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EFFICIENT FORMULA OF THE COLORED KAUFFMAN BRACKETS

(submitted by M. Malakhaltsev)

ABSTRACT. In this paper, we introduce a formula for the homogeneous linear recursive relations of the colored Kauffman brackets, which is more efficient than the formula in [G2].

1. MOTIVATION

In this paper, we discuss the "reducibility" of recursive relations. Assume that for a sequence $\{F_i\}_{i\in\mathbb{Z}}$ in an integral domain R there exists a non-empty finite subset S in $\mathbb{Z}$ such that

$$\sum_{i\in S} c_i F_i = 0, \text{ where } c_i(\neq 0) \in R \text{ for any } i \in S.$$

Then the above relation, called a homogeneous linear recursive relation of $\{F_i\}_{i\in\mathbb{Z}}$, is said to be reducible if there exist a non-empty proper subset $S' \subset S$ such that

$$\sum_{i\in S'} d_i F_i = 0, \text{ where } d_i(\neq 0) \in R \text{ for any } i \in S'.$$

If there does not exist such a proper subset S', then the recursive relation is said to be irreducible.

Let us focus on the following homogeneous linear recursive relation of the colored Kauffman brackets $\{\kappa_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}[t, t^{-1}]$ without details:

Theorem 1.1 (Gelca [G2]). *If* $\mathrm{Ker}(\pi_t)$ *for a knot* K *in* S^3 *has a non-zero element* $s := \sum_{i=1}^{k} c_i \cdot (p_i, q_i)_T$, *then* $\{\kappa_n(K_0)\}_{n \in \mathbb{Z}}$ *has the following homogeneous linear recursive relation:*

$$
\begin{aligned}
\widetilde{K_n^s}(K_0) \;\; &:= \;\; \sum_{i=1}^{k} c_i t^{(2n+p_i)q_i} \left\{ (-t^2)^{q_i} \kappa_{n+p_i}(K_0) - (-t^{-2})^{q_i} \kappa_{n+p_i-2}(K_0) \right\} \\
&+ \;\; \sum_{i=1}^{k} c_i t^{-(2n-p_i)q_i} \left\{ (-t^2)^{q_i} \kappa_{-n+p_i}(K_0) - (-t^{-2})^{q_i} \kappa_{-n+p_i-2}(K_0) \right\} = 0,
\end{aligned}
$$

where K_0 *is a framed knot in* S^3 *with 0-framing such that the core of* K_0 *is isotopic to* K. *(* K_0 *is uniquely determined up to isotopy.)*

In fact, for a knot satisfying $\mathrm{Ker}(\pi_t) \neq 0$, all the recursive relations of the colored Kauffman brackets derived from non-zero elements of $\mathrm{Ker}(\pi_t)$ represent defining polynomials of a "noncommutative" $SL(2, \mathbb{C})$-character variety. Moreover the recursive relations include the information of the A-polynomial. (Refer to [GL, N] for details of these topics and related researches.) In these sense, Theorem 1.1 is very interesting, and so we now focus on Theorem 1.1. Note that it is still unknown if $\mathrm{Ker}(\pi_t) \neq 0$ for any knot.

Now, the formula in Theorem 1.1 is in fact reducible. Namely, all the recursive relations given by the formula in Theorem 1.1 are reducible. Indeed, we can get a more efficient formula as follows:

Theorem 1.2. *Under the same notations and the conditions as in Theorem* 1.1, *the following homogeneous linear recursive relation holds:*

$$
K_n^s(K_0) := \sum_{i=1}^{k} c_i t^{-p_i q_i} \left\{ (-t^{2(n+p_i)+2})^{q_i} \kappa_{n+p_i}(K_0) + (-t^{2(n-p_i)+2})^{-q_i} \kappa_{n-p_i}(K_0) \right\} = 0.
$$

We will first review some concepts needed later through Subsections 2.1 and 2.3, prove Theorem 1.2 in Subsections 2.4 and 2.5, and show the efficiency of the above formula in Subsection 2.6.

2. FORMULA IN THEOREM 1.2 AND ITS EFFICIENCY

2.1. **Glossary.** In this paper, we will often consider gluings of 3-manifolds with at least one torus boundary. For convenience, we would like to introduce "the canonical gluing" of such 3-manifolds. Let M_i, $i \in \{1, 2\}$, be a 3-manifold with at least one torus boundary T_i^2. Fix a longitude λ_i and a meridian μ_i of T_i^2. (In the case of the exterior of a knot in a 3-sphere S^3, we fix a preferred longitude of the knot as a longitude of the torus boundary.) Then a gluing of M_1 to M_2 along the tori T_1^2, T_2^2 is said to be canonical (in terms of λ_i's and μ_i's) if λ_1 and μ_1 are glued to λ_2 and μ_2 respectively.

