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LIFTINGS OF LINEAR VECTOR FIELDS TO PRODUCT PRESERVING GAUGE BUNDLE FUNCTORS ON VECTOR BUNDLES

(submitted by B. N. Shapukov)

Abstract. All natural operators lifting linear vector fields to product preserving gauge bundle functors on vector bundles are classified. Some relevant properties of Weil modules are studied, too.

Introduction

Let $F\colon \mathcal{VB} \to \mathcal{FM}$ be a covariant functor from the category $\mathcal{VB}$ of all vector bundles and their vector bundle homomorphisms into the category $\mathcal{FM}$ of fibered manifolds and their fiber maps. Let $B_{\mathcal{VB}}\colon \mathcal{VB} \to \mathcal{M}f$ and $B_{\mathcal{FM}}\colon \mathcal{FM} \to \mathcal{M}f$ be the respective base functors. A *gauge bundle functor* on $\mathcal{VB}$ is a functor F satisfying

G I. (*Base-preservation*) $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}}$

G II. (*Locality*) for every inclusion of an open vector subbundle $i_{E|U}\colon E|U \to E$, $F(E|U)$ is the restriction $p_E^{-1}(U)$ of $p_E\colon FE \to B_{\mathcal{VB}}(E)$ over U and $Fi_{E|U}$ is the inclusion $p_E^{-1}(U) \to FE$

Given two gauge bundle functors F_1, F_2 on $\mathcal{VB}$, by a *natural transformation* $\tau\colon F_1 \to F_2$ we mean a system of base preserving fibered maps $\tau_E\colon F_1E \to F_2E$ for every vector bundle E satisfying $F_2f \circ \tau_E = \tau_{\bar{E}} \circ F_1 f$ for every vector bundle homomorphism $f\colon E \to \bar{E}$. If proj_1 and proj_2 are projections in $\mathcal{VB}$ of $E_1 \times E_2$ to E_1 and E_2, respectively, then we say that a gauge bundle functor F on $\mathcal{VB}$ is *product preserving* if $FE_1 \xleftarrow{F\,\mathrm{proj}_1} F(E_1 \times E_2) \xrightarrow{F\,\mathrm{proj}_2} FE_2$ is a product diagram in $\mathcal{FM}$.

The structure of the paper is the following. The properties of Weil modules are studied in the Section 1. We refer to [4], [5], where Weil modules are

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also used. We publicize a number of examples of Weil modules in 1.3 and generalize the classical result from the Lie group theory about the identification of algebra of derivations and Lie algebra of the group of automorphisms in 1.6. The canonical vector fields on vector bundles with m-dimensional base and n-dimensional fibers are classified in Section 2. The Section 3 contains the main results of the paper. Namely, the theorem classifying all natural operators lifting linear vector fields to product preserving gauge bundle functors on vector bundles is expressed in 3.5.

All manifolds in the paper are assumed to be Hausdorff, finite-dimensional, without boundary and of class C^∞. All maps are assumed to be of class C^∞.

1. Weil modules

1.1. We denote by $\mathcal{WA}$ the category with Weil algebras as objects and algebra homomorphisms of Weil algebras as morphisms. Then the classical result of Kainz and Michor, Luciano and Eck reads as follows (see [2]).

Theorem A. *Product preserving bundle functors from the category $\mathcal{M}f$ of manifolds into the category $\mathcal{FM}$ of fibered manifolds are in bijection with objects of $\mathcal{WA}$ and natural transformations between two such functors are in bijection with the morphisms of $\mathcal{WA}$.*

In particular, for a given Weil algebra A we can construct a product preserving bundle functor T^A by

$$T^A M := \bigcup_{z\in M} \{\phi \mid \phi \in \mathrm{Hom}(C^\infty_z(M), A)\},$$

where M is an object of $\mathcal{M}f$, $C^\infty_z(M)$ is the algebra of germs of smooth functions from M into $\mathbb{R}$ in $z \in M$ and $\mathrm{Hom}(C^\infty_z(M), A)$ is the set of all algebra homomorphisms from $C^\infty_z(M)$ into A.

1.2. Let A be a Weil algebra and V a A-module. If V is finite-dimensional as a real vector space, we name it the *Weil A-module*. More precisely, Weil A-module is a triple $(A, V, *)$, where A is an object of $\mathcal{WA}$, V is a finite-dimensional real vector space, and $*\colon A \times V \to V$ is a bilinear mapping endowing V with a structure of A-module; we write av instead $*(a, v)$ (for $a \in A$, $v \in V$). If there is no confusion, we write (A, V) or only V instead $(A, V, *)$ as well.

