + \dots + a_{u_s} = b_0 + b_{v_1} + \dots + b_{v_t} = b_0 + c_{v_1} + \dots + c_{v_t} - b_0(d_{v_1} + \dots + d_{v_t})$$

$$b_0 = \frac{a_0 + a_{u_1} + \dots + a_{u_s} - c_{v_1} - \dots - c_{v_t}}{1 - (d_{v_1} + \dots + d_{v_t})}.$$

To conclude the proof we note that the number of sets $\{u_1, \dots, u_s\}$ and $\{v_1, \dots, v_t\}$ is at most 2^{2k} and there are at most 2^{k^2} choices for B_k and $\underline{a}$, respectively. Finally, for given B_k, $\underline{a}$, b_0, $1 \le u_1 < \dots < u_s \le k$ and $1 \le v_1 < \dots < v_t \le k$, the vector $\mathbf{B}^{(k,n)}$ is determined uniquely. □

In order to prove the theorems we need two lemmas from probability theory (see e.g. [1] p. 41, 95-98.). Let X_i be the indicator function of the event A_i and $S_N = X_1 + \dots + X_N$. For indices i, j write $i \sim j$ if $i \ne j$ and the events A_i, A_j are dependant. We set $\Gamma = \sum_{i\sim j} \Pr\{A_i \cap A_j\}$ (the sum over ordered pairs).

Lemma 2.4. *If $E(S_n) \to \infty$ and $\Gamma = o(E(S_n)^2)$, then $X > 0$ a.e.*

In many instances, we would like to bound the probability that none of the bad events B_i, $i \in I$, occur. If the events are mutually independent, then $\Pr\{\cap_{i\in I}\overline{B_i}\} = \prod_{i\in I} \Pr\{\overline{B_i}\}$. When the B_i are "mostly" independent, the Janson's inequality allows us, sometimes, to say that these two quantities are "nearly" equal. Let Ω be a finite set and R be a random subset of Ω given by $\Pr\{r \in R\} = p_r$, these events being mutually independent over $r \in \Omega$. Let E_i, $i \in I$ be subsets of Ω, where I a finite index set. Let B_i be the event $E_i \subset R$. Let X_i be the indicator random variable for B_i and $X = \sum_{i\in I} X_i$ be the number of E_is contained in R. The event $\cap_{i\in I}\overline{B_i}$ and $X = 0$ are then identical. For $i, j \in I$, we write $i \sim j$ if $i \neq j$ and $E_i \cap E_j \neq \emptyset$. We define $\Delta = \sum_{i\sim j} \Pr\{B_i \cap B_j\}$, here the sum is over ordered pairs. We set $M = \prod_{i\in I} \Pr\{\overline{B_i}\}$.

Lemma 2.5 (Janson's inequality)**.** *Let $\varepsilon \in]0,1[$ and let $B_i, i \in I, \Delta, M$ be as above and assume that $Pr\{B_i\} \leq \varepsilon$ for all i. Then*

$$M \leq Pr\{\cap_{i\in I}\overline{B_i}\} \leq M e^{\frac{1}{1-\varepsilon}\frac{\Delta}{2}}.$$

Proof of Theorem 1.5. Let $\mathbf{H}_{\mathbf{A_1}^{(k,n)}}, \ldots, \mathbf{H}_{\mathbf{A_N}^{(k,n)}}$ be the distinct non-degenerate k-cubes in $[1,n]$. Let B_i be the event $\mathbf{H}_{\mathbf{A_i}^{(k,n)}} \subset S(n, cn^{-\frac{k+1}{2^k}})$. Then $\Pr\{B_i\} = c^{2^k} n^{-k-1} = o(1)$ and $N = (1+o(1))\binom{n}{k+1}\frac{1}{k!}$. It is enough to prove

$$\Delta = \sum_{i\sim j} \Pr\{B_i \cap B_j\} = o(1)$$

since then Janson's inequality implies

$$\begin{aligned}
&\Pr\{S(n, cn^{-\frac{k+1}{2^k}}) \text{ does not contain any } k\text{-cubes}\} \\
&\qquad = \Pr\{\cap_{i=1}^N \overline{B_i}\} \\
&\qquad = (1+o(1))(1-(cn^{-\frac{k+1}{2^k}})^{2^k})^{(1+o(1))\binom{n}{k+1}\frac{1}{k!}} \\
&\qquad = (1+o(1))e^{-\frac{c^{2^k}}{(k+1)!k!}}.
\end{aligned}$$