For an arbitrary compact orientable 3-manifold M, a framed link in M is an embedding of the disjoint union of some annuli into M. The framing of a framed link is presented by the blackboard framing in the case where M is a 3-sphere, a

knot complement or a solid torus. The framing is done by the torus framing in the case where M is a cylinder $T^2 \times I$. Here by a framed link in $T^2 \times I$ with 0-framing in terms of the torus framing, we mean a framed link isotopic to an embedding of the disjoint union of some annuli into the torus $T^2 \times \{\frac{1}{2}\}$.

For convenience, we fix a longitude λ and a meridian μ of a torus T^2, and fix a preferred longitude and a meridian of a knot K in S^3 throughout this paper. Note that λ and μ naturally induce a longitude $\lambda(c)$ and a meridian $\mu(c)$ of $T^2 \times \{c\}$ for any $c \in I$.

2.2. **KBSM.** We mention the Kauffman bracket skein module (KBSM for short) needed later. (Refer to [B, BL, HP, P1, P2] for details.) For an arbitrary compact orientable 3-manifold M, the Kauffman bracket skein module $\mathcal{K}_t(M)$ is defined by the quotient of the $\mathbb{C}[t, t^{-1}]$-module $\mathbb{C}[t, t^{-1}]\mathcal{L}_M$ generated by all isotopy classes of framed links in M (including the empty link ϕ) by the $\mathbb{C}[t, t^{-1}]$-submodule generated by all possible elements as follows:

PSfrag replacements PSfrag replacements PSfrag replacements $-t$ $-t^{-1}$,
$\text{Int}(B^3)$ $\text{Int}(B^3)$ $\text{Int}(B^3)$

PSfrag replacements $L \sqcup$ $-(-t^2 - t^{-2})L$, for any framed link L in M,
$\text{Int}(B^3)$

where the three drawings of the first line in the above depictions express framed links identically embedded in M, except in an open ball $\text{Int}(B^3)$.

For a framed knot K_f in M and a positive integer n, let $(K_f)^n$ be the framed link consisting of n parallel copies of K_f. Then we define the element $T_n(K_f)$ of $\mathcal{K}_t(M)$ as follows:

$$T_n(K_f) = K_f \cdot T_{n-1}(K_f) - T_{n-2}(K_f),$$

$$T_1(K_f) = K_f, \ T_0(K_f) = 2 \cdot \phi, \ T_{-n}(K_f) = T_n(K_f).$$

Also we define the element $S_n(K_f)$ of $\mathcal{K}_t(M)$ as follows:

$$S_n(K_f) = K_f \cdot S_{n-1}(K_f) - S_{n-2}(K_f),$$

$$S_1(K_f) = K_f, \ S_0(K_f) = 1 \cdot \phi, \ S_{-n}(K_f) = -S_{n-2}(K_f).$$

Then focus on the following theorem in [P1].

Theorem 2.1 (Przytycki [P1]). *Let F be an orientable surface, and let I be an interval $[0, 1]$. Then the KBSM $\mathcal{K}_t(F \times I)$ is the free $\mathbb{C}[t, t^{-1}]$-module generated by all the isotopy classes of framed links in $F \times I$ (including the empty link) isotopic to embeddings of the disjoint union of some annuli into F with no trivial component.*

Regarding a solid torus $D^2 \times S^1$ as a cylinder $(S^1 \times I) \times I$, we see that $\mathcal{K}_t(D^2 \times S^1)$ is free as $\mathbb{C}[t, t^{-1}]$-module with basis (representatives)

$$\{T_n(\alpha)| \ n \in \mathbb{Z}_{\geq 0}\},$$

where α is an embedded annulus in $(S^1 \times I) \times I$ isotopic to $(S^1 \times [\frac{1}{3}, \frac{2}{3}]) \times \{\frac{1}{2}\}$. Let (p, q) for coprime integers p, q be a framed knot in $T^2 \times I$ with 0-framing whose core is isotopic to the simple closed curve of slope p/q on $T^2 \times \{\frac{1}{2}\}$. (Note that the curve of slope p/q means one homologous to $p[\lambda(\frac{1}{2})] + q[\mu(\frac{1}{2})]$ in $H_1(T^2 \times \{\frac{1}{2}\})$.) Then it also follows from Theorem 2.1 that $\mathcal{K}_t(T^2 \times I)$ is free as $\mathbb{C}[t, t^{-1}]$-module with basis (representatives)

$$\{(np, nq)_T := T_n((p, q))|p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}, \gcd(p, q) = 1, n \in \mathbb{Z}_{\geq 0}\}.$$