For a clear understanding, the roman font (Hom, End, Aut, ...) is used for algebra homomorphisms and the italic font (*Hom*, *End*, *Aut*, ...) for homomorphisms of vector spaces from now on. If we write no subscript, we think the homomorphisms over $\mathbb{R}$. If A, B be two Weil algebras, V a Weil A-module and W a Weil B-module, then we can consider a homomorphism ψ from V to W as a homomorphism of $\mathbb{R}$-modules (real vector spaces) and write $\psi \in \mathit{Hom}(V, W)$. Apart from that, we can consider an algebra homomorphism ϕ from A to B, i.e. $\phi \in \mathrm{Hom}(A, B)$, and a map ψ from V to W satisfying $\psi(v + w) = \psi(v) + \psi(w)$, $\psi(av) = \phi(a)\psi(v)$ for all $a \in A$, $v, w \in V$. In this case we write $\psi \in \mathit{Hom}_{A,B}(V, W)$. (If $A = B$, we write $\mathit{Hom}_A(V, W)$ instead $\mathit{Hom}_{A,A}(V, W)$.) Of course, $\psi \in \mathit{Hom}_{A,B}(V, W) \implies \psi \in \mathit{Hom}(V, W)$. It is evident that, for a given ψ, the relation $\psi(av) = \phi(a)\psi(v)$ is not satisfied for an arbitrary $\phi \in \mathrm{Hom}(A, B)$; if it is satisfied, we say that ψ *is over* ϕ. We write a pair of homomorphisms $\phi \in \mathrm{Hom}(A, B)$, $\psi \in \mathit{Hom}_{A,B}(V, W)$, where ψ is over ϕ, exclusively in brackets $\langle\rangle$, i.e. $\langle\phi, \psi\rangle$.

We obtain the category $\mathcal{WM}$, objects of which are Weil modules, and morphisms of which are pairs $\langle\phi, \psi\rangle$ consisting of a homomorphism $\phi \in \mathrm{Hom}(A, B)$, $\psi \in \mathit{Hom}_{A,B}(V, W)$, where ψ is over ϕ.

1.3. We introduce some examples of Weil modules.

(i) An arbitrary ideal $\mathfrak{i}$ of a Weil algebra A is a Weil A-module. In particular, A is a Weil A-module.

(ii) If $\mathfrak{i}$ is an ideal of a Weil algebra A and V a Weil A-module, then $\mathfrak{i}V$ is a Weil A-module.

(iii) If V and W are Weil A-modules of a Weil algebra A, the $V \oplus W$ is a Weil A-module. Further, by a *free Weil A-module* we mean a Weil A-module V in the form $V = \bigoplus_{i=1}^{n} V_i$, where every V_i is isomorphic with the Weil A-module A.

(iv) If A and B are Weil algebras and if A is a subalgebra of B, then B is a Weil A-module, in which $*$ operates usually (not necessarily) as product in B.

(v) If $\phi \in \mathrm{Hom}(B, A)$ is a homomorphism of Weil algebras and if V is a Weil A-module, then V is also B-module as we can define bv as $\phi(b)v$ (for $b \in B$ and $v \in V$). In particular, for a Weil algebra with the order $\mathrm{ord}(A) = r$ and with the width $\mathrm{w}(A) = k$ there is an epimorphism $\pi_A\colon \mathbb{D}^r_k \to A$ and that is why every A-module is a $\mathbb{D}^r_k$-module as well.

(vi) If U is a vector space and if V is a Weil A-module, then $\mathit{Hom}(U, V)$ is a Weil A-module.

(vii) If V and W are Weil A-modules, then $\mathit{Hom}_A(V, W)$ is a Weil A-module.

(viii) If A and B are Weil algebras and if V is a A-module and W is a B-module, then $V \otimes W$ is a Weil $(A \otimes B)$-module.