It remains to verify that $\sum_{i\sim j}\Pr\{B_i\cap B_j\}=o(1)$. We split this sum according to the ranks in the following way

$$\sum_{i\sim j}\Pr\{B_i\cap B_j\}=\sum_{s=0}^{k}\sum_{t=0}^{k}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(s,t)}}\Pr\{B_i\cap B_j\}$$

$$=2\sum_{s=1}^{k}\sum_{t=0}^{s-1}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(s,t)}}\Pr\{B_i\cap B_j\}$$

$$+\sum_{r=0}^{k-1}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(r,r)\\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|=2^r}}\Pr\{B_i\cap B_j\}$$

$$+\sum_{r=1}^{k-1}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(r,r)\\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|<2^r}}\Pr\{B_i\cap B_j\}$$

$$+\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(k,k)\\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|\le 2^{k-1}}}\Pr\{B_i\cap B_j\}+\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(k,k)\\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|>2^{k-1}}}\Pr\{B_i\cap B_j\}.$$

The first sum can be estimated by Lemmas 2 and 2.3(1)

$$\sum_{s=1}^{k}\sum_{t=0}^{s-1}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(s,t)}}\Pr\{B_i\cap B_j\}$$
$$\le\sum_{s=1}^{k}\sum_{t=0}^{s-1}2^{2k^2}\binom{n}{k+1}n^{k+1-s}(cn^{-\frac{k+1}{2^k}})^{2\cdot 2^k-2^t}$$
$$=n^{o(1)}\sum_{s=1}^{k}n^{2^{s-1}\frac{k+1}{2^k}-s}=n^{o(1)}(n^{\frac{k+1}{2^k}-1}+n^{\frac{k+1}{2}-k})=o(1),$$

since the sequence $a_s=2^{s-1}\frac{k+1}{2^k}-s$ is decreasing for $1\le s\le k+1-\log_2(k+1)$ and increasing for $k+1-\log_2(k+1)<s\le k$.

To estimate the second sum we apply Lemma 2.3(2)

$$\sum_{r=0}^{k-1} \sum_{\substack{i\sim j \\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(r,r) \\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|=2^r}} \Pr\{B_i\cap B_j\}$$

$$\leq \sum_{r=0}^{k-1} 2^{2k^2}\binom{n}{k+1} n^{k-r}(cn^{-\frac{k+1}{2^k}})^{2\cdot 2^k-2^r}$$

$$= n^{-1+o(1)}\sum_{r=0}^{k-1} n^{2^r\frac{k+1}{2^k}-r} = n^{-1+o(1)}(n^{\frac{k+1}{2^k}}+n^{\frac{k+1}{2}-(k-1)}) = o(1).$$

The third sum can be bounded using Lemma 2.3(1):

$$\sum_{r=1}^{k-1} \sum_{\substack{i\sim j \\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(r,r) \\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|<2^r}} \Pr\{B_i\cap B_j\}$$

$$\leq \sum_{r=1}^{k-1} 2^{2k^2}\binom{n}{k+1} n^{k+1-r}(cn^{-\frac{k+1}{2^k}})^{2\cdot 2^k-2^r+1}$$

$$\leq n^{o(1)-\frac{k+1}{2^k}}\sum_{r=1}^{k-1} n^{2^r\frac{k+1}{2^k}-r} = n^{o(1)-\frac{k+1}{2^k}}(n^{2\frac{k+1}{2^k}-1}+n^{\frac{k+1}{2}-(k-1)}) = o(1).$$

Similarly, for the fourth sum we apply Lemma 2.3(1)

$$\sum_{\substack{i\sim j \\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(k,k) \\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|\leq 2^{k-1}}} \Pr\{B_i\cap B_j\} \leq n^{o(1)}n^{k+2}(cn^{-\frac{k+1}{2^k}})^{1.5\cdot 2^k} = o(1).$$

To estimate the fifth sum we note that $|\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cup\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}| \geq 2^k+1$. It follows from Lemma 2.3(3) that

$$\sum_{\substack{i\sim j \\ r(\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}},\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}})=(k,k) \\ |\mathbf{H}_{\mathbf{A}_\mathbf{i}^{(k,n)}}\cap\mathbf{H}_{\mathbf{A}_\mathbf{j}^{(k,n)}}|>2^{k-1}}} \Pr\{B_i\cap B_j\} \leq 2^{2k^2+2k}n^{k+1}(cn^{-\frac{k+1}{2^k}})^{2^k+1} = o(1),$$

which completes the proof. □

Proof of Theorem 1.6. Let $\epsilon > 0$ and for simplicity let $D_n = D_n(\epsilon)$ and $E_n = E_n(\epsilon)$. In the proof we use the estimations

$$2^{2^{D_n}} \le 2^{2^{\log_2\log_2 n+\log_2\log_2\log_2 n+\frac{(1-\epsilon)\log_2\log_2\log_2 n}{\log 2\log_2\log_2 n}}} = n^{\log_2\log_2 n+(1-\epsilon+o(1))\log_2\log_2\log_2 n}$$

and

$$2^{2^{E_n+1}} \ge 2^{2^{\log_2\log_2 n+\log_2\log_2\log_2 n+\frac{(1+\epsilon)\log_2\log_2\log_2 n}{\log 2\log_2\log_2 n}}} = n^{\log_2\log_2 n+(1+\epsilon+o(1))\log_2\log_2\log_2 n}$$