2.3. **Colored Kauffman bracket.** The colored Kauffman bracket is an invariant of framed knots in S^3 defined as follows. For a framed knot K_f in S^3 and a non-negative integer n, consider an element $S_n(K_f)$ in $\mathcal{K}_t(S^3) = \mathbb{C}[t, t^{-1}]$. Then the n-th colored Kauffman bracket $\kappa_n(K_f)$ of a framed knot K_f is defined as the element $S_n(K_f)$. Namely, $\kappa_n(K_f) := S_n(K_f)$. Note that the equation $S_{-n}(K_f) = -S_{n-2}(K_f)$ naturally induces $\kappa_{-n}(K_f) = -\kappa_{n-2}(K_f)$. Here S_n corresponds to the Jones-Wenzl idempotent or "the magic element". (Refer to [FG, L].)

2.4. **Efficient formula.** As stated in the first section, the formula in Theorem 1.1 is reducible, which fact will be observed in Subsection 2.6. Indeed, we can polish it as seen in Theorem 1.2. (It is still unknown if the formula in Theorem 1.2 is irreducible.) In this subsection, we give a proof of Theorem 1.2.

We first review some propositions and concepts needed later. For a knot K in a 3-sphere S^3 let $N(K)$ be an open tubular neighborhood of K in S^3, and let E_K be the exterior $S^3 - N(K)$ of K. In [G2] a method is introduced to get a homogeneous linear recursive relation of the colored Kauffman brackets $\{\kappa_n(K_0)\}_{n \in \mathbb{Z}}$. The method is based on the kernel of the homomorphism as $\mathbb{C}[t, t^{-1}]$-module

$$\pi_t : \mathcal{K}_t(T^2 \times I) \to \mathcal{K}_t(E_K),$$

induced by the canonical gluing (see Subsection 2.1) of a cylinder $T^2 \times I$ to the exterior E_K along $T^2 \times \{1\}$ and ∂E_K. Indeed, the gluing induces a bihomomorphism
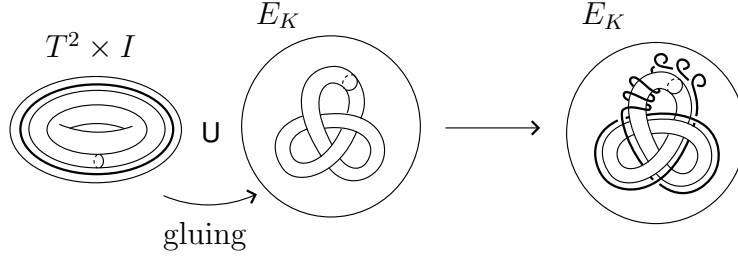
$$C_E : \mathcal{K}_t(T^2 \times I) \times \mathcal{K}_t(E_K) \to \mathcal{K}_t(E_K).$$

We simply denote by $a * b$ the image $C_E(a, b)$ of $(a, b) \in \mathcal{K}_t(T^2 \times I) \times \mathcal{K}_t(E_K)$. Then the homomorphism $\pi_t : \mathcal{K}_t(T^2 \times I) \to \mathcal{K}_t(E_K)$ is defined by $\pi_t((p, q)_T) = (p, q)_T * \phi$.

Now, consider the bihomomorphism

$$C_S : \mathcal{K}_t(D^2 \times S^1) \times \mathcal{K}_t(T^2 \times I) \to \mathcal{K}_t(D^2 \times S^1)$$

induced by the canonical gluing (see Subsection 2.1) of $T^2 \times I$ to $D^2 \times S^1$ along $T^2 \times \{0\}$ and $\partial(D^2 \times S^1)$. We also simply denote by $c * b$ the image $C_S(c, b)$ of $(c, b) \in \mathcal{K}_t(D^2 \times S^1) \times \mathcal{K}_t(T^2 \times I)$. Then we get the following formula.