1.4. The second author have generalized Theorem A as follows (see [3]).

Theorem B. *Product preserving gauge bundle functors from the category $\mathcal{VB}$ of vector bundles into the category $\mathcal{FM}$ of fibered manifolds are in bijection with objects of $\mathcal{WM}$ and natural transformations between two such functors are in bijection with the morphisms of $\mathcal{WM}$.*

In particular, for a given Weil module V we can construct a product preserving bundle functor $T^{A,V}$ by

$$T^{A,V}E := \bigcup_{z \in M} \{\langle\phi, \psi\rangle \,|\, \phi \in \mathrm{Hom}(C^\infty_z(M), A), \psi \in \mathit{Hom}_{C^\infty_z(M),A}(C^{\infty,\,\mathrm{f.l.}}_z(E), V)\},$$

where E is an object of $\mathcal{VB}$, $C^\infty_z(M)$ is the algebra of germs of smooth functions from M into $\mathbb{R}$ in $z \in M$, $C^{\infty,\,\mathrm{f.l.}}_z(E)$ is the $C^\infty_z(M)$-module of germs of smooth fiber linear functions from E into $\mathbb{R}$ in $z \in M$, $\mathrm{Hom}(C^\infty_z(M), A)$ is the set of all algebra homomorphisms from $C^\infty_z(M)$ into A and $\mathit{Hom}_{C^\infty_z(M),A}(C^{\infty,\,\mathrm{f.l.}}_z(E), V)$ is the set of all module homomorphisms from $C^{\infty,\,\mathrm{f.l.}}_z(E)$ into V.

1.5. Let A be a Weil algebra. The group $\mathrm{Aut}(A)$ of automorphisms of A is a closed subgroup of the Lie group $\mathrm{GL}(A)$, so therefore $\mathrm{Aut}(A)$ is a Lie group. The Lie algebra of $\mathrm{GL}(A)$ is $\mathfrak{gl}(A) = End(A)$ and the Lie algebra of $\mathrm{Aut}(A)$ is

$$\mathfrak{aut}(A) = \{D \in End(A) \mid \exp(tD) \in \mathrm{Aut}(A)\ \forall t \in \mathbb{R}\}.$$

The *algebra of derivations* of A is defined as

$$\mathrm{Der}(A) = \{D \in End(A) \mid D(ab) = D(a)b + aD(b)\ \forall a, b \in A\}.$$

In the classical Lie group theory, the identification $\mathrm{Der}(A) = \mathfrak{aut}(A)$ is proved.

1.6. Let A be a Weil algebra and V a Weil A-module. The group $\mathrm{Aut}(A, V)$ consists of pairs $\langle \phi, \psi \rangle$, where $\phi \in \mathrm{Aut}(A)$ and $\psi \in Aut_A(V)$; $\mathrm{Aut}(A, V)$ is a closed subgroup of the Lie group $\mathrm{GL}(A) \times \mathrm{GL}(V)$, so therefore $\mathrm{Aut}(A, V)$ is a Lie group. The Lie algebra of $\mathrm{Aut}(A, V)$ is

$$\mathfrak{aut}(A, V) = \{(D, \Delta) \in End(A) \times End(V) \mid \langle \exp(tD), \exp(t\Delta) \rangle \in \mathrm{Aut}(A) \times Aut_A(V)\ \forall t \in \mathbb{R}\}.$$

Further, the *algebra of derivations* of (A, V) is

$$\mathrm{Der}(A, V) = \{(D, \Delta) \in End(A) \times End(V) \mid D(ab) = D(a)b + aD(b)\ \forall a, b \in A,\ \Delta(av) = D(a)v + a\Delta(v)\ \forall a \in A, v \in V\}.$$

We improve the classical result from 1.5 by the following assertion. Another useful generalization is e.g. in [6].

Proposition 1. *$\mathrm{Der}(A, V) = \mathfrak{aut}(A, V)$.*

Proof. Step 1. Let us consider $(D, \Delta) \in \mathfrak{aut}(A, V)$. The condition $\exp(tD) \in \mathrm{Aut}(A)$ means

$$\exp(tD)(ab) = \exp(tD)(a) \exp(tD)(b).$$

The differentiating with respect to t gives

$$\exp(tD)D(ab) = \exp(tD)D(a) \exp(tD)(b) + \exp(tD)(a) \exp(tD)D(b),$$

or when evaluated at $t = 0$,

$$D(ab) = D(a)b + aD(b).$$

Similarly, $\exp(t\Delta)$ is over $\exp(tD)$ means

$$\exp(t\Delta)(av) = \exp(tD)(a) \exp(t\Delta)(v).$$

The differentiating with respect to t gives

$$\exp(t\Delta)\Delta(av) = \exp(tD)D(a) \exp(t\Delta)(v) + \exp(tD)(a) \exp(t\Delta)\Delta(v),$$

or when evaluated at $t = 0$,

$$\Delta(av) = D(a)v + a\Delta(v).$$

We have proved $(D, \Delta) \in \mathfrak{aut}(A, V) \implies (D, \Delta) \in \mathrm{Der}(A, V)$.

