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Koji Matsumoto, Adela Mihai, and Dorotea Naitza

SUBMANIFOLDS OF AN EVEN-DIMENSIONAL MANIFOLD STRUCTURED BY A $\mathcal{T}$-PARALLEL CONNECTION

(submitted by B. N. Shapukov)

Abstract. Even-dimensional manifolds N structured by a $\mathcal{T}$-parallel connection have been defined and studied in [DR], [MRV].

In the present paper, we assume that N carries a (1,1)-tensor field J of square -1 and we consider an immersion $x : M \to N$. It is proved that any such M is a CR-product [B] and one may decompose M as $M = M_D \times M_{D^\perp}$, where M_D is an invariant submanifold of M and $M_{D^\perp}$ is an antiinvariant submanifold of M.

Some other properties regarding the immersion $x : M \to N$ are discussed.

1. Preliminaries

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g (∇ is the Levi-Civita connection).

Let ΓTM and $\flat : TM \to T^*M$ be the set of sections of the tangent bundle TM and the musical isomorphism [P] defined by g, respectively. Following [P], we set

$$A^q(M, TM) = Hom(\Lambda^q TM, TM)$$

and notice that elements of $A^q(M, TM)$ are vector valued q-forms.

If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M. Since ∇ is symmetric, one has $d^\nabla(dp) = 0$.

Let $\mathcal{O} = vect\{e_A | A = 1, ..., 2m\}$ be an adapted local field of orthonormal frames on M and $\mathcal{O}^* = covect\{\omega^A\}$ be its associate coframe. One has

$$dp = \omega^A \otimes e_A \qquad (1.1)$$

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and Cartan's structure equations, written in indexless manner, are:

$$\nabla e = \theta \otimes e, \tag{1.2}$$

$$d\omega = -\theta \wedge \omega, \tag{1.3}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.4}$$

In the above equations, θ, resp. Θ, are the local connection forms in TM and the curvature forms on M, respectively.

2. CR-products

Manifolds structured by a $\mathcal{T}$-parallel connection have been initiated by [R1] and several papers have treated such types of manifolds, as for instance [MRV], [DR].

Let $N(J, \Omega, g)$ be a $2m$-dimensional C^∞-manifold endowed with a $(1,1)$-tensor field J such that $J^2 = -1$ and a 2-form Ω of rank $2m$ and let $\mathcal{T}(T^A)$, $A = 1, ...2m$, be a globally defined vector field of components T^A.

There, by reference [DR], one says that N is structured by a $\mathcal{T}$-*parallel connection* if the connection forms θ^A_B and the vectors e_A of the orthonormal vector basis $\mathcal{O} = \{e_A\}$ satisfy

$$\theta^A_B = g(\mathcal{T}, e_B \wedge e_A) = T^B\omega^A - T^A\omega^B \tag{2.1}$$

($\wedge$: the wedge product of vector fields).

It follows from (2.1)

$$\nabla_{\mathcal{T}} e_A = 0, \tag{2.2}$$

which shows that all the vectors of $\mathcal{O}$ are $\mathcal{T}$-parallel and this legitimates the definition of the structure of N.

In addition, it is shown in [DR] that one has

$$\nabla e_A = T^A dp - \omega^A \otimes \mathcal{T} \tag{2.3}$$

and

$$dT^A = f\omega^A, \ f \in \Lambda^0 N. \tag{2.4}$$

It has been proved in [DR] that the forms ω^A of the cobasis $\{\omega^A\} = \mathcal{O}^*$ satisfy

$$d\omega^A = \alpha \wedge \omega^A, \tag{2.5}$$

where

$$\alpha = \mathcal{T}^\flat. \tag{2.6}$$

In the present paper we will study submanifolds M of N.

Recall now the following definition [B]:

A submanifold M of a manifold N endowed with a $(1,1)$-tensor field J, $J^2 = -1$, is defined to be a CR-*submanifold* of N if there exists on M a differentiable distribution $D : p \to D_p \subset T_pM$ satisfying the following conditions:

i) D is invariant (or holomorphic), i.e. $JD_p = D_p$;

ii) the complementary orthogonal distribution $D^\perp : p \to D^\perp_p \subset T_pM$ is antiinvariant, i.e. $JD^\perp_p \subset T^\perp_p M$.

We define the following distributions:

$$D_p = T_pM \cap J(T_pM),$$
$$D_p^{\perp} = \{Z \in T_pM |\, g(Z,X) = 0, \forall X \in D_p\}.$$

By a standard calculation, it follows that

$$N_J(Z, Z') = 0,$$

for any $Z \in \Gamma D$, $Z' \in \Gamma D^{\perp}$, where N_J is the Nijenhuis tensor of J.

This result affirms that any submanifold M of a manifold N structured by a $\mathcal{T}$-parallel connection is a CR-submanifold (see [B]).

