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TAUBERIAN CONDITIONS FOR L^1-CONVERGENCE OF MODIFIED COMPLEX TRIGONOMETRIC SUMS

(submitted by F. G. Avkhadiev)

Abstract. An L^1–convergence property of the complex form $g_n(c,t) = S_n(c,t) - [c_nE_n(t) + c_{-n}E_{-n}(t)]$ of the modified sums introduced by Garrett and Stanojević [3] is established and a necessary and sufficient condition for L^1-convergence of Fourier series is obtained.

1. Introduction.

Let $S_n(f,t)$ and $\sigma_n(f,t)$ denotes the n^{th} partial sum and n^{th} Cesàro means of the Fourier series $\sum\limits_{|n|<\infty} c_n e^{\iota nt}, t \in T = \frac{R}{2\pi Z}$ respectively. Define $\triangle\hat{f}(n)$as follows: for $n > 0$, $\triangle\hat{f}(n) = \hat{f}(n) - \hat{f}(n+1)$ and $\triangle\hat{f}(-n) = \hat{f}(-n) - \hat{f}(-n-1)$. If the trigonometric series is the Fourier of some $f \in L^1(T)$, we shall write $c_n = \hat{f}(n)$ for all n, and $S_n(c,t) = S_n(f,t) = S_n(f)$. Also $g_n(c,t) = g_n(f,t) = g_n(f)$.

Many authors have defined L^1-convergence classes in terms of the conditions on the sequences of the Fourier coefficients as there exists an integrable function on T whose Fourier series does not converge to itself in L^1-norm. An L^1-convergence class is a class of Fourier coefficients $\{\hat{f}(n)\}$for which

$$|S_n(f,t) - f(t)| = o(1) \qquad (n \to \infty), \tag{1.1}$$

if and only if $\hat{f}(n)\log|n| = o(1) \quad (|n| \to \infty)$.

The following is the well-known L^1-convergence class for Fourier series:

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} \left|\Delta\hat{f}(k)\right|^p = 0. \qquad (1 < p \le 2) \tag{1.2}$$

The above condition (1.2) is a Tauberian condition of Hardy-Karamata kind and is weaker than those considered by Fomin [2], Kolmogorov [4], Littlewood [5] and Telyakovskii [7]. Stanojevic [6] proved the following Tauberian

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Theorem for L^1-convergence of Fourier series of complex valued Lebesgue integrable functions on $T = \frac{R}{2\pi Z}$.

Theorem[3]. Let $f \sim \sum\limits_{|n|<\infty} c_n e^{\iota nt}$ be a Fourier series of $f \in L^1(T)$, whose coefficients satisfy

$$\frac{1}{n}\sum_{k=1}^{n}\left|\hat{f}(k)-\hat{f}(-k)\right|\log k = o(1) \quad (n\to\infty), \tag{1.3}$$

$$\lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}\left|\triangle\left(\hat{f}(k)-\hat{f}(-k)\right)\right|\log k = 0. \tag{1.4}$$

If for some $1 < p \leq 2$,

$$\lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}k^{p-1}\left|\Delta\hat{f}(k)\right|^p = 0, \tag{1.5}$$

then $|S_n(f) - f| = o(1) \quad (n \to \infty)$, if and only if

$$\left|\hat{f}(n)E_n(t)+\hat{f}(-n)E_{-n}(t)\right| = o(1) \quad (n\to\infty) \tag{1.6}$$

where $E_n(t) = \sum\limits_{k=0}^{n} e^{\iota nt}$.

Sequences satisfying conditions (1.3) and (1.4) are called asymptotically even. In the case of even or odd coefficients, condition (1.6) is equivalent with $\hat{f}(n)\log n = o(1) \quad (n \to \infty)$.

The object of this paper is to study the L^1-convergence of the complex form

$$g_n(c,t) = S_n(c,t) - \left[\hat{f}(n)E_n(t)+\hat{f}(-n)E_{-n}(t)\right] \tag{1.7}$$

of the modified sums introduced by Garrett and Stanojević [3] and to obtain the above mentioned theorem of Stanojević [6] without the notion of asymptotic evenness.