Step 2. Let us consider $(D, \Delta) \in \mathrm{Der}(A, V)$. Take $P(t) := \exp(tD)(ab)$, $S(t) := \exp(tD)(a) \exp(tD)(b)$. Then

$$\frac{d}{dt}P(t) = \frac{d}{dt}\exp(tD)(ab) = \exp(tD)D(ab) = D\exp(tD)(ab) = DP(t)$$

and

$$P(0) = ab.$$

Also,

$$\begin{aligned}\frac{d}{dt}S(t) &= \frac{d}{dt}(\exp(tD)(a)\exp(tD)(b)) = \exp(tD)D(a)\exp(tD)(b) + \exp(tD)(a)\exp(tD)D(b)\\ &= D\exp(tD)(a)\exp(tD)(b) + \exp(tD)(a)D\exp(tD)(b)D\exp(tD)(a)\exp(tD)(b)\\ &= DS(t)\end{aligned}$$

and

$$S(0) = ab.$$

We have

$$\begin{aligned}\frac{d}{dt}P(t) = DP(t) \quad &\text{and} \quad P(0) = ab,\\ \frac{d}{dt}S(t) = DS(t) \quad &\text{and} \quad S(0) = ab.\end{aligned}$$

It means $P(t)$, $S(t)$ satisfy the same first order differential equations and initial conditions. Hence $P(t) = S(t)$.

Now, take $\Pi(t) := \exp(t\Delta)(av)$, $\Sigma(t) := \exp(tD)(a)\exp(t\Delta)(v)$. Then

$$\frac{d}{dt}\Pi(t) = \frac{d}{dt}\exp(t\Delta)(av) = \exp(t\Delta)\Delta(av)\Delta\exp(t\Delta)(av) = \Delta\Pi(t)$$

and

$$\Pi(0) = av.$$

Also,

$$\begin{aligned}\frac{d}{dt}\Sigma(t) &= \frac{d}{dt}(\exp(tD)(a)\exp(t\Delta)(v)) = \exp(tD)D(a)\exp(t\Delta)(v) + \exp(tD)(a)\exp(t\Delta)\Delta(v)\\ &= D\exp(tD)(a)\exp(t\Delta)(v) + \exp(tD)(a)\Delta\exp(t\Delta)(v)\\ &= \Delta\exp(tD)(a)\exp(t\Delta)(v) = \Delta\Sigma(t)\end{aligned}$$

and

$$\Sigma(0) = av.$$

We have

$$\begin{aligned}\frac{d}{dt}\Pi(t) = \Delta\Pi(t) \quad &\text{and} \quad \Pi(0) = av,\\ \frac{d}{dt}\Sigma(t) = \Delta\Sigma(t) \quad &\text{and} \quad \Sigma(0) = av.\end{aligned}$$

It means $\Pi(t)$, $\Sigma(t)$ satisfy the same first order differential equations and initial conditions. Hence $\Pi(t) = \Sigma(t)$. We have proved $(D, \Delta) \in \mathrm{Der}(A, V) \implies (D, \Delta) \in \mathfrak{aut}(A, V)$, too. □

1.7. Now, we derive the result about the important case $V = A$. For it, we define the *algebra of fissions* of A as

$$\mathrm{Fis}(A) = \{F \in End(A) \mid F(ab) = aF(b)\ \forall a, b \in A\}.$$

Then the following assertion holds.

Proposition 2. $\mathrm{Der}(A, A) = \{(D, D + F) \mid D \in \mathrm{Der}(A), F \in \mathrm{Fis}(A)\}$.

Proof. By the definition,

$$\mathrm{Der}(A,A) = \{(D,\Delta) \in End(A) \times End(A) \mid \quad D(ab) = D(a)b + aD(b), \\ \Delta(ab) = D(a)b + a\Delta(b)\ \forall a,b \in A\}.$$

It follows

$$D(ab) - aD(b) = \Delta(ab) - a\Delta(b).$$

Let $F := \Delta - D$. Then $F \in End(A)$ and relation above reads as $F(ab) = aF(b)$.