FIGURE 1. The image $\pi_t((1,0)_T)$

Proposition 2.1 (Gelca [G2]). *For elements $T_n(\alpha) \in \mathcal{K}_t(D^2 \times S^1)$ and $(p,q)_T \in \mathcal{K}_t(T^2 \times I)$, the following holds:*

$$
\begin{aligned}
T_n(\alpha) * (p,q)_T &= t^{(2n+p)q} \left\{ (-t^2)^q S_{n+p}(\alpha) - (-t^{-2})^q S_{n+p-2}(\alpha) \right\} \\
&+ t^{-(2n-p)q} \left\{ (-t^2)^q S_{-n+p}(\alpha) - (-t^{-2})^q S_{-n+p-2}(\alpha) \right\}.
\end{aligned}
$$

In fact, the above equation can be simplified as follows:

Proposition 2.2. *For elements $S_n(\alpha) \in \mathcal{K}_t(D^2 \times S^1)$ and $(p,q)_T \in \mathcal{K}_t(T^2 \times I)$ the following holds:*

$$
S_n(\alpha) * (p,q)_T = t^{-pq} \left\{ (-t^{2(n+p)+2})^q S_{n+p}(\alpha) + (-t^{2(n-p)+2})^{-q} S_{n-p}(\alpha) \right\}.
$$

By Proposition 2.2, we can easily prove Theorem 1.2. We review Gelca's construction given in [G2] to prove the theorem. For any knot K in S^3, consider the pairing

$$
\langle\,,\,\rangle : \mathcal{K}_t(D^2 \times S^1) \times \mathcal{K}_t(E_K) \to \mathcal{K}_t(S^3) = \mathbb{C}[t, t^{-1}],
$$

naturally induced by the 1/0-Dehn filling on E_K. By the above pairing we can represent the n-th colored Kauffman bracket $\kappa_n(K_0)$ of K_0 with 0-framing as follows:

$$
\langle S_n(\alpha), \phi \rangle = \kappa_n(K_0).
$$

(Refer to Subsection 2.3.) Here we see immediately that for any elements $u \in \mathcal{K}_t(D^2 \times S^1)$, $v \in \mathcal{K}_t(E_K)$ and $w \in \mathcal{K}_t(T^2 \times I)$,

$$
\langle u * w, v \rangle = \langle u, w * v \rangle.
$$

Let us consider the case where $u = S_n(\alpha)$, $v = \phi$ and $w = \sum_{i=1}^{k} c_i(p_i, q_i)_T \in \mathrm{Ker}(\pi_t)$ in the above equation. Then by Proposition 2.2 we get the following recursive relation of $\{\kappa_n(K_0) = \langle S_n(\alpha), \phi \rangle\}_{n \in \mathbb{Z}}$:

$$
\sum_{i=1}^{k} c_i t^{-p_i q_i} \left\{ (-t^{2(n+p_i)+2})^{q_i} \langle S_{n+p_i}(\alpha), \phi \rangle + (-t^{2(n-p_i)+2})^{-q_i} \langle S_{n-p_i}(\alpha), \phi \rangle \right\} = 0.
$$

This completes the proof of Theorem 1.2.

2.5. **Proof of Proposition** 2.2. In this subsection, we give a proof of Proposition 2.2. We first see that $T_n(\alpha) = S_n(\alpha) - S_{n-2}(\alpha)$ by induction. Therefore the following holds by Proposition 2.1:

$$
\begin{aligned}
(S_n(\alpha) - S_{n-2}(\alpha)) * (p,q)_T &= t^{(2n+p)q} \left\{ (-t^2)^q S_{n+p}(\alpha) - (-t^{-2})^q S_{n+p-2}(\alpha) \right\} \\
&+ t^{-(2n-p)q} \left\{ (-t^2)^q S_{-n+p}(\alpha) - (-t^{-2})^q S_{-n+p-2}(\alpha) \right\}.
\end{aligned}
$$