In order to verify Theorem 1.6 we have to show that

$$\lim_{n\to\infty}\Pr\{S(n,\frac{1}{2}) \text{ contains a } D_n\text{-cube}\} = 1 \tag{2}$$

and

$$\lim_{n\to\infty}\Pr\{S(n,\frac{1}{2}) \text{ contains an } (E_n+1)\text{-cube}\} = 0. \tag{3}$$

To prove the limit in (4) let $\mathbf{H}_{\mathbf{A_1}^{(D_n,n)}},\dots,\mathbf{H}_{\mathbf{A_N}^{(D_n,n)}}$ be the different non-degenerate D_n-cubes in $[1,n]$, B_i be the event $H_{\mathbf{A_i}^{(D_n,n)}} \subset S(n,\frac{1}{2})$, X_i be the indicator random variable for B_i and $S_N = X_1+\ldots+X_N$ be the number of $\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}} \subset S(n,\frac{1}{2})$. The linearity of expectation gives by Lemma 2.1 and inequality (2)

$$\begin{aligned}E(S_N) &= NE(X_i) = (1+o(1))\binom{n}{D_n+1}\frac{1}{D_n!}2^{-2^{D_n}}\\ &\ge n^{\log_2\log_2 n+(1+o(1))\log_2\log_2\log_2 n}n^{-\log_2\log_2 n-(1-\epsilon+o(1))\log_2\log_2\log_2 n}\\ &= n^{(\epsilon+o(1))\log_2\log_2\log_2 n}.\end{aligned}$$

Therefore $E(S_N)\to\infty$, as $n\to\infty$. By Lemma 2.4 it remains to prove that

$$\sum_{i\sim j}\Pr\{B_i\cap B_j\} = o(E(S_N)^2)$$

where $i\sim j$ means that the events B_i, B_j are not independent i.e. the cubes $\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}}, \mathbf{H}_{\mathbf{A_j}^{(D_n,n)}}$ have common vertices. We split this sum according to the ranks

$$\sum_{i\sim j}\Pr\{B_i\cap B_j\} = \sum_{s=0}^{D_n}\sum_{t=0}^{D_n}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}},\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}})=(s,t)}}\Pr\{B_i\cap B_j\}$$

$$\leq \sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}},\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}})=(0,0)}}\Pr\{B_i\cap B_j\} + 2\sum_{s=1}^{D_n}\sum_{t=0}^{s}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}},\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}})=(s,t)}}\Pr\{B_i\cap B_j\}.$$

The condition $r(\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}},\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}})=(0,0)$ implies that

$$|\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}}\cup\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}}| = 2^{D_n+1}-1,$$

thus by Lemma 2.3(2)

$$\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}},\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}})=(0,0)}}\Pr\{B_i\cap B_j\} \leq 2^{2D_n^2}\binom{n}{D_n+1}n^{D_n}2^{-2^{D_N+1}+1}$$

$$= o\left(\left(\binom{n}{D_n+1}\frac{1}{D_n!}2^{-2^{D_n}}\right)^2\right)$$

$$= o(E(S_N)^2).$$

In the light of Lemmas 2 and 2.3(1) the second term in (6) can be estimated as

$$\sum_{s=1}^{D_n}\sum_{t=0}^{s}\sum_{\substack{i\sim j\\ r(\mathbf{H}_{\mathbf{A_i}^{(D_n,n)}},\mathbf{H}_{\mathbf{A_j}^{(D_n,n)}})=(s,t)}}\Pr\{B_i\cap B_j\}$$

$$\leq \sum_{s=1}^{D_n}\sum_{t=0}^{s}\binom{n}{D_n+1}2^{2D_n^2}n^{D_n+1-s}2^{-2\cdot 2^{D_n}+2^t}$$

$$= \left(\binom{n}{D_n+1}\frac{1}{D_n!}2^{-2^{D_n}}\right)^2 n^{o(1)}\sum_{s=1}^{D_n}\sum_{t=0}^{s}\frac{2^{2^t}}{n^s}$$

$$= \left(\binom{n}{D_n+1}\frac{1}{D_n!}2^{-2^{D_n}}\right)^2 n^{o(1)}\sum_{s=1}^{D_n}\frac{2^{2^s}}{n^s}.$$