Conversely, if we take $D \in \mathrm{Der}(A)$ and $F \in \mathrm{Fis}(A)$, then for $\Delta := D + F$

$$\begin{aligned} \Delta(ab) &= D(ab) + F(ab) = D(a)b + aD(b) + aF(b) = D(a)b + aD(b) + a(\Delta(b) - D(b)) \\ &= D(a)b + a\Delta(b) \end{aligned}$$

is satisfied. □

1.8. We illustrate the previous result on the Weil algebra $A = \mathbb{D} = \mathbb{R}[t]/\langle t^2\rangle$. The elements A can be expressed as

$$k_1 + k_2t, \quad \text{where } k_1, k_2 \in \mathbb{R}.$$

It is not difficult derive that every $D \in \mathrm{Der}(A)$ has a form

$$k_1 + k_2t \mapsto qk_2t, \qquad q \in \mathbb{R},$$

and every $F \in \mathrm{Fis}(A)$ has a form

$$k_1 + k_2t \mapsto qk_1 + (rk_1 + qk_2)t, \qquad q, r \in \mathbb{R}.$$

2. Absolute operators

2.1. We denote by $\mathcal{VB}_{m,n}$ the category of vector bundles with m-dimensional base and n-dimensional fibers together with local vector bundle isomorphisms.

Proposition 3. *Let $m, n \in \mathbb{N}$ and let $F\colon \mathcal{VB} \to \mathcal{FM}$ be a product preserving gauge bundle functor. Every natural transformation $\tau^\diamond\colon F|_{\mathcal{VB}_{m,n}} \to F|_{\mathcal{VB}_{m,n}}$ can be extended uniquely to a natural transformation $\tau\colon F \to F$.*

Proof. Step 1. Let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on the trivial vector bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, which is an object of $\mathcal{VB}_{m,n}$. Let us consider a natural transformation $\tau^\diamond\colon F|_{\mathcal{VB}_{m,n}} \to F|_{\mathcal{VB}_{m,n}}$. As we have $F(\mathbb{R}^m \times \mathbb{R}^n) \cong A^m \times V^n$, the transformation $\tau^\diamond$ corresponds with the map

$$\mathsf{T}\colon A^m \times V^n \to A^m \times V^n,$$

$$\begin{aligned} \mathsf{T}\colon (a^1, \dots, a^m, v^1, \dots, v^n) \mapsto \quad & (\phi^1(a^1, \dots, a^m, v^1, \dots, v^n), \dots, \phi^m(a^1, \dots, a^m, v^1, \dots, v^n), \\ & \psi^1(a^1, \dots, a^m, v^1, \dots, v^n), \dots, \psi^n(a^1, \dots, a^m, v^1, \dots, v^n)), \end{aligned}$$

where $a^1, \dots, a^m \in A$, $v^1, \dots, v^n \in V$. Thus, by the invariance of $\tau^\diamond$ with respect to the homotheties

$$(x^1, \dots, x^m, y^1, \dots, y^n) \mapsto (K_1x^1, \dots, K_mx^m, L_1y^1, \dots, L_ny^n),$$

$(K_1, \dots, K_m, L_1, \dots, L_n \in \mathbb{R}^+)$ which represent morphisms of $\mathcal{VB}_{m,n}$, we have obtained the homogeneity conditions

$$\begin{array}{rcl} K_1\phi^1(a^1,\dots,a^m,v^1,\dots,v^n) &=& \phi^1(K_1a^1,\dots,K_ma^m,L_1v^1,\dots,L_nv^n) \\ & \dots & \\ K_m\phi^m(a^1,\dots,a^m,v^1,\dots,v^n) &=& \phi^m(K_1a^1,\dots,K_ma^m,L_1v^1,\dots,L_nv^n) \\ L_1\psi^1(a^1,\dots,a^m,v^1,\dots,v^n) &=& \psi^1(K_1a^1,\dots,K_ma^m,L_1v^1,\dots,L_nv^n) \\ & \dots & \\ L_n\psi^n(a^1,\dots,a^m,v^1,\dots,v^n) &=& \psi^n(K_1a^1,\dots,K_ma^m,L_1v^1,\dots,L_nv^n). \end{array}$$

Inasmuch the homogeneous function theorem (see e.g. [2]) holds, ϕ_i depends linearly on a_i $(i = 1, \dots, m)$, ψ_j depends linearly on v_j $(j = 1, \dots, n)$ and they do not depend on the remaining a's and v's. As to permissible permutations in $A^m \times V^n$, we have that $\phi_1 = \dots = \phi_m =: \phi$ belongs to *End*(A) and $\psi_1 = \dots = \psi_m =: \psi$ belongs to *End*(V).