2. Lemma.

We shall use the following Lemma for the proof of our result:

Lemma[1]. For each non-negative integer n, there holds

$$\left|\hat{f}(n)E_n(t)+\hat{f}(-n)E_{-n}(t)\right| = o(1) \quad (n\to\infty)$$

if and only if

$$\hat{f}(n)\log|n| = o(1) \quad (|n|\to\infty),$$

where $\{\hat{f}(n)\}$ is a complex null sequence.

3. Main Result

The main result of this paper is the following theorem:

Theorem. Let $f \sim \sum\limits_{|n|<\infty} c_n e^{\iota nt}$ be a Fourier series of $\hat{f} \in L^1(T)$.

If for some $1 < p \leq 2$,

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} \left|\Delta \hat{f}(k)\right|^p = 0, \tag{3.1}$$

then

(*i*) $|g_n(f,t) - f(t)| = o(1) \quad (n \to \infty)$.

(*ii*) $|S_n(f,t) - f(t)| = o(1) \quad (n \to \infty)$, if and only if

$$\hat{f}(n) \log |n| = o(1) \quad (|n| \to \infty).$$

Here and in the sequel, $[.]$ means the greatest integral part and $|.|$denotes $L^1(T)$-norm:

$|f| = \frac{1}{\pi}\int\limits_0^\pi |f(t)|\, dt$

We draw three corollaries of the above theorem:

Corollary 1. If $f \in L^1(T)$ and for some $1 < p \leq 2$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |k|^p \left|\Delta \hat{f}(k)\right|^p \tag{3.2}$$

exists and is finite, then we have both part (i) and (ii) of the theorem.

Corollary 2. If $f \in L^1(T)$ and for some $1 < p \leq 2$,

$$\sum_{|k|=1}^{\infty} |k|^{p-1} \left|\Delta \hat{f}(k)\right|^p < \infty, \tag{3.3}$$

then we have both part (i) and (ii) of the theorem.

It is well known that (3.3) implies the existence of (3.2), and the latter implies (3.1).

Corollary 3. If $f \in L^1(T)$ and for some $1 < p \leq 2$,

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{k=|n|}^{[\lambda n]} \left(\frac{[\lambda n] - k}{[\lambda n] - n}\right)^p |k|^{p-1} \left|\Delta \hat{f}(k)\right|^p = 0, \tag{3.4}$$

then

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} |g_n(f,t) - f(t)| = o(1) \quad (|n| \to \infty),$$

and $|S_n(f,t) - f(t)| = o(1) \quad (|n| \to \infty)$, if and only if

$$\hat{f}(n) \log |n| = o(1) \quad (|n| \to \infty).$$

Also **corollary 1** extends for coefficient sequences satisfying (3.4) instead of (3.1).

Proof of Theorem.

Let $\lambda > 1$ and $n > 1$, then we have

$V_n^\lambda(f,t) - f(t) = \frac{[\lambda n]+1}{[\lambda n]-n} \left(\sigma_{[\lambda n]}(f,t) - f(t)\right) - \frac{n+1}{[\lambda n]-n} \left(\sigma_n(f,t) - f(t)\right)$

where $V_n^\lambda(f,t)=\frac{1}{[\lambda n]-n}\sum\limits_{k=n+1}^{[\lambda n]} S_k(f,t)$ is the generalized de Ia Vallée-Poussin means.

And $\sigma_n(f,t)=\frac{1}{n+1}\sum\limits_{k=0}^{n} S_k(f,t)$

Since $|\sigma_n(f,t)-f(t)|=o(1)\quad(|n|\to\infty)$, then it follows that $\left|V_n^\lambda(f,t)-f(t)\right|=o(1)\quad(|n|\to\infty)$.

