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D-GAP FUNCTIONS AND DESCENT METHODS FOR A CLASS OF MONOTONE EQUILIBRIUM PROBLEMS

(submitted by A. V. Lapin)

ABSTRACT. We consider a general class of monotone equilibrium problems, which involve nonsmooth convex functions, in a real Banach space. We combine the D-gap function approach and regularization techniques and suggest a descent type algorithm to find solutions to the initial problem.

1. INTRODUCTION

Let U be a nonempty closed and convex subset of a real reflexive Banach space E, $\psi : E \times E \to R$ an equilibrium bifunction, i.e. $\psi(u, u) = 0$ for every $u \in U$. In addition, we assume that $\psi(u, \cdot)$ is convex and lower semicontinuous for each $u \in E$.

We consider equilibrium problem (EP for short) of the form: Find a point $u^* \in U$ such that

$$\psi(u^*, v) \geq 0 \qquad \forall v \in U. \tag{1}$$

It is well known that equilibrium problems represent rather general and suitable format for the formulation and investigation of various complex problems arising in Economics, Mathematical Physics, Transportation, Operations Research and other fields. Moreover, they are closely related to other general problems of Nonlinear Analysis such as fixed point, optimization, complementarity and variational inequality ones. For this reason, various aspects of EPs were investigated by many researchers (see e.g. [1]–[4] and references therein).

One of the most popular approaches to solve various problems of Nonlinear Analysis is to convert them into a suitable optimization problem with the help of so-called gap functions. In particular, in [5] this approach was applied for smooth EPs in a Banach space setting, in [6],[7] we applied this approach to the problem (1) involving a non-smooth function, and we presented descent methods, converging strongly to a solution. However, these methods need

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additional strong monotonicity properties to ensure convergence. In [8], it was suggested to combine the descent methods with regularization technics for solving monotone variational inequalities. In this paper, combining so the descent and regularization techniques, we will construct a converging method for EPs in a common monotone case.

2. Regularization of equilibrium problems

First, we recall several well-known properties of EPs; see e.g. [1]–[3].

The equilibrium bifunction ψ is said to be

(i) *monotone*, if, for all $u, v \in E$, we have

$$\psi(u, v) + \psi(v, u) \leq 0;$$

(ii) *strictly monotone*, if, for all $u, v \in E, u \neq v$, we have

$$\psi(u, v) + \psi(v, u) < 0;$$

(iii) *strongly monotone with constant τ*, if, for all $u, v \in E$, we have

$$\psi(u, v) + \psi(v, u) \leq -\tau \|u - v\|^2.$$

Proposition 2.1. *(i) If $\psi(\cdot, v)$ is hemicontinuous for each $u \in U$ and ψ is monotone, then the solutions set of EP(1) coincides with that of the dual problem*

$$v^* \in U : \qquad \psi(u, v^*) \leq 0 \qquad \forall u \in U \qquad (2)$$

and it is convex and closed.

(ii) If ψ is strictly monotone, then EP(1) has at most one solution.

(iii) If $\psi(\cdot, v)$ is hemicontinuous for each $v \in U$ and ψ is strongly monotone, then EP(1) has a unique solution.

In this section, in addition to the assumptions of Section 1, we shall suppose that ψ is monotone, and that $\psi(\cdot, v)$ is hemicontinuous for each $v \in U$. We denote by U^* the solutions set of EP(1).

Suppose that there exists an equilibrium bifunction $\varphi : E \times E \to R$ which possesses the following properties:

(a) φ is strongly monotone with constant $\tau > 0$,

(b) for all $u, v \in E$ it holds that $\varphi(u, v) \leq \|u\| \|v - u\|$;

(c) $\varphi(\cdot, v)$ is hemicontinuous for each $v \in E$ and $\varphi(u, \cdot)$ is convex and lower semicontinuous for each $u \in E$.

For instance, if there exists a hemicontinuous strongly monotone operator $B : E \to E^*$ such that $\|B\| \leq 1$, we can set

$$\varphi(u, v) = \langle B(u), v - u \rangle. \qquad (3)$$

If E is a Hilbert space, the simplest choice for B is the identity map, i.e

$$\varphi(u, v) = \langle u, v - u \rangle.$$

We now consider the perturbed problem: Find a point $u_\varepsilon \in U$ such that

$$\psi(u_\varepsilon, v) + \varepsilon \varphi(u_\varepsilon, v) \geq 0 \qquad \forall v \in U, \qquad (4)$$

where ε is a positive parameter.