Step 2. We need to prove that $\phi \in \mathrm{End}(A)$ and ψ is over ϕ. As T is invariant with respect to the morphism of $\mathcal{VB}_{m,n}$

$$(x^1,\dots,x^m,y^1,\dots,y^n) \mapsto (x^1+(x^1)^2,x^2,\dots,x^m,y^1,\dots,y^n),$$

we have

$$\phi(a+a^2) = \phi(a) + (\phi(a))^2$$

and the linearity of ϕ yields

$$\phi(a^2) = (\phi(a))^2.$$

It gives for $a = b + c$

$$\phi(b^2+2bc+c^2) = \phi((b+c)^2) = (\phi(b+c))^2 = (\phi(b)+\phi(c))^2,$$

whence

$$\phi(bc) = \phi(b)\phi(c)$$

is satisfied for all $b, c \in A$ and that is why $\phi \in \mathrm{End}(A)$.

Further, as T is invariant with respect to the morphism of $\mathcal{VB}_{m,n}$

$$(x^1,\dots,x^m,y^1,\dots,y^n) \mapsto (x^1,\dots,x^m,y^1,y^1+x^1y^1,y^2,\dots,y^n),$$

we have

$$\psi(v+av) = \psi(v) + \phi(a)\psi(v)$$

and the linearity of ψ yields

$$\psi(av) = \phi(a)\psi(v)$$

and that is why ψ is over ϕ.

Thus, $\langle\phi,\psi\rangle$ is a morphism of the category $\mathcal{WM}$.

Step 3. We denote by $\tau\colon F \to F$ the natural transformation which correspond with $\langle\phi,\psi\rangle$. Clearly, $\tau^\diamond$ is the restriction of τ. If $\bar\tau\colon F \to F$ is another natural transformation, restriction of which is $\tau^\diamond$, then $\bar\tau = \tau$, because τ and $\bar\tau$ determine the same morphism $\langle\phi,\psi\rangle$. $\square$

In particular, the corollary of [3] and the previous proposition is, that natural transformations of $F|_{\mathcal{VB}_{m,n}}$ into itself correspond exactly to $\mathrm{Aut}(A,V)$.

2.2. Let us consider $(D,\Delta)\in \mathrm{Der}(A,V)=\mathfrak{aut}(A,V)$. Then $\langle \exp(tD),\exp(t\Delta)\rangle$ is a 1-parameter subgroup of $\mathrm{Aut}(A,V)$, i.e. it can be considered as a 1-parameter subgroup of natural transformations $F|_{\mathcal{VB}_{m,n}}\to F|_{\mathcal{VB}_{m,n}}$. If Y is an object of $\mathcal{VB}_{m,n}$, then $\langle \exp(tD),\exp(t\Delta)\rangle$ defines the flow on FY of a vector field on FM, and accordingly, this vector field. We denote it by $\mathbf{op}(D,\Delta)$. So, the vector field $\mathbf{op}(D,\Delta)$ on FY is the canonical $\mathcal{VB}_{m,n}$-invariant.

Proposition 4. *Let $m,n\in\mathbb{N}$ and let $F\colon \mathcal{VB}\to\mathcal{FM}$ be a product preserving gauge bundle functor with the corresponding Weil module (A,V). Then every canonical vector field (or absolute operator) on F have a form $\mathbf{op}(D,\Delta)$ for some $(D,\Delta)\in\mathrm{Der}(A,V)$.*

Proof. Let us consider such a vector field Ξ; the flow of Ξ on FY for an object Y of $\mathcal{VB}_{m,n}$ is $\mathcal{VB}_{m,n}$-invariant. We denote the flow by Fl^{Ξ}_{τ}. Using homotheties, we can easily show that there exists $u\in FY$ such that FY is the orbit of a neighborhood $U\subset FY$ of u. This yields Ξ is complete, i.e. Fl^{Ξ}_{τ} is global. Hence Fl^{Ξ}_{τ} corresponds to some 1-parameter subgroup in $\mathrm{Aut}(A,V)$. Thus, the corresponding pair $\langle D,\Delta\rangle\in\mathrm{Der}(A,V)$ is determined. Clearly, $\Xi=\mathbf{op}(D,\Delta)$. □

