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A new lower bound for $\|(3/2)^k\|$

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Résumé. Nous démontrons que pour tout entier k supérieur à une constante K effectivement calculable, la distance de $(3/2)^k$ à l'entier le plus proche est minorée par $0{,}5803^k$.

Abstract. We prove that, for all integers k exceeding some effectively computable number K, the distance from $(3/2)^k$ to the nearest integer is greater than 0.5803^k.

1. Historical overview of the problem

Let $\lfloor\cdot\rfloor$ and $\{\cdot\}$ denote the integer and fractional parts of a number, respectively. It is known [12] that the inequality $\{(3/2)^k\} \le 1-(3/4)^k$ for $k \ge 6$ implies the explicit formula $g(k) = 2^k + \lfloor(3/2)^k\rfloor - 2$ for the least integer $g = g(k)$ such that every positive integer can be expressed as a sum of at most g positive kth powers (Waring's problem). K. Mahler [10] used Ridout's extension of Roth's theorem to show that the inequality $\|(3/2)^k\| \le C^k$, where $\|x\| = \min(\{x\}, 1-\{x\})$ is the distance from $x \in \mathbb{R}$ to the nearest integer, has finitely many solutions in integers k for any $C < 1$. The particular case $C = 3/4$ gives one the above value of $g(k)$ for all $k \ge K$, where K is a certain absolute but ineffective constant. The first non-trivial (i.e., $C > 1/2$) and effective (in terms of K) estimate of the form

$$\left\|\left(\frac{3}{2}\right)^k\right\| > C^k \qquad \text{for} \quad k \ge K, \tag{1}$$

with $C = 2^{-(1-10^{-64})}$, was proved by A. Baker and J. Coates [1] by applying effective estimates of linear forms in logarithms in the p-adic case. F. Beukers [4] improved on this result by showing that inequality (1) is valid with $C = 2^{-0.9} = 0.5358\ldots$ for $k \ge K = 5000$ (although his proof yielded the better choice $C = 0.5637\ldots$ if one did not require an explicit evaluation of the effective bound for K). Beukers' proof relied on explicit Padé approximations to a tail of the binomial series $(1-z)^m = \sum_{n=0}^{m}\binom{m}{n}(-z)^n$ and was later used by A. Dubickas [7] and L. Habsieger [8] to derive inequality (1) with $C = 0.5769$ and 0.5770, respectively. The latter work also includes

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the estimate $\|(3/2)^k\| > 0.57434^k$ for $k \geq 5$ using computations from [6] and [9].

By modifying Beukers' construction, namely, considering Padé approximations to a tail of the series

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \binom{m+n}{m} z^n \tag{2}$$

and studying the explicit p-adic order of the binomial coefficients involved, we are able to prove

Theorem 1. *The following estimate is valid:*

$$\left\| \left(\frac{3}{2}\right)^k \right\| > 0.5803^k = 2^{-k \cdot 0.78512916\ldots} \qquad \textit{for} \quad k \geq K,$$

where K is a certain effective constant.

2. Hypergeometric background

The binomial series on the left-hand side of (2) is a special case of the generalized hypergeometric series

$$ {}_{q+1}F_q\left(\begin{matrix} A_0, A_1, \ldots, A_q \\ B_1, \ldots, B_q \end{matrix} \,\middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(A_0)_k (A_1)_k \cdots (A_q)_k}{k! (B_1)_k \cdots (B_q)_k} z^k, \tag{3}$$

where

$$(A)_k = \frac{\Gamma(A+k)}{\Gamma(A)} = \begin{cases} A(A+1)\cdots(A+k-1) & \text{if } k \geq 1, \\ 1 & \text{if } k = 0, \end{cases}$$

denotes the Pochhammer symbol (or shifted factorial). The series in (3) converges in the disc $|z| < 1$, and if one of the parameters $A_0, A_1, \ldots, A_q$ is a non-positive integer (i.e., the series terminates) the definition of the hypergeometric series is valid for all $z \in \mathbb{C}$.

In what follows we will often use the ${}_{q+1}F_q$-notation. We will require two classical facts from the theory of generalized hypergeometric series: the Pfaff–Saalschütz summation formula

$$ {}_3F_2\left(\begin{matrix} -n, A, B \\ C, 1+A+B-C-n \end{matrix} \,\middle|\, 1 \right) = \frac{(C-A)_n (C-B)_n}{(C)_n (C-A-B)_n} \tag{4}$$

(see, e.g., [11], p. 49, (2.3.1.3)) and the Euler–Pochhammer integral for the Gauss ${}_2F_1$-series

$$ {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, z \right) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1}(1-zt)^{-A}\,\mathrm{d}t, \tag{5}$$

provided $\operatorname{Re} C > \operatorname{Re} B > 0$ (see, e.g., [11], p. 20, (1.6.6)). Formula (5) is valid for $|z| < 1$ and also for any $z \in \mathbb{C}$ if A is a non-positive integer.