Set $\Phi(u,v) = \psi(u,v) + \varepsilon\varphi(u,v)$, then Φ is an equilibrium bifunction, which is strongly monotone with constant $\varepsilon\tau$, moreover, $\Phi(\cdot,v)$ is hemicontinuous for each $v \in E$ and $\Phi(u,\cdot)$ is convex and lower semicontinuous for each $u \in E$. Applying now Proposition 2.1 with $\psi = \Phi$, we conclude that EP(4) has always a unique solution.

Theorem 2.1. *If $U^* \neq \emptyset$, then $\{u_\varepsilon\} \longrightarrow u_n^*$ as $\varepsilon \longrightarrow 0$, where*

$$u_n^* \in U^* : \qquad \varphi(u_n^*, w) \geq 0 \qquad \forall w \in U^*. \tag{5}$$

Proof. First we note that EP(5) has the unique solution u_n^* due to Proposition 2.1. If $u^* \in U^*$, then

$$\psi(u^*, u_\varepsilon) \geq 0 \qquad \text{and} \qquad \psi(u_\varepsilon, u^*) + \varepsilon\varphi(u_\varepsilon, u^*) \geq 0.$$

Adding these inequalities gives

$$\varepsilon\varphi(u_\varepsilon, u^*) \geq -[\psi(u^*, u_\varepsilon) + \psi(u_\varepsilon, u^*)] \geq 0.$$

Since φ is strongly monotone it follows that

$$-\varepsilon\varphi(u^*, u_\varepsilon) = -\varepsilon[\varphi(u_\varepsilon, u^*) + \varphi(u^*, u_\varepsilon)] + \varepsilon\varphi(u_\varepsilon, u^*) \geq \varepsilon\tau\|u_\varepsilon - u^*\|^2, \tag{6}$$

and

$$\|u^*\| \geq \tau\|u_\varepsilon - u^*\|.$$

So, the sequence $\{u_\varepsilon\}$ is bounded, hence it has weak limit points. Note that in view of Proposition 2.1 (i),

$$\Phi(u, u_\varepsilon) = \psi(u, u_\varepsilon) + \varepsilon\varphi(u, u_\varepsilon) \leq 0 \quad \forall u \in U.$$

Since $\Phi(u,\cdot)$ is convex and lower semicontinuous, it is weakly lower semicontinuous and, for any limit point u' of $\{u_\varepsilon\}$, we have

$$0 \geq \lim_{\varepsilon \to 0}[\psi(u, u_\varepsilon) + \varepsilon\varphi(u, u_\varepsilon)] = \lim_{\varepsilon \to 0}\psi(u, u_\varepsilon) = \psi(u, u') \quad \forall u \in U,$$

i.e. u' solves problem (2), and, in view of Proposition 2.1 (i), it solves EP (1) too. So, all the weak limits of $\{u_\varepsilon\}$ are contained in U^*. Using (6) with $u^* = u_n^*$, i.e.

$$-\varphi(u_n^*, u_\varepsilon) \geq \tau\|u_\varepsilon - u_n^*\|^2,$$

we have by setting $\varepsilon \to 0$

$$0 \geq -\varphi(u_n^*, u') \geq \tau\|u' - u_n^*\|^2 \geq 0,$$

where u' is any limit point of $\{u_\varepsilon\}$. Therefore, $\lim_{\varepsilon \to 0} u_\varepsilon = u_n^*$, as desired. $\qquad\square$

The result above is clearly an extension of the known convergence properties of the Browder-Tikhonov approximations from variational inequalities; see e.g. [9].

3. Gap and D-gap functions

From now on, we shall consider EP(1) where

$$\psi(u,v) = h(u,v) + f(u) - f(v),$$

$h : E \times E \to R$ is a differentiable equilibrium bifunction, $h(u, \cdot)$ is convex for each $u \in E$ and $f : E \to R$ is a convex continuous, but not necessarily differentiable function. In other words, the problem is to find $u^* \in U$ such that

$$h(u^*, v) + f(u^*) - f(v) \geq 0 \qquad \forall v \in U \tag{7}$$

In what follows we shall consider the simplified variant (3) for the perturbation bifunction φ and we shall suppose that $B : E \to E^*$ is a linear continuous operator which is strongly monotone with constant $\tau > 0$ and $\|B\| \leq 1$. Then, the perturbed EP is formulated as follows: Find $u_\varepsilon \in U$ such that

$$h(u_\varepsilon, v) + f(u_\varepsilon) - f(v) + \varepsilon \langle Bu_\varepsilon, v - u_\varepsilon \rangle \geq 0 \qquad \forall v \in U, \tag{8}$$

where $\varepsilon > 0$ is a given number.