Consequently it is sufficient to prove that

$$\lim_{\lambda\downarrow 1}\overline{\lim_{|n|\to\infty}}\left|V_n^\lambda(f,t)-g_n(f,t)\right|=0. \tag{3.5}$$

Elementary calculation gives

$$V_n^\lambda(f,t)-S_n(f,t)=\sum_{|k|=n+1}^{[\lambda n]}\frac{[\lambda n]-|k|+1}{[\lambda n]-n}\hat{f}(k)e^{\iota kt}$$

By (1.7), we have

$$\begin{aligned} V_n^\lambda(f,t)-g_n(f,t)=&\sum_{|k|=n+1}^{[\lambda n]}\frac{[\lambda n]-|k|+1}{[\lambda n]-n}\hat{f}(k)e^{\iota kt}\\ &+\hat{f}(n)E_n(t)+\hat{f}(-n)E_{-n}(t)\end{aligned} \tag{3.6}$$

By using summation by parts, we get

$\sum\limits_{k=n+1}^{[\lambda n]}\frac{[\lambda n]-k+1}{[\lambda n]-n}\hat{f}(k)e^{\iota kt}$

$=\sum\limits_{k=n+1}^{[\lambda n]-1}\triangle\left(\frac{[\lambda n]-k+1}{[\lambda n]-n}\hat{f}(k)\right)E_k(t)+\frac{1}{[\lambda n]-n}\hat{f}([\lambda n])E_{[\lambda n]}(t)-\hat{f}(n+1)E_n(t)$

$=\sum\limits_{k=n+1}^{[\lambda n]-1}\frac{[\lambda n]-k}{[\lambda n]-n}\triangle\hat{f}(k)E_k(t)+\frac{1}{[\lambda n]-n}\hat{f}(k)E_k(t)$

$\qquad+\frac{1}{[\lambda n]-n}\hat{f}([\lambda n])E_{[\lambda n]}(t)-\hat{f}(n+1)E_n(t)$

$=\sum\limits_{k=n}^{[\lambda n]}\frac{[\lambda n]-k}{[\lambda n]-n}\triangle\hat{f}(k)E_k(t)+\frac{1}{[\lambda n]-n}\sum\limits_{k=n+1}^{[\lambda n]}\hat{f}(k)E_k(t)-\hat{f}(n)E_n(t)$

Similarly

$\sum\limits_{k=n+1}^{[\lambda n]}\frac{[\lambda n]-k+1}{[\lambda n]-n}\hat{f}(-k)e^{-\iota kt}$

$=\sum\limits_{k=n}^{[\lambda n]}\frac{[\lambda n]-k}{[\lambda n]-n}\triangle\hat{f}(-k)E_{-k}(t)+\frac{1}{[\lambda n]-n}\sum\limits_{k=n+1}^{[\lambda n]}\hat{f}(-k)E_{-k}(t)-\hat{f}(-n)E_{-n}(t)$

Therefore

$V_n^\lambda(f,t)-g_n(f,t)$

$=\sum\limits_{|k|=n}^{[\lambda n]}\frac{[\lambda n]-|k|}{[\lambda n]-n}\triangle\hat{f}(k)E_k(t)+\frac{1}{[\lambda n]-n}\sum\limits_{|k|=n+1}^{[\lambda n]}\hat{f}(k)E_k(t)$

Note that

$$\left|V_n^\lambda(f,t) - g_n(f,t)\right| = \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right\}\left|V_n^\lambda(f,t) - g_n(f,t)\right| dt$$
$$= I_1 + I_2$$

For the first integral, we have the following estimate from (3.5)

$$I_1 \le \frac{1}{n}\sum_{|k|=n+1}^{[\lambda n]} \frac{[\lambda n]-|k|+1}{[\lambda n]-n}\left|\hat{f}(k)\right| \le \frac{1}{n}\sum_{|k|=n+1}^{[\lambda n]}\left|\hat{f}(k)\right| = o(1) \quad (|n|\to\infty),$$

since $\{\hat{f}(n)\} = o(1) \quad (|n|\to\infty)$.