3. Padé approximations of the shifted binomial series

Fix two positive integers a and b satisfying $2a \le b$. Formula (2) yields

$$(6)\quad \begin{aligned} \left(\frac{3}{2}\right)^{3(b+1)} &= \left(\frac{27}{8}\right)^{b+1} = 3^{b+1}\left(1-\frac{1}{9}\right)^{-(b+1)} \\ &= 3^{b+1}\sum_{k=0}^{\infty}\binom{b+k}{b}\left(\frac{1}{9}\right)^{k} = 3^{b-2a+1}\sum_{k=0}^{\infty}\binom{b+k}{b}3^{2(a-k)} \\ &= \text{an integer} + 3^{b-2a+1}\sum_{k=a}^{\infty}\binom{b+k}{b}3^{2(a-k)} \\ &\equiv 3^{b-2a+1}\sum_{\nu=0}^{\infty}\binom{a+b+\nu}{b}3^{-2\nu} \pmod{\mathbb{Z}}. \end{aligned}$$

This motivates (cf. [4]) constructing Padé approximations to the function

$$(7)\quad F(z) = F(a,b;z) = \sum_{\nu=0}^{\infty}\binom{a+b+\nu}{b}z^{\nu} = \binom{a+b}{b}\sum_{\nu=0}^{\infty}\frac{(a+b+1)_{\nu}}{(a+1)_{\nu}}z^{\nu}$$

and applying them with the choice $z = 1/9$.

Remark 1. The connection of $F(a,b;z)$ with Beukers' auxiliary series $H(\widetilde{a},\widetilde{b};z)$ from [4] is as follows:

$$F(a,b;z) = z^{-a}\left((1-z)^{-b-1} - \sum_{k=0}^{a-1}\binom{b+k}{b}z^{k}\right) = H(-a-b-1,a;z).$$

Although Beukers considers $H(\widetilde{a},\widetilde{b};z)$ only for $\widetilde{a},\widetilde{b} \in \mathbb{N}$, his construction remains valid for any $\widetilde{a} \in \mathbb{C}$, $\widetilde{b} \in \mathbb{N}$ and $|z| < 1$. However the diagonal Padé approximations, used in [4] for $H(z)$ and used below for $F(z)$, are different.

Taking an arbitrary integer n satisfying $n \le b$, we follow the general recipe of [5], [13]. Consider the polynomial

$$(8)\quad \begin{aligned} Q_n(x) &= \binom{a+b+n}{a+b}\,{}_2F_1\!\left(\left.\begin{matrix}-n,\ a+n\\ a+b+1\end{matrix}\,\right|\,x\right) \\ &= \sum_{\mu=0}^{n}\binom{a+n-1+\mu}{\mu}\binom{a+b+n}{n-\mu}(-x)^{\mu} = \sum_{\mu=0}^{n}q_{\mu}x^{\mu} \in \mathbb{Z}[x] \end{aligned}$$

of degree n. Then

$$Q_n(z^{-1})F(z) = \sum_{\mu=0}^{n} q_{n-\mu} z^{\mu-n} \cdot \sum_{\nu=0}^{\infty} \binom{a+b+\nu}{b} z^{\nu} \tag{9}$$
$$= \sum_{l=0}^{\infty} z^{l-n} \sum_{\substack{\mu=0 \\ \mu\le l}}^{n} q_{n-\mu} \binom{a+b+l-\mu}{b}$$
$$= \sum_{l=0}^{n-1} r_l z^{l-n} + \sum_{l=n}^{\infty} r_l z^{l-n} = P_n(z^{-1}) + R_n(z).$$

Here the polynomial

$$P_n(x) = \sum_{l=0}^{n-1} r_l x^{n-l} \in \mathbb{Z}[x], \qquad \text{where} \quad r_l = \sum_{\mu=0}^{l} q_{n-\mu} \binom{a+b+l-\mu}{b}, \tag{10}$$

has degree at most n, while the coefficients of the remainder

$$R_n(z) = \sum_{l=n}^{\infty} r_l z^{l-n}$$

are of the following form:

$$r_l = \sum_{\mu=0}^{n} q_{n-\mu} \binom{a+b+l-\mu}{b}$$
$$= \sum_{\mu=0}^{n} (-1)^{n-\mu} \binom{a+2n-1-\mu}{n-\mu} \binom{a+b+n}{\mu} \binom{a+b+l-\mu}{b}$$
$$= (-1)^n \frac{(a+b+n)!}{(a+n-1)!n!b!} \sum_{\mu=0}^{n} (-1)^{\mu} \binom{n}{\mu} \frac{(a+2n-1-\mu)!(a+b+l-\mu)!}{(a+l-\mu)!(a+b+n-\mu)!}$$
$$= (-1)^n \frac{(a+b+n)!}{(a+n-1)!n!b!}$$
$$\times \frac{(a+2n-1)!(a+b+l)!}{(a+l)!(a+b+n)!} \sum_{\mu=0}^{n} \frac{(-n)_{\mu}(-a-l)_{\mu}(-a-b-n)_{\mu}}{\mu!(-a-2n+1)_{\mu}(-a-b-l)_{\mu}}$$
$$= (-1)^n \frac{(a+2n-1)!(a+b+l)!}{(a+n-1)!(a+l)!n!b!} \cdot {}_3F_2\left(\begin{matrix} -n,\ -a-l,\ -a-b-n \\ -a-2n+1,\ -a-b-l \end{matrix} \,\middle|\, 1\right).$$

If we apply (4) with the choice $A = -a-l$, $B = -a-b-n$ and $C = -a-b-l$, we obtain

$$r_l = (-1)^n \frac{(a+2n-1)!(a+b+l)!}{(a+n-1)!(a+l)!n!b!} \cdot \frac{(-b)_n(n-l)_n}{(-a-b-l)_n(a+n)_n}.$$

The assumed condition $n \le b$ guarantees that the coefficients r_l do not vanish identically (otherwise $(-b)_n = 0$). Moreover, $(n-l)_n = 0$ for l ranging over the set $n \le l \le 2n-1$, therefore $r_l = 0$ for those l, while

$$\begin{aligned} r_l &= \frac{(a+2n-1)!(a+b+l)!}{(a+n-1)!(a+l)!n!b!} \\ &\quad \times \frac{b!/(b-n)! \cdot (l-n)!/(l-2n)!}{(a+b+l)!/(a+b+l-n)! \cdot (a+2n-1)!/(a+n-1)!} \\ &= \frac{(a+b+l-n)!(l-n)!}{n!(b-n)!(a+l)!(l-2n)!} \qquad \text{for} \quad l \ge 2n. \end{aligned}$$

Finally,

$$\begin{aligned} R_n(z) &= \sum_{l=2n}^{\infty} r_l z^{l-n} = z^n \sum_{\nu=0}^{\infty} r_{\nu+2n} z^{\nu} \\ &= z^n \frac{1}{n!(b-n)!} \sum_{\nu=0}^{\infty} \frac{(a+b+n+\nu)!(n+\nu)!}{\nu!(a+2n+\nu)!} z^{\nu} \\ &= z^n \binom{a+b+n}{b-n} \cdot {}_2F_1\left(\left.\begin{matrix} a+b+n+1,\ n+1 \\ a+2n+1 \end{matrix}\right| z\right). \end{aligned} \tag{11}$$

Using the integral (5) for the polynomial (8) and remainder (11) we arrive at

Lemma 1. *The following representations are valid:*

$$Q_n(z^{-1}) = \frac{(a+b+n)!}{(a+n-1)!n!(b-n)!} \int_0^1 t^{a+n-1}(1-t)^{b-n}(1-z^{-1}t)^n \, \mathrm{d}t$$

and

$$R_n(z) = \frac{(a+b+n)!}{(a+n-1)!n!(b-n)!} z^n \int_0^1 t^n(1-t)^{a+n-1}(1-zt)^{-(a+b+n+1)} \, \mathrm{d}t.$$

We will also require linear independence of a pair of neighbouring Padé approximants, which is the subject of

Lemma 2. *We have*

$$Q_{n+1}(x)P_n(x) - Q_n(x)P_{n+1}(x) = (-1)^n \binom{a+2n+1}{a+n} \binom{a+b+n}{b-n} x. \tag{12}$$