We denote by U^* the solutions sets of EP(7). From Proposition 2.1 it follows that EP(8) has always a unique solution. Moreover, in view of Theorem 2.1, if $U^* \neq \emptyset$, then $\{u_\varepsilon\} \longrightarrow u_n^*$ as $\varepsilon \longrightarrow 0$, where

$$u_n^* \in U^* : \qquad \langle Bu_n^*, w - u_n^* \rangle \geq 0 \qquad \forall w \in U^*.$$

For brevity, we set $h_\varepsilon(u,v) = h(u,v) + \varepsilon \langle Bu, v - u \rangle$. Thus, h_ε is the cost bifunction in the regularized EP. Clearly, h_ε is strongly monotone. Therefore, in order to solve EP(8) with fixed ε, we can apply the D-gap function approach from [6].

Set

$$\Phi_\alpha^{(\varepsilon)}(u,v) = -h_\varepsilon(u,v) - f(v) + f(u) - 0.5\alpha \|v - u\|^2$$

and

$$\mu_\alpha^{(\varepsilon)}(u) = \sup_{v \in U} \Phi_\alpha^{(\varepsilon)}(u,v) = \Phi_\alpha^{(\varepsilon)}(u, v_\alpha^{(\varepsilon)}(u)), \tag{9}$$

where α is a fixed number. The function $\mu_\alpha^{(\varepsilon)}$ can serve as a gap function for EP(8). Note that the inner maximization problem in (9) always has the unique solution $v_\alpha^{(\varepsilon)}(u)$, since $\Phi_\alpha^{(\varepsilon)}(u, \cdot)$ is strongly concave and continuous.

The optimality condition for EP(8) and for the inner problem in (9) can be formulated in the form of the mixed variational inequalities.

Proposition 3.1. [6, Propositions 2.1 and 2.3] *(i) An element u_ε solves EP(8) if and only if $u_\varepsilon \in U$ and*

$$\langle \nabla_v h_\varepsilon(u_\varepsilon, u_\varepsilon), v - u_\varepsilon \rangle + f(v) - f(u_\varepsilon) \geq 0 \quad \forall v \in U. \tag{10}$$

(ii) For all $u' \in U$ it holds that

$$\begin{aligned}\langle \nabla_v h_\varepsilon(u', v_\alpha(u')) + \alpha(v_\alpha(u') - u'), v - v_\alpha(u') \rangle \ + \\ f(v) - f(v_\alpha(u')) \ \geq \ 0 \quad \forall v \in U.\end{aligned} \tag{11}$$

The basic properties of the gap function $\mu_\alpha^{(\varepsilon)}$ can be obtained by using the similar results from [6, Proposition 2.4].

Proposition 3.2. *(i) It holds that $\mu_\alpha^{(\varepsilon)}(u) \geq 0$ for every $u \in U$.*
(ii) If $\mu_\alpha^{(\varepsilon)}(u) = 0$ and $u \in U$, then $u \in U_\varepsilon^*$.
(iii) The following properties are equivalent:
  (a) $u \in U_\varepsilon^*$; (b) $u = v_\alpha^{(\varepsilon)}(u)$; (c) $\mu_\alpha^{(\varepsilon)}(u) = 0$ and $u \in U$.

Therefore, EP(8) can be in principle replaced with the constrained optimization problem:

$$\min_{u \in U} \to \mu_\alpha^{(\varepsilon)}(u).$$

The gap function $\mu_\alpha^{(\varepsilon)}$ is however non-smooth in general and this fact may create additional difficulties in developing a suitable descent method. In order to obtain a smooth problem we turn to so-called D-gap functions.

Let us now consider the function

$$\psi_{\alpha\beta}^{(\varepsilon)}(u) = \mu_\alpha^{(\varepsilon)}(u) - \mu_\beta^{(\varepsilon)}(u),$$

where $0 < \alpha < \beta$. In order to obtain the basic properties of $\psi_{\alpha\beta}^{(\varepsilon)}$ we need the following auxiliary assertion.