Finally, the function $f(x) = \frac{2^{2^x}}{n^x}$ decreases on $(-\infty, \log_2 \log n - 2\log_2 \log 2]$ and increases on $[\log_2 \log n - 2\log_2 \log 2, \infty)$, therefore by (2)

$$\sum_{s=1}^{D_n} \frac{2^{2^s}}{n^r} = n^{o(1)}\left(\frac{4}{n} + \frac{2^{2^{D_n}}}{n^{D_n}}\right) = n^{-1+o(1)},$$

which proves the limit in (4).

In order to prove the limit in (5) let $\mathbf{H}_{\mathbf{C}_1^{(E_n+1,n)}}, \dots, \mathbf{H}_{\mathbf{C}_K^{(E_n+1,n)}}$ be the distinct $(E_n{+}1)$-cubes in $[1,n]$ and let F_i be the event $\mathbf{H}_{\mathbf{C}_i^{(E_n+1,n)}} \subset S(n, \frac{1}{2})$. By (3) we have

$$\Pr\{S_n \text{ contains an } (E_n+1)\text{-cube}\} = \Pr\{\cup_{i=1}^{K} F_i\} \le \sum_{i=1}^{K} \Pr\{F_i\} \le$$

$$\binom{n}{E_n+2} 2^{-2^{E_n+1}} \le \frac{n^{\log_2 \log_2 n + (1+o(1))\log_2 \log_2 \log_2 n}}{n^{\log_2 \log_2 n + (1+\epsilon+o(1))\log_2 \log_2 \log_2 n}} = o(1),$$

which completes the proof. □

3. Concluding remarks

The aim of this paper is to study non-degenerate Hilbert cubes in a random sequence. A natural problem would be to give analogous theorems for Hilbert cubes, where degenerate cubes are allowed. In this situation the dominant terms may come from arithmetic progressions. An AP_{k+1} forms a k-cube. One can prove by the Janson inequality (see Lemma 2.5) that for a fixed $k \ge 2$

$$\lim_{n\to\infty} \Pr\{S(n, cn^{-\frac{2}{k+1}}) \text{ contains no } AP_{k+1}\} = e^{-\frac{c^{k+1}}{2k}}.$$

An easy argument shows (using Janson's inequality again) that for all $c > 0$, with $p_n = cn^{-2/5}$

$$\lim_{n\to\infty} \Pr\{S(n, p_n) \text{ contains no 4-cubes}\} = e^{-\frac{c^5}{8}}.$$

Conjecture 3.1. *For $k \ge 4$*

$$\lim_{n\to\infty} Pr\{S(n, cn^{-\frac{2}{k+1}}) \textit{ contains no } k\textit{-cubes}\} = e^{-\frac{c^{k+1}}{2k}}.$$

A simple calculation implies that in the random sequence $S(n, \frac{1}{2})$ the length of the longest arithmetic progression is a.e. nearly $2\log_2 n$, therefore it contains a Hilbert cube of dimension $(2-\epsilon)\log_2 n$.

Conjecture 3.2. *For every $\epsilon > 0$*

$$\lim_{n\to\infty} \Pr \left\{ \begin{matrix} \text{the maximal dimension of Hilbert cubes} \\ \text{in } S(n, \frac{1}{2}) \text{ is } < (2+\epsilon)\log_2 n \end{matrix} \right\} = 1.$$

N. Hegyvári (see [5]) studied the special case where the generating elements of Hilbert cubes are distinct. He proved that in this situation the maximal dimension of Hilbert cubes is a.e. between $c_1 \log n$ and $c_2 \log n \log\log n$. In this problem the lower bound seems to be the correct magnitude.

References

[1] N. Alon, J. Spencer, *The Probabilistic Method.* Wiley-Interscience, Series in Discrete Math. and Optimization, 1992.

[2] A. Godbole, S. Janson, N. Locantore, R. Rapoport, *Random Sidon Seqence.* J. Number Theory **75** (1999), no. 1, 7–22.

[3] D. S. Gunderson, V. Rödl, *Extremal problems for Affine Cubes of Integers.* Combin. Probab. Comput **7** (1998), no. 1, 65–79.

[4] R. L. Graham, B. L. Rothchild, J. Spencer, *Ramsey Theory.* Wiley-Interscience, Series in Discrete Math. and Optimization, 1990.

[5] N. Hegyvári, *On the dimension of the Hilbert cubes.* J. Number Theory **77** (1999), no. 2, 326–330.

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