Proof. Clearly, the left-hand side in (12) is a polynomial; its constant term is 0 since $P_n(0) = P_{n+1}(0) = 0$ by (10). On the other hand,

$$\begin{aligned} &Q_{n+1}(z^{-1})P_n(z^{-1}) - Q_n(z^{-1})P_{n+1}(z^{-1}) \\ &\quad = Q_{n+1}(z^{-1})\big(Q_n(z^{-1})F(z) - R_n(z)\big) \\ &\qquad - Q_n(z^{-1})\big(Q_{n+1}(z^{-1})F(z) - R_{n+1}(z)\big) \\ &\quad = Q_n(z^{-1})R_{n+1}(z) - Q_{n+1}(z^{-1})R_n(z), \end{aligned}$$

and from (8), (11) we conclude that the only negative power of z originates with the last summand:

$$\begin{aligned} &- Q_{n+1}(z^{-1})R_n(z) \\ &\quad = (-1)^n \binom{a+2n+1}{a+n} z^{-n-1}\bigl(1+O(z)\bigr) \cdot \binom{a+b+n}{b-n} z^n \bigl(1+O(z)\bigr) \\ &\quad = (-1)^n \binom{a+2n+1}{a+n}\binom{a+b+n}{b-n}\frac{1}{z} + O(1) \qquad \text{as} \quad z \to 0. \qquad \square \end{aligned}$$

4. Arithmetic constituents

We begin this section by noting that, for any prime $p > \sqrt{N}$,

$$\operatorname{ord}_p N! = \left\lfloor \frac{N}{p} \right\rfloor \qquad \text{and} \qquad \operatorname{ord}_p N = \lambda\biggl(\frac{N}{p}\biggr),$$

where

$$\lambda(x) = 1 - \{x\} - \{-x\} = 1 + \lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

For primes $p > \sqrt{a+b+n}$, let

$$\begin{aligned} e_p &= \min_{\mu\in\mathbb{Z}} \biggl(-\biggl\{-\frac{a+n}{p}\biggr\} + \biggl\{-\frac{a+n+\mu}{p}\biggr\} + \biggl\{\frac{\mu}{p}\biggr\} \\ &\qquad - \biggl\{\frac{a+b+n}{p}\biggr\} + \biggl\{\frac{a+b+\mu}{p}\biggr\} + \biggl\{\frac{n-\mu}{p}\biggr\} \biggr) \\ &= \min_{\mu\in\mathbb{Z}} \biggl(\biggl\lfloor -\frac{a+n}{p} \biggr\rfloor - \biggl\lfloor -\frac{a+n+\mu}{p} \biggr\rfloor - \biggl\lfloor \frac{\mu}{p} \biggr\rfloor \\ &\qquad + \biggl\lfloor \frac{a+b+n}{p} \biggr\rfloor - \biggl\lfloor \frac{a+b+\mu}{p} \biggr\rfloor - \biggl\lfloor \frac{n-\mu}{p} \biggr\rfloor \biggr) \\ &\le \min_{0\le\mu\le n} \operatorname{ord}_p \frac{a+n}{a+n+\mu} \binom{a+n+\mu}{\mu}\binom{a+b+n}{n-\mu} \\ &= \min_{0\le\mu\le n} \operatorname{ord}_p \binom{a+n-1+\mu}{\mu}\binom{a+b+n}{n-\mu} \end{aligned} \tag{13}$$

and

$$\begin{aligned} e'_p &= \min_{\mu\in\mathbb{Z}} \biggl(-\biggl\{\frac{a+n+\mu}{p}\biggr\} + \biggl\{\frac{a+n}{p}\biggr\} + \biggl\{\frac{\mu}{p}\biggr\} \\ &\qquad - \biggl\{\frac{a+b+n}{p}\biggr\} + \biggl\{\frac{a+b+\mu}{p}\biggr\} + \biggl\{\frac{n-\mu}{p}\biggr\} \biggr) \end{aligned} \tag{14}$$

$$
\begin{aligned}
&= \min_{\mu\in\mathbb{Z}}\biggl(\biggl\lfloor\frac{a+n+\mu}{p}\biggr\rfloor - \biggl\lfloor\frac{a+n}{p}\biggr\rfloor - \biggl\lfloor\frac{\mu}{p}\biggr\rfloor \\
&\qquad + \biggl\lfloor\frac{a+b+n}{p}\biggr\rfloor - \biggl\lfloor\frac{a+b+\mu}{p}\biggr\rfloor - \biggl\lfloor\frac{n-\mu}{p}\biggr\rfloor\biggr) \\
&\le \min_{0\le\mu\le n} \operatorname{ord}_p \binom{a+n+\mu}{\mu}\binom{a+b+n}{n-\mu}.
\end{aligned}
$$

Set

(15)
$$\Phi = \Phi(a,b,n) = \prod_{p>\sqrt{a+b+n}} p^{e_p} \qquad \text{and} \qquad \Phi' = \Phi'(a,b,n) = \prod_{p>\sqrt{a+b+n}} p^{e'_p}.$$

From (8), (10) and (13), (15) we deduce

Lemma 3. *The following inclusions are valid:*

$$\Phi^{-1}\cdot\binom{a+n-1+\mu}{\mu}\binom{a+b+n}{n-\mu} \in \mathbb{Z} \qquad \textit{for} \quad \mu = 0,1,\ldots,n,$$

hence

$$\Phi^{-1}Q_n(x) \in \mathbb{Z}[x] \qquad \textit{and} \qquad \Phi^{-1}P_n(x) \in \mathbb{Z}[x].$$