Proposition 3.3. *For every $u \in E$, we have*

$$\|u - v_\beta^{(\varepsilon)}(u)\|^2 \leq 2\psi_{\alpha\beta}^{(\varepsilon)}(u)/(\beta - \alpha) \leq \|u - v_\alpha^{(\varepsilon)}(u)\|^2 \qquad (12)$$

Proof. By definition,

$$\begin{aligned}
\psi_{\alpha\beta}^{(\varepsilon)}(u) &= \Phi_\alpha^{(\varepsilon)}(u, v_\alpha^{(\varepsilon)}(u)) - \Phi_\beta^{(\varepsilon)}(u, v_\beta^{(\varepsilon)}(u)) \\
&\geq \Phi_\alpha^{(\varepsilon)}(u, v_\beta^{(\varepsilon)}(u)) - \Phi_\beta^{(\varepsilon)}(u, v_\beta^{(\varepsilon)}(u)) \geq (\beta - \alpha)\|u - v_\beta^{(\varepsilon)}(u)\|^2/2,
\end{aligned}$$

i.e. the left inequality in (12) holds. Similarly, we obtain the right inequality in (12). $\qquad\square$

The next proposition, which follows directly from Propositions 3.2 and 3.3, says that the D-gap function $\psi_{\alpha\beta}^{(\varepsilon)}$ possesses the gap properties not only on the feasible set U but over the whole space too.

Proposition 3.4. *(i) It holds that $\psi_{\alpha\beta}^{(\varepsilon)}(u) \geq 0$ for every $u \in E$.*
(ii) It is true that $\psi_{\alpha\beta}^{(\varepsilon)}(u) = 0 \Longleftrightarrow u \in U_\varepsilon^*$.

Thus, the perturbed EP(8) is equivalent to the unconstrained minimization problem

$$\min_{u \in E} \to \psi_{\alpha\beta}^{(\varepsilon)}(u). \qquad (13)$$

However, this problem can have in principle local minima which differ from the global ones. For this reason, it is more suitable to replace EP(8) with the problem of finding a stationary point of problem (13) that is

$$\nabla \psi_{\alpha\beta}^{(\varepsilon)}(u) = 0.$$

This result needs certain additional assumptions.

(A1) *The map $\nabla_v h(\cdot, \cdot)$ is uniformly Lipschitz continuous with constant L_h.*

(A2) *The map $\nabla_u h(\cdot, \cdot)$ is continuous.*

(A3) *The map $\nabla_u h(u, \cdot)$ is monotone for each fixed $u \in E$.*

Note that it follows from (A1) that $\nabla_v h_\varepsilon(\cdot, \cdot)$ is uniformly Lipschitz continuous with constant $L'_h = L_h + 1$. Also, it follows from (A3) that $\nabla_u h_\varepsilon(u, \cdot)$ is strongly monotone for each fixed $u \in E$.

For each $u \in U$, we set

$$H(u) = \nabla_v h(u, v)|_{v=u},$$

then

$$H_\varepsilon(u) = \nabla_v h_\varepsilon(u, v)|_{v=u} = H(u) + \varepsilon B u.$$

Since h is monotone, then so is H (see [11, Proposition 2.1.17]). Moreover, it follows that the bifunction h_ε is strongly monotone with constant $\varepsilon\tau$ and so is the map $H_\varepsilon : E \to E^*$.

The following results are crucial for applying D-gap functions to nonsmooth EPs of form (7).

Proposition 3.5. *(i) Let (A1) and (A2) hold. Then the function $\psi^{(\varepsilon)}_{\alpha\beta}$ is continuously differentiable and*

$$\begin{aligned}
\nabla \psi^{(\varepsilon)}_{\alpha\beta}(u) &= \nabla_u h_\varepsilon(u, v^{(\varepsilon)}_\beta(u)) - \nabla_u h_\varepsilon(u, v^{(\varepsilon)}_\alpha(u)) \\
&+ \beta(u - v^{(\varepsilon)}_\beta(u)) - \alpha(u - v^{(\varepsilon)}_\alpha(u)).
\end{aligned}$$

(ii) Let (A1),(A2) and (A3) hold. Then $\nabla \psi^{(\varepsilon)}_{\alpha\beta}(u) = 0$ implies that $u \in U^*_\varepsilon$.