3. Linear vector fields

3.1. We recall the following noted Kolář's result. (see [1]).
Theorem C. *Let $F=T^A$ be a Weil bundle. All natural operators transforming vector field on a manifold M into a vector field on FM (i.e. natural operators $T\rightsquigarrow TT^A$) are of the form*

$$\mathcal{F}^{(a)}+\mathbf{op}(D),$$

where $\mathcal{F}^{(a)}:=\ \mathrm{af}(a)\circ\mathcal{F}$ ($\mathrm{af}(a)$ is a natural affinor determined by $a\in A$) and $D\in\mathrm{Der}(A)$.

3.2. Let $F=T^{A,V}$ be a product preserving gauge bundle functor on vector bundles. Let $Y\to M$ be an object of $\mathcal{VB}_{m,n}$. Then $TY\to TM$ is a vector bundle. A vector field $X\colon Y\to TY$ is called *linear*, if it represents a homomorphism of vector bundles $Y\to M$ and $TY\to TM$. Equivalently, the flow Fl^X_t is a $\mathcal{VB}_{m,n}$-morphism. In the remaining text, we shall study natural operators transforming linear vector fields on a vector bundle Y with m-dimensional base and n-dimensional fibers into a vector fields on FY. Respecting the notation of [2], we write such natural operators as $T_{\mathrm{lin}|\mathcal{VB}_{m,n}}\rightsquigarrow TT^{A,V}$.

3.3. Now, we construct for $F=T^{A,V}$ the flow operator $\mathcal{F}\colon T_{\mathrm{lin}|\mathcal{VB}_{m,n}}\rightsquigarrow TT^{A,V}$. Let $Y\to M$ be an object of $\mathcal{VB}_{m,n}$. Let us consider a linear vector field X on Y. Applying F to the flow Fl^X_t, we obtain the flow $F(Fl^X_t)$ of a vector field on FY. This corresponding vector field on FY determines the operator $\mathcal{F}$ by

$$\mathcal{F}X := \frac{\partial}{\partial t}\Big|_0 F(Fl_t^X).$$

The natural operator $\mathcal{F}$ is called the *flow operator* of $F|_{\mathcal{VB}_{m,n}}$.

3.4. The construction of the *flow operator* of $F|_{\mathcal{VB}_{m,n}}$ can be generalized by the following way. The linear vector field X on Y can be assumed to be complete thanks to locality. The flow Fl^X of X can be considered as the morphism of vector bundles:

$$\begin{tikzcd} \mathbb{R}\times Y \arrow[r, "Fl^X"'] \arrow[d, "\mathrm{id}_{\mathbb{R}}\times p"'] & Y \arrow[d, "p"] \\ \mathbb{R}\times M \arrow[r] & M \end{tikzcd}$$

Applying F, we obtain

$$F(Fl^X)\colon A\times FY \to FY.$$

The flow condition

$$Fl^X(t_1+t_2, y) = Fl^X(t_1, Fl^X(t_2, y))$$

$(t_1, t_2 \in \mathbb{R}, y \in Y)$ transforms to

$$F(Fl^X)(a_1+a_2, w) = F(Fl^X)(a_1, Fl^X(a_2, w))$$

$(a_1, a_2 \in A, w \in FY)$. Hence

$$\Phi_t^{X,a} := F(Fl^X)(at, \quad)$$

is a flow on FY. The corresponding vector field on FY determines the natural operator $\mathcal{F}^{(a)}\colon T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$.

Clearly, $\mathcal{F}^{(1)} = \mathcal{F}$ and the map

$$a \mapsto \mathcal{F}^{(a)}$$

belongs to $Mon(A, S)$, where S is the vector space of all linear natural operators $T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$.

3.5. Let $F = T^{A,V}$ be a product preserving gauge bundle functor on vector bundles. We present the following assertion analogous to Theorem C.

Theorem 1. *All natural operators transforming linear vector field on a vector bundle Y with m-dimensional base and n-dimensional fibers into a vector field on FY (i.e. natural operators $T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{A,V}$) are of the form*

$$\mathcal{L} = \mathcal{F}^{(a)} + \mathbf{op}(D, \Delta),$$

where $(D, \Delta) \in \mathrm{Der}(A, V)$.