Supplementary arithmetic information for the case n replaced by $n+1$ is given in

Lemma 4. *The following inclusions are valid:*

(16)
$$(n+1)\Phi'^{-1}\cdot\binom{a+n+\mu}{\mu}\binom{a+b+n+1}{n+1-\mu} \in \mathbb{Z} \qquad \textit{for} \quad \mu = 0,1,\ldots,n+1,$$

hence

$$(n+1)\Phi'^{-1}Q_{n+1}(x) \in \mathbb{Z}[x] \qquad \textit{and} \qquad (n+1)\Phi'^{-1}P_{n+1}(x) \in \mathbb{Z}[x].$$

Proof. Write

$$\binom{a+n+\mu}{\mu}\binom{a+b+n+1}{n+1-\mu} = \binom{a+n+\mu}{\mu}\binom{a+b+n}{n-\mu}\cdot\frac{a+b+n+1}{n+1-\mu}.$$

Therefore, if $p \nmid n+1-\mu$ then

(17)
$$\operatorname{ord}_p\binom{a+n+\mu}{\mu}\binom{a+b+n+1}{n+1-\mu} \ge \operatorname{ord}_p\binom{a+n+\mu}{\mu}\binom{a+b+n}{n-\mu} \ge e'_p;$$

otherwise $\mu \equiv n+1 \pmod p$, hence $\mu/p - (n+1)/p \in \mathbb{Z}$ yielding

$$\begin{aligned}
(18) \qquad & \operatorname{ord}_p \binom{a+n+\mu}{\mu}\binom{a+b+n+1}{n+1-\mu} \\
&= -\left\{\frac{a+n+\mu}{p}\right\} + \left\{\frac{a+n}{p}\right\} + \left\{\frac{\mu}{p}\right\} \\
&\quad - \left\{\frac{a+b+n+1}{p}\right\} + \left\{\frac{a+b+\mu}{p}\right\} + \left\{\frac{n+1-\mu}{p}\right\} \\
&= -\left\{\frac{a+2n+1}{p}\right\} + \left\{\frac{a+n}{p}\right\} + \left\{\frac{n+1}{p}\right\} \\
&= \operatorname{ord}_p \binom{a+2n+1}{n+1} = \operatorname{ord}_p \binom{a+2n}{n} + \operatorname{ord}_p \frac{a+2n+1}{n+1} \\
&= \operatorname{ord}_p \binom{a+n+\mu}{\mu}\binom{a+b+n}{n-\mu}\bigg|_{\mu=n} + \operatorname{ord}_p \frac{a+2n+1}{n+1} \\
&\ge e_p' - \operatorname{ord}_p(n+1).
\end{aligned}$$

Combination of (17) and (18) gives us the required inclusions (16). □

5. Proof of theorem 1

The parameters a, b and n will now depend on an increasing parameter $m \in \mathbb{N}$ in the following way:

$$a = \alpha m, \qquad b = \beta m, \qquad n = \gamma m \quad \text{or} \quad n = \gamma m + 1,$$

where the choice of the positive integers α, β and γ, satisfying $2\alpha \le \beta$ and $\gamma < \beta$, is discussed later. Then Lemma 1 and Laplace's method give us

(19)

$$\begin{aligned}
C_0(z) &= \lim_{m\to\infty} \frac{\log|R_n(z)|}{m} \\
&= (\alpha+\beta+\gamma)\log(\alpha+\beta+\gamma) - (\alpha+\gamma)\log(\alpha+\gamma) \\
&\quad - \gamma\log\gamma - (\beta-\gamma)\log(\beta-\gamma) + \gamma\log|z| \\
&\quad + \max_{0\le t\le 1} \operatorname{Re}\bigl(\gamma\log t + (\alpha+\gamma)\log(1-t) - (\alpha+\beta+\gamma)\log(1-zt)\bigr)
\end{aligned}$$

and

(20)

$$\begin{aligned}
C_1(z) &= \lim_{m\to\infty} \frac{\log|Q_n(z^{-1})|}{m} \\
&= (\alpha+\beta+\gamma)\log(\alpha+\beta+\gamma) - (\alpha+\gamma)\log(\alpha+\gamma) \\
&\quad - \gamma\log\gamma - (\beta-\gamma)\log(\beta-\gamma) \\
&\quad + \max_{0\le t\le 1} \operatorname{Re}\bigl((\alpha+\gamma)\log t + (\beta-\gamma)\log(1-t) + \gamma\log(1-z^{-1}t)\bigr).
\end{aligned}$$