The proofs of this results follow directly from Theorems 4.1 and 5.1 in [6], respectively.

The smoothing property of D-gap functions for mixed variational inequalities was noticed by Konnov [10].

Assertion (ii) shows that the initial non-smooth and constrained problem (8) can be replaced with the problem of finding a stationary point of a continuously differentiable function $\psi^{(\varepsilon)}_{\alpha\beta}$. This problem can be solved with the help of the usual unconstrained optimization methods. We intend to describe such a method in the next section.

4. Solution method

We first establish an error bound with the help of the D-gap function. Let us introduce the additional condition.

(A4) *The map $\nabla_u h(\cdot, \cdot)$ is uniformly Lipschitz continuous on each bounded subset of $E \times E$.*

Lemma 4.1. *Let (A1) and (A4) hold. Then*

$$\|u' - u_\varepsilon\| \le \hat{\gamma}\|u' - v^{(\varepsilon)}_\beta(u')\| \qquad \forall u' \in E, \tag{14}$$

where $\hat{\gamma} = (\beta + 2L'_h)/(\varepsilon\tau)$, u_ε solves EP(8).

Proof. Take an arbitrary point $u' \in E$. For brevity, set $v' = v_\beta^{(\varepsilon)}(u')$. Adding (10) with $v = v'$ and (11) with $\alpha = \beta$, $v = u_\varepsilon$ gives

$$
\begin{aligned}
0 &\leq \langle \nabla_v h_\varepsilon(u_\varepsilon, u_\varepsilon) - \nabla_v h_\varepsilon(u', v'), v' - u_\varepsilon \rangle + \beta \langle v' - u', u_\varepsilon - v' \rangle \\
&= \langle \nabla_v h_\varepsilon(u_\varepsilon, u_\varepsilon) - \nabla_v h_\varepsilon(u', u'), v' - u_\varepsilon \rangle \\
&+ \langle \nabla_v h_\varepsilon(u', u') - \nabla_v h_\varepsilon(u', v'), v' - u_\varepsilon \rangle + \beta \langle v' - u', u_\varepsilon - v' \rangle.
\end{aligned}
$$

Since H is monotone, we have

$$
\begin{aligned}
\varepsilon \tau \| u_\varepsilon - u' \|^2 &\leq \langle \nabla_v h_\varepsilon(u_\varepsilon, u_\varepsilon) - \nabla_v h_\varepsilon(u', u'), u^* - u' \rangle \\
&\leq \langle \nabla_v h_\varepsilon(u_\varepsilon, u_\varepsilon) - \nabla_v h_\varepsilon(u', u'), v' - u' \rangle \\
&+ \langle \nabla_v h_\varepsilon(u', u') - \nabla_v h_\varepsilon(u', v'), u' - u_\varepsilon \rangle \\
&+ \langle \nabla_v h_\varepsilon(u', u') - \nabla_v h_\varepsilon(u', v'), v' - u' \rangle \\
&+ \beta \langle v' - u', u_\varepsilon - u' \rangle + \beta \langle v' - u', u' - v' \rangle.
\end{aligned}
$$

Since $\nabla_v h_\varepsilon(u', \cdot)$ is monotone, it follows that

$$
\langle \nabla_v h_\varepsilon(u', u') - \nabla_v h_\varepsilon(u', v'), v' - u' \rangle \leq 0.
$$

Besides, it is clear that $\beta \langle v' - u', u' - v' \rangle \leq 0$. Taking into account both the inequalities and (A1),(A4), we obtain

$$
\begin{aligned}
\varepsilon \tau \| u^* - u' \|^2 &\leq L_h' \| u^* - u' \| \| v' - u' \| \\
&+ L_h' \| u^* - u' \| \| v' - u' \| + \beta \| u^* - u' \| \| v' - u' \|,
\end{aligned}
$$

i.e. (14) holds with $\hat{\gamma} = (\beta + 2L_h')/(\varepsilon \tau)$. $\qquad \square$

From this proposition, we can get the following global error result for EP(8).

Theorem 4.1. *Let (A1) and (A4) hold. Then there exists a constant $\tilde{\gamma} > 0$ which is independent of ε and such that*

$$
\varepsilon^2 \| u - u_\varepsilon \|^2 \leq \tilde{\gamma} \psi_{\alpha\beta}^{(\varepsilon)}(u) \quad \forall u \in U,
$$

where u_ε solves EP(8).