Therefore (3.2) holds if and only if

$$\lim_{\lambda\downarrow 1}\overline{\lim_{|n|\to\infty}}\int_{\frac{\pi}{n}}^{\pi}\left|V_n^\lambda(f,t) - g_n(f,t)\right| dt = 0.$$

To estimate I_2

$$V_n^\lambda(f,t) - g_n(f,t) = \sum_{|k|=n}^{[\lambda n]}\frac{[\lambda n]-|k|}{[\lambda n]-n}\triangle\hat{f}(k)E_k(t) + \frac{1}{[\lambda n]-n}\sum_{|k|=n+1}^{[\lambda n]}\hat{f}(k)E_k(t)$$
$$= I_{n1} + I_{n2}$$

After applying Hölder-inequality and then the Hausdorff-Young inequality to I_{n2}, we have

$$I_{n2} \le C_p\left(\frac{n}{[\lambda n]-n}\right)^{\frac{1}{q}}\left(\frac{1}{[\lambda n]-n}\sum_{|k|=n+1}^{[\lambda n]}\left|\hat{f}(k)\right|^p\right)^{\frac{1}{p}}$$

Similarly

$$I_{n1} \le B_p\left(\sum_{|k|=n}^{[\lambda n]}|k|^{p-1}\left|\triangle\hat{f}(k)\right|^p\right)^{\frac{1}{p}}$$

where C_p and B_p are absolute constants depending on p, and $\frac{1}{p}+\frac{1}{q}=1$.

Since$\{\hat{f}(n)\}$ is a null sequence and $\lambda > 1$, we have

$$\lim_{\lambda\downarrow 1}\overline{\lim_{|n|\to\infty}} I_{n2} = 0.$$

Hence

$$\begin{aligned}
|g_n(f,t) - f(t)| &\le \left|g_n(f,t) - V_n^\lambda(f,t)\right| + \left|V_n^\lambda(f,t) - f(t)\right| \\
&\le \frac{[\lambda n]+1}{[\lambda n]-n}\left|\left(\sigma_{[\lambda n]}(f,t) - f(t)\right| + \frac{n+1}{[\lambda n]-n}\left|(\sigma_n(f,t) - f(t)\right|\right. \\
&\qquad + B_p\left(\sum_{|k|=n}^{[\lambda n]}|k|^{p-1}\left|\triangle\hat{f}(k)\right|^p\right)^{\frac{1}{p}}
\end{aligned} \tag{3.7}$$

Also as $f \in L^1(T)$, it follows that $|\sigma_n(f,t) - f(t)| = o(1) \quad (|n|\to\infty)$.

Taking *lim sup* of both sides of (3.6), we have

$$\overline{\lim_{n\to\infty}}|g_n(f,t) - f(t)| \le B_p\overline{\lim_{n\to\infty}}\left(\sum_{|k|=n}^{[\lambda n]}|k|^{p-1}\left|\triangle\hat{f}(k)\right|^p\right)^{\frac{1}{p}}.$$

By taking *lim* as $\lambda\downarrow 1$ and by condition (3.1), we obtain

$$\lim_{\lambda\downarrow 1}\overline{\lim_{|n|\to\infty}}|g_n(f,t) - f(t)| = 0.$$

$(ii)\quad |S_n(f,t) - f(t)| \le |S_n(f,t) - g_n(f,t)| + |g_n(f,t) - f(t)|$
$= |g_n(f,t) - f(t)| + \left|\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t)\right|,$
$\left|\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t)\right| = |g_n(f,t) - S_n(f,t)|$
$\le |g_n(f,t) - f(t)| + |S_n(f,t) - f(t)|$
Since $|g_n(f,t) - f(t)| = o(1)$, $(|n| \to \infty)$ by (i) and by Lemma,

$$\left|\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t)\right| = o(1) \quad (n \to \infty),$$

if and only if $\hat{f}(n)\log|n| = o(1) \quad (|n| \to \infty)$.

Therefore $|S_n(f,t) - f(t)| = o(1) \quad (|n| \to \infty)$, if and only if $\hat{f}(n)\log|n| = o(1) \quad (|n| \to \infty)$.

This proves part (ii)

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