In addition, from (13)–(15) and the prime number theorem we deduce that

$$
\begin{aligned}
C_2 &= \lim_{m\to\infty} \frac{\log \Phi(\alpha m, \beta m, \gamma m)}{m} = \int_0^1 \varphi(x)\,\mathrm{d}\psi(x), \\
C_2' &= \lim_{m\to\infty} \frac{\log \Phi'(\alpha m, \beta m, \gamma m)}{m} = \int_0^1 \varphi'(x)\,\mathrm{d}\psi(x),
\end{aligned} \tag{21}
$$

where $\psi(x)$ is the logarithmic derivative of the gamma function and the 1-periodic functions $\varphi(x)$ and $\varphi'(x)$ are defined as follows:

$$
\begin{aligned}
\varphi(x) &= \min_{0\le y<1} \widehat{\varphi}(x,y), \qquad \varphi'(x) = \min_{0\le y<1} \widehat{\varphi}'(x,y), \\
\widehat{\varphi}(x,y) &= -\{-(\alpha+\gamma)x\} + \{-(\alpha+\gamma)x - y\} + \{y\} \\
&\quad - \{(\alpha+\beta+\gamma)x\} + \{(\alpha+\beta)x + y\} + \{\gamma x - y\}, \\
\widehat{\varphi}'(x,y) &= -\{(\alpha+\gamma)x + y\} + \{(\alpha+\gamma)x\} + \{y\} \\
&\quad - \{(\alpha+\beta+\gamma)x\} + \{(\alpha+\beta)x + y\} + \{\gamma x - y\}.
\end{aligned}
$$

Lemma 5. *The functions $\varphi(x)$ and $\varphi'(x)$ differ on a set of measure* 0, *hence*

$$
C_2 = C_2'. \tag{22}
$$

Remark 2. The following proof is due to the anonymous referee. In an earlier version of this article, we verify directly assumption (22) for our specific choice of the integer parameters α, β and γ.

Proof. First note that $\widehat{\varphi}(x,0) = \widehat{\varphi}'(x,0) \in \{0,1\}$, which implies $\varphi(x) \in \{0,1\}$ and $\varphi'(x) \in \{0,1\}$.

From the definition we see that

$$
\Delta(x,y) = \widehat{\varphi}(x,y) - \widehat{\varphi}'(x,y) = \lambda((\alpha+\gamma)x) - \lambda((\alpha+\gamma)x + y).
$$

Assume that $(\alpha+\gamma)x \notin \mathbb{Z}$; we plainly obtain $\Delta(x,y) = -\lambda((\alpha+\gamma)x+y) \le 0$, hence $\Delta(x,y) = 0$ unless $y = -\{(\alpha+\gamma)x\}$. Therefore the only possibility to get $\varphi(x) \ne \varphi'(x)$ is the following one:

$$
\begin{aligned}
\widehat{\varphi}(x,y) \ge 1 \quad &\text{for } y \ne -\{(\alpha+\gamma)x\} \qquad \text{and} \\
\widehat{\varphi}(x,y) = 0 \quad &\text{for } y = -\{(\alpha+\gamma)x\}.
\end{aligned} \tag{23}
$$

Since $\widehat{\varphi}'(x,0) \ge 1$, we have $\{(\alpha+\beta)x\} + \{\gamma x\} = 1$; furthermore,

$$
\begin{aligned}
&\{(\alpha+\beta)x + y\} + \{\gamma x - y\} - \{(\alpha+\beta+\gamma)x\} \\
&\quad = \begin{cases} 1 & \text{if } 0 \le y < 1 - \{(\alpha+\beta)x\}, \\ 0 & \text{if } 1 - \{(\alpha+\beta)x\} \le y < \{\gamma x\}, \\ 1 & \text{if } \{\gamma x\} \le y < 1, \end{cases}
\end{aligned}
$$

and

$$\{-(\alpha+\gamma)x-y\}+\{y\}-\{-(\alpha+\gamma)x\}=\begin{cases}0 & \text{if } 0\le y\le\{-(\alpha+\gamma)x\},\\ 1 & \text{if } \{-(\alpha+\gamma)x\}<y<1.\end{cases}$$

The above conditions (23) on $\widehat{\varphi}(x,y)$ imply $1-\{(\alpha+\beta)x\}=\{-(\alpha+\gamma)x\}$ or, equivalently, $(\beta-\gamma)x\in\mathbb{Z}$.