Proof. Let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on the trivial vector bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, which is the standard object of $\mathcal{VB}_{m,n}$. Let $\tilde{x}^1, \dots, \tilde{x}^m \colon F(\mathbb{R}^m \times \mathbb{R}^n) \to A$, $\tilde{y}^1, \dots, \tilde{y}^n \colon F(\mathbb{R}^m \times \mathbb{R}^n) \to V$ be the induced coordinates (cf. Example 1 in [3]). Let $\lambda_\alpha \colon A \to \mathbb{R}$, $\alpha = 1, \dots, \dim_A$, $\mu_\beta \colon V \to \mathbb{R}$, $\beta = 1, \dots, \dim_V$ be a basis of the dual vector spaces A^* and V^*. Then

$$x^{i,\alpha} := \lambda_\alpha \circ \tilde{x}^i$$

together with

$$y^{j,\beta} := \mu_\beta \circ \tilde{y}^j$$

($i = 1, \dots, m$, $j = 1, \dots, n$) form a coordinate system on $F(\mathbb{R}^m \times \mathbb{R}^n)$. Let $\frac{\partial}{\partial x^1}$ be the standard linear vector field on $\mathbb{R}^m \times \mathbb{R}^n$. Since any non-vanishing linear vector field can be expressed locally as $\frac{\partial}{\partial x^1}$ in a suitable vector bundle coordinates, then $\mathcal{L}$ is uniquely determined by $\mathcal{L}(\frac{\partial}{\partial x^1})$.

We can write

$$\mathcal{L}(t\frac{\partial}{\partial x^1}) \sum_{i,\alpha} f_{i,\alpha}(t, a, v) \frac{\partial}{\partial x^{i,\alpha}}(a, v) + \sum_{j,\beta} g_{j,\beta}(t, a, v) \frac{\partial}{\partial y^{j,\beta}}(a, v).$$

Using the invariance of $\mathcal{L}$ with respect to the homotheties $K \, \mathrm{id}_{\mathbb{R}^m \times \mathbb{R}^n}$, we obtain the homogeneity conditions

$$\begin{aligned} f_{i,\alpha}(Kt, Ka, Kv) &= K f_{i,\alpha}(t, a, v) \\ g_{j,\beta}(Kt, Ka, Kv) &= K g_{j,\beta}(t, a, v). \end{aligned}$$

Inasmuch the homogeneous function theorem holds, each $f_{i,\alpha}$, $g_{j,\beta}$ depend linearly on t, a, v. Further, $\mathcal{L}(0)$ corresponds to an absolute vector. So, $\mathcal{L}(0) = \mathbf{op}(D, \Delta)$ for some $(D, \Delta) \in \mathrm{Der}(A, V)$. Replacing $\mathcal{L}$ by $\mathcal{L} - \mathcal{L}(0)$, we can assume that $\mathcal{L}(0) = 0$, i.e.

$$\begin{aligned} f_{i,\alpha}(0, a, v) &= 0 \\ g_{j,\beta}(0, a, v) &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} f_{i,\alpha}(t, a, v) &= A_{i,\alpha} t \\ g_{j,\beta}(t, a, v) &= B_{j,\beta} t \end{aligned}$$

for some $A_{i,\alpha}, B_{j,\beta} \in \mathbb{R}$. Further, using the invariance of $\mathcal{L}$ with respect to the homotheties

$$(x^1, \dots, x^m, y^1, \dots, y^n) \mapsto (x^1, Kx^2, \dots, Kx^m, Ky^1, \dots, Ky^n)$$

($K \neq 0$), we obtain the conditions

$$\begin{aligned} A_{i,\alpha} &= 0 \qquad \text{for } i = 2, \dots, m \\ B_{j,\beta} &= 0 \qquad \text{for } j = 1, \dots, n. \end{aligned}$$

Hence $\mathcal{L}$ is uniquely determined by $A_{1,\alpha}$, $\alpha = 1, \dots, \dim A$. It implies that the vector space S of all linear natural operators $T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$ with $\mathcal{L}(0) = 0$ has the dimension less or equal $\dim A$.

Of course, the vector space $\{\mathcal{F}^{(a)}\}_{a \in A}$ has the dimension exactly $\dim A$. It follows S identifies with $\{\mathcal{F}^{(a)}\}_{a \in A}$ and then $\mathcal{L} = \mathcal{F}^{(a)}$ for some $a \in A$. $\square$

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