By Frobenius theorem it is easily seen that both distributions D and $D^{\perp}$ are integrable. Hence, following [B], such a CR-submanifold is defined as CR-product. In consequence of this fact, M is locally a Riemannian product $M = M_D \times M_{D^{\perp}}$ (M_D is a leaf of D and $M_{D^{\perp}}$ is a leaf of $D^{\perp}$).

Consider now an m'-dimensional submanifold M of N and denote by q the dimension of the normal space corresponding to M, i.e. $m^{'} + q = 2m$. Then if a, b denote tangential indices and h^r_{ab} the components of the second fundamental form, the mean curvature vector field H associated with the immersion $x : M \to N$ is expressed by

$$H = \frac{\sum h^r_{aa}\ e_r}{m'}. \tag{2.7}$$

We recall that H is an extrinsic invariant.

By (2.1), one has

$$h^r_{aa} = g(h(e_a, e_a), e_r) = g(\nabla_{e_a} e_a, e_r) = g(\theta^C_a(e_a)e_C, e_r) =$$
$$= \theta^r_a(e_a) = T^a\omega^r(e_a) - T^r\omega^a(e_a) = -m'T^r.$$

Then

$$H = -\sum_r T^r e_r = -\mathcal{T}^{\perp}. \tag{2.8}$$

This says that, up to sign, H is expressed by the normal component of the structure vector field $\mathcal{T}$.

Operating now on H by the covariant differential ∇ and taking account of (2.4), one infers

$$\nabla_{e_a} H = -\nabla_{e_a}(T^r e_r) = -e_a(T^r)e_r - T^r\nabla_{e_a} e_r = \tag{2.9}$$
$$= -e_a(T^r)e_r - T^r\theta^b_r(e_a)e_b = -dT^r(e_a)e_r - \sum_r (T^r)^2 e_a = -\|\mathcal{T}^{\perp}\|^2 e_a.$$

It follows at once from (2.9)

$$g(A_H e_a, e_b) = -g(\nabla_{e_A} H, e_b) = \|\mathcal{T}^{\perp}\|^2\delta_{ab}. \tag{2.10}$$

Hence, following a known definition [Ch], one may say that the immersion $x : M \to N$ is *pseudo-umbilical.* Moreover, since the second fundamental forms h^r are, as is known [Ch], expressed by

$$h^r_{ab} = h^r(e_a, e_b) = g(h(e_a, e_b), e_r) = g(\nabla_{e_a} e_b, e_r) = \theta^r_b(e_a) = -T^r\delta_{ab}, \tag{2.11}$$

or equivalently

$$h(X, Y) = g(X, Y)H, \tag{2.12}$$

for any vector fields X and Y tangent to M.

This says that the immersion $x : M \to N$ is also *totally umbilical.*

Recall now that in general for any immersion $x : M \to N$ the curvature 2-forms Θ^r_a, Θ^r_s are called the *transversal* and the *vertical curvature forms* [R2], respectively.

Taking use of (2.1) and (2.3), one finds with the help of the structure equations (1.3)

$$\Theta^r_a = 0,\ \Theta^r_s = 0. \tag{2.13}$$

This shows that the transversal and the vertical curvature forms associated with the immersion $x : M \to N$ vanish. In the same order of ideas one derives that the curvature forms Θ^a_b of the CR-submanifold are given by

$$\Theta^a_b = -(2f + \|\mathcal{T}\|^2)\omega^a \wedge \omega^b. \tag{2.14}$$

Hence, following a well-known formula, the above relation affirms the relevant fact that the CR-submanifold M is a space form (see [KN], [YK]).

This also agrees the fact that $2f + \|\mathcal{T}\|^2 = const.$ (see [DR]).

Summing up, we state the following

Theorem. *Let $x : M \to N$ be an immersion of a submanifold M in a $2m$-dimensional manifold N carrying a $(1,1)$-tensor field J of square -1, structured by a $\mathcal{T}$-parallel connection.*

Then any such submanifold M is a CR-product and one may write

$$M = M_D \times M_{D^\perp},$$

where M_D is an invariant submanifold of M and $M_{D^\perp}$ is an antiinvariant submanifold of M.

In addition, one has the following properties:

(i) *If $\mathcal{T}$ means the structure vector field on N, then the mean curvature vector field H associated with the immersion $x : M \to N$ is expressed by*

$$H = -\mathcal{T}^\perp,$$

where $\mathcal{T}^\perp$ represents the normal component of $\mathcal{T}$;

(ii) *The immersion $x : M \to N$ is pseudo-umbilical and totally umbilical; in particular, $x' : M_{D^\perp} \to N$ is antiinvariant pseudo-umbilical and antiinvariant totally umbilical*;

(iii) *M is a space form submanifold of N.*

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Department of Mathematics, Faculty of Education, Yamagata University, 990-8560 Yamagata, Japan
E-mail address: ej192@kdw.kj.yamagata-u.ac.jp

Faculty of Mathematics, Str. Academiei 14, 70109 Bucharest, Romania
E-mail address: adela@geometry.math.unibuc.ro

Istituto di Matematica, Facoltà di Economia, Via dei Verdi 75, 98100 Messina, Italia

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