Finally, the sets $\{x\in\mathbb{R}:(\alpha+\gamma)x\in\mathbb{Z}\}$ and $\{x\in\mathbb{R}:(\beta-\gamma)x\in\mathbb{Z}\}$ have measure 0, thus proving the required assertion. □

Our final aim is estimating the absolute value of ε_k from below, where

$$\left(\frac{3}{2}\right)^k=M_k+\varepsilon_k,\qquad M_k\in\mathbb{Z},\quad 0<|\varepsilon_k|<\frac{1}{2}.$$

Write $k\ge 3$ in the form $k=3(\beta m+1)+j$ with non-negative integers m and $j<3\beta$. Multiply both sides of (9) by $\widetilde{\Phi}^{-1}3^{b-2a+j+1}$, where $\widetilde{\Phi}=\Phi(\alpha m,\beta m,\gamma m)$ if $n=\gamma m$ and $\widetilde{\Phi}=\Phi'(\alpha m,\beta m,\gamma m)/(\gamma m+1)$ if $n=\gamma m+1$, and substitute $z=1/9$:

$$Q_n(9)\widetilde{\Phi}^{-1}2^j\cdot\left(\frac{3}{2}\right)^j 3^{b-2a+1}F\left(a,b;\frac{1}{9}\right)=P_n(9)\widetilde{\Phi}^{-1}3^{b-2a+j+1}+R_n\left(\frac{1}{9}\right)\widetilde{\Phi}^{-1}3^{b-2a+j+1}. \tag{24}$$

From (6), (7) we see that

$$\left(\frac{3}{2}\right)^j 3^{b-2a+1}F\left(a,b;\frac{1}{9}\right)\equiv\left(\frac{3}{2}\right)^{3(b+1)+j}\pmod{\mathbb{Z}}=\left(\frac{3}{2}\right)^k,$$

hence the left-hand side equals $M'_k+\varepsilon_k$ for some $M'_k\in\mathbb{Z}$ and we may write equality (24) in the form

$$Q_n(9)\widetilde{\Phi}^{-1}2^j\cdot\varepsilon_k=M''_k+R_n\left(\frac{1}{9}\right)\widetilde{\Phi}^{-1}3^{b-2a+j+1}, \tag{25}$$

where

$$M''_k=P_n(9)\widetilde{\Phi}^{-1}3^{b-2a+j+1}-Q_n(9)\widetilde{\Phi}^{-1}2^jM'_k\in\mathbb{Z}$$

by Lemmas 3 and 4. Lemma 2 guarantees that, for at least one of $n=\gamma m$ or $\gamma m+1$, we have $M''_k\ne 0$; we make the corresponding choice of n. Assuming furthermore that

$$C_0\left(\frac{1}{9}\right)-C_2+(\beta-2\alpha)\log 3<0, \tag{26}$$

from (19) and (21) we obtain

$$\left|R_n\left(\frac{1}{9}\right)\widetilde{\Phi}^{-1}3^{b-2a+j+1}\right|<\frac{1}{2}\qquad\text{for all}\quad m\ge N_1,$$

where $N_1 > 0$ is an effective absolute constant. Therefore, by (25) and $|M_k''| \geq 1$ we have

$$|Q_n(9)\widetilde{\Phi}^{-1}2^j| \cdot |\varepsilon_k| \geq |M_k''| - \left|R_n\left(\frac{1}{9}\right)\widetilde{\Phi}^{-1}3^{b-2a+j+1}\right| > \frac{1}{2},$$

hence from (19), (20) we conclude that

$$|\varepsilon_k| > \frac{\widetilde{\Phi}}{2^{j+1}|Q_n(9)|} \geq \frac{\widetilde{\Phi}}{2^{3\beta}|Q_n(9)|} > e^{-m(C_1(1/9)-C_2+\delta)}$$

for any $\delta > 0$ and $m > N_2(\delta)$, provided that $C_1(1/9) - C_2 + \delta > 0$; here $N_2(\delta)$ depends effectively on δ. Finally, since $k > 3\beta m$, we obtain the estimate

$$|\varepsilon_k| > e^{-k(C_1(1/9)-C_2+\delta)/(3\beta)} \tag{27}$$

valid for all $k \geq K_0(\delta)$, where the constant $K_0(\delta)$ may be determined in terms of $\max(N_1, N_2(\delta))$.