Proof. Combining Proposition 3.3 and Lemma 4.1, we have

$$
\varepsilon^2 \| u - u_\varepsilon \|^2 \leq [2(\beta + L_h')^2/(\tau^2(\beta - \alpha)] \psi_{\alpha\beta}(u) = \tilde{\gamma} \psi_{\alpha\beta}(u) \qquad \forall u \in E,
$$

as desired. $\qquad \square$

Set

$$
r(u) = v_\alpha^{(\varepsilon)}(u) - v_\beta^{(\varepsilon)}(u),
$$

$$
s(u) = \alpha[u - v_\alpha^{(\varepsilon)}(u)] - \beta[u - v_\beta^{(\varepsilon)}(u)].
$$

Now we state an algorithm for EP(8), which can be viewed as an application of the algorithm from [6].

Algorithm.

Step 0: Select an initial point $u^0 \in E$ and parameters $\rho > 0, \theta \in (0,1), \gamma > 0$. Set $k = 0$.
Step 1: If $\psi_{\alpha\beta}^{(\varepsilon)}(u^k) = 0$, then stop, u^k is a solution of EP.
Step 2: Set $d^k = r(u^k) + \rho s(u^k)$.
Step 3: Compute m as the smallest nonnegative integer such that

$$
\psi_{\alpha\beta}^{(\varepsilon)}(u^k + \theta^m d^k) - \psi_{\alpha\beta}^{(\varepsilon)}(u^k) \leq -\theta^m \gamma (\| r(u^k) \| + \rho \| s(u^k) \|)^2.
$$

Step 4: Set $\lambda_k = \theta^m, u^{k+1} = u^k + \lambda_k d^k, k = k + 1$ and go to Step 1.

A convergence result for this algorithm follows directly from Theorem 6.2 in [6].

Theorem 4.2. *Let (A1), (A3) and (A4) hold. Then the sequence $\{u^k\}$, generated by the algorithm with $\rho \in (0, \tilde{\rho})$ and $\gamma \leq \varepsilon\tau/2$, converges strongly to a unique solution of EP(8).*

Now we can describe a solution method for monotone EP(7).

Choose a number $\theta > 0$, a positive sequence $\{\varepsilon_l\} \searrow 0$ and a point $z^0 \in E$. For each $l = 1, 2, \ldots$ we have a point $z^{l-1} \in E$ and set $\varepsilon = \varepsilon_l$ and $u^0 = z^{l-1}$. Afterwards we apply the algorithm above and obtain a point

$$u^k \in \{u \in E \mid \psi_{\alpha\beta}^{(\varepsilon)}(u) \leq \psi_{\alpha\beta}^{(\varepsilon)}(u^0)\}$$

such that

$$\psi_{\alpha\beta}^{(\varepsilon)}(u^k) \leq \varepsilon^{(2+\theta)}. \tag{15}$$

Then we set $z^l = u^k$ and $l = l + 1$.

Theorem 4.3. *Let all the assumptions of Theorem 4.2 hold. If EP(7) is solvable, then*

$$\lim_{l \to \infty} z^l = u_n^*.$$

Proof. Using Theorem 4.1, we have

$$\varepsilon_l^2 \|z^l - u_{\varepsilon_l}\|^2 \leq \tilde{\gamma}\psi_{\alpha\beta}^{(\varepsilon)}(z^l).$$

In view of (15) we now obtain

$$\varepsilon_l^2 \|z^l - u_{\varepsilon_l}\|^2 \leq \tilde{\gamma}^2 \varepsilon_l^{2+\theta}.$$

It follows that

$$\|z^l - u_n^*\| \leq \|z^l - u_{\varepsilon_l}\| + \|u_{\varepsilon_l} - u_n^*\| \leq \|u_{\varepsilon_l} - u_n^*\| + \tilde{\gamma}\sqrt{\varepsilon_l^\theta},$$

Due to Theorem 2.1 $\{u_{\varepsilon_l}\} \longrightarrow u_n^*$, hence we obtain $\lim_{l \to \infty} z^l = u_n^*$, as desired. $\square$

Note that many problems in Mathematical Physics and various saddle point problems involve non strictly monotone bifunctions; see e.g. [1], [12], [13]. Hence, our approach can be applied for rather broad classes of problems.

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