Taking $\alpha = \gamma = 9$ and $\beta = 19$ (which is the optimal choice of the integer parameters α, β, γ, at least under the restriction $\beta \leq 100$) we find that

$$C_0\left(\frac{1}{9}\right) = 3.28973907\ldots, \qquad C_1\left(\frac{1}{9}\right) = 35.48665992\ldots,$$

and

$$\varphi(x) = \begin{cases} 1 & \text{if } \{x\} \in \left[\frac{2}{37}, \frac{1}{18}\right] \cup \left[\frac{3}{37}, \frac{1}{10}\right) \cup \left[\frac{4}{37}, \frac{1}{9}\right) \cup \left[\frac{6}{37}, \frac{1}{6}\right] \cup \left[\frac{7}{37}, \frac{1}{5}\right) \\ & \quad \cup \left[\frac{8}{37}, \frac{2}{9}\right) \cup \left[\frac{10}{37}, \frac{5}{18}\right] \cup \left[\frac{11}{37}, \frac{3}{10}\right) \cup \left[\frac{12}{37}, \frac{1}{3}\right) \cup \left[\frac{14}{37}, \frac{7}{18}\right] \\ & \quad \cup \left[\frac{16}{37}, \frac{4}{9}\right) \cup \left[\frac{18}{37}, \frac{1}{2}\right) \cup \left[\frac{20}{37}, \frac{5}{9}\right) \cup \left[\frac{22}{37}, \frac{3}{5}\right) \cup \left[\frac{24}{37}, \frac{2}{3}\right) \\ & \quad \cup \left[\frac{28}{37}, \frac{7}{9}\right) \cup \left[\frac{32}{37}, \frac{8}{9}\right) \cup \left[\frac{36}{37}, 1\right), \\ 0 & \text{otherwise,} \end{cases}$$

hence

$$C_2 = C_2' = 4.46695926\ldots .$$

Using these computations we verify (26),

$$C_0\left(\frac{1}{9}\right) - C_2 + (\beta - 2\alpha)\log 3 = -0.07860790\ldots,$$

and find that with $\delta = 0.00027320432\ldots$

$$e^{-(C_1(1/9)-C_2+\delta)/(3\beta)} = 0.5803.$$

This result, in view of (27), completes the proof of Theorem 1. □

6. Related results

The above construction allows us to prove similar results for the sequences $\|(4/3)^k\|$ and $\|(5/4)^k\|$ using the representations

$$(28)\qquad \left(\frac{4}{3}\right)^{2(b+1)} = 2^{b+1}\left(1+\frac{1}{8}\right)^{-(b+1)} \equiv (-1)^a 2^{b-3a+1} F\left(a,b;-\frac{1}{8}\right) \pmod{\mathbb{Z}}, \quad \text{where}\quad 3a\le b,$$

and

$$(29)\qquad \left(\frac{5}{4}\right)^{7b+3} = 2\cdot 5^b\left(1+\frac{3}{125}\right)^{-(2b+1)} \equiv (-1)^a 2\cdot 3^a\cdot 5^{b-3a} F\left(a,2b;-\frac{3}{125}\right) \pmod{\mathbb{Z}}, \quad \text{where}\quad 3a\le b.$$

Namely, taking $a=5m$, $b=15m$, $n=6m(+1)$ in case (28) and $a=3m$, $b=9m$, $n=7m(+1)$ in case (29) and repeating the arguments of Section 5, we arrive at

Theorem 2. *The following estimates are valid:*

$$\left\|\left(\frac{4}{3}\right)^k\right\| > 0.4914^k = 3^{-k\cdot 0.64672207\ldots} \qquad \text{for}\quad k\ge K_1,$$

$$\left\|\left(\frac{5}{4}\right)^k\right\| > 0.5152^k = 4^{-k\cdot 0.47839775\ldots} \qquad \text{for}\quad k\ge K_2,$$

where K_1, K_2 are certain effective constants.

The general case of the sequence $\|(1+1/N)^k\|$ for an integer $N\ge 5$ may be treated as in [4] and [2] by using the representation

$$\left(\frac{N+1}{N}\right)^{b+1} = \left(1-\frac{1}{N+1}\right)^{-(b+1)} \equiv F\left(0,b;\frac{1}{N+1}\right) \pmod{\mathbb{Z}}.$$

The best result in this direction belongs to M. Bennett [2]: $\|(1+1/N)^k\| > 3^{-k}$ for $4\le N\le k3^k$.

Remark 3. As mentioned by the anonymous referee, our result for $\|(4/3)^k\|$ is of special interest. It completes Bennett's result [3] on the order of the additive basis $\{1, N^k, (N+1)^k, (N+2)^k, \dots\}$ for $N=3$ (case $N=2$ corresponds to the classical Waring's problem); to solve this problem one needs the bound $\|(4/3)^k\| > (4/9)^k$ for $k\ge 6$. Thus we remain verification of the bound in the range $6\le k\le K_1$.

We would like to conclude this note by mentioning that a stronger argument is required to obtain the effective estimate $\|(3/2)^k\| > (3/4)^k$ and its relatives.

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