Recall $S_n(\alpha) = -S_{-n-2}(\alpha)$. Hence the above equation is transformed as follows:

$$
\begin{aligned}
(S_n(\alpha) - S_{n-2}(\alpha)) * (p,q)_T &= t^{(2n+p)q} \left\{ (-t^2)^q S_{n+p}(\alpha) - (-t^{-2})^q S_{n-2+p}(\alpha) \right\} \\
&+ t^{-(2n-p)q} \left\{ -(-t^2)^q S_{n-2-p}(\alpha) + (-t^{-2})^q S_{n-p}(\alpha) \right\} \\
&= (-t^{2n+p+2})^q S_{n+p}(\alpha) + (-t^{2n-p+2})^{-q} S_{n-p}(\alpha) \\
&- \left\{ (-t^{2n+p-2})^q S_{n-2+p}(\alpha) + (-t^{2n-p-2})^{-q} S_{n-2-p}(\alpha) \right\}.
\end{aligned}
$$

Here let us put $V_n := (-t^{2n+p+2})^q S_{n+p}(\alpha) + (-t^{2n-p+2})^{-q} S_{n-p}(\alpha)$. Then the above transformation immediately derives the following recursive relation:

$$
S_n(\alpha) * (p,q)_T - V_n = S_{n-2}(\alpha) * (p,q)_T - V_{n-2}.
$$

Therefore it suffices to show the equation $S_n(\alpha) * (p,q)_T = V_n$ in the case where n is -1 and 0 for proving Proposition 2.2. If n is -1, then we have

$$
\begin{aligned}
S_{-1}(\alpha) * (p,q)_T - V_{-1} &= (-t^p)^q S_{p-1}(\alpha) + (-t^{-p})^{-q} S_{-p-1}(\alpha) \\
&= (-t^p)^q S_{p-1}(\alpha) - (-t^p)^q S_{p-1}(\alpha) \\
&= 0.
\end{aligned}
$$

If n is 0, then

$$
\begin{aligned}
S_0(\alpha) * (p,q)_T - V_0 &= \frac{1}{2}(S_0(\alpha) - S_{-2}(\alpha)) * (p,q)_T \\
&- (-t^{p+2})^q S_p(\alpha) - (-t^{-p+2})^{-q} S_{-p}(\alpha) \\
&= \frac{1}{2} T_0(\alpha) * (p,q)_T - (-t^{p+2})^q S_p(\alpha) - (-t^{-p+2})^{-q} S_{-p}(\alpha).
\end{aligned}
$$

By Proposition 2.1, we have

$$
\begin{aligned}
\frac{1}{2} T_0(\alpha) * (p,q)_T &= t^{pq} \left\{ (-t^2)^q S_p(\alpha) - (-t^{-2})^q S_{p-2}(\alpha) \right\} \\
&= (-t^{p+2})^q S_p(\alpha) + (-t^{-p+2})^{-q} S_{-p}(\alpha).
\end{aligned}
$$

This shows that $S_0(\alpha) * (p,q)_T - V_0 = 0$ and completes the proof of Proposition 2.2.

2.6. **Efficiency of the formula in Theorem** 1.2. In this subsection, we show the efficiency of the formula $K_n^s(K_0) = 0$ in Theorem 1.2 comparing $\widetilde{K_n^s}(K_0) = 0$ in Theorem 1.1.

According to [G1], for the left-handed trefoil,

$$
s' := (1,-5)_T - t^{-8}(1,-1)_T + t^{-3}(0,5)_T - t(0,1)_T \in \mathcal{K}_t(T^2 \times I)
$$

is in the kernel of π_t. We pick up this element to show the efficiency. Let K_0 be the left-handed trefoil with 0-framing. Then the element s gives rise to the following recursive relation of $\{\kappa_n(K_0)\}_{n\in\mathbb{Z}}$ via the formula $\widetilde{K_n^{s'}}(K_0)=0$ in Theorem 1.1:

$$
\begin{aligned}
\widetilde{K_n^{s'}}(K_0) &= (-t^{-10n-15} + t^{-2n-11})\kappa_{n+1}(K_0) \\
&+ (-t^{10n+7} - t^{-10n-13} + t^{2n+3} + t^{-2n-1})\kappa_n(K_0) \\
&+ (t^{-10n+5} - t^{10n+5} - t^{-2n-7} + t^{2n-7})\kappa_{n-1}(K_0) \\
&+ (t^{10n-13} + t^{-10n+7} - t^{2n-1} - t^{-2n+3})\kappa_{n-2}(K_0) \\
&+ (t^{10n-15} - t^{2n-11})\kappa_{n-3}(K_0) = 0.
\end{aligned}
$$

On the other hand, s gives rise to the following recursive relation of $\{\kappa_n(K_0)\}_{n\in\mathbb{Z}}$ via the formula $K_n^{s'}(K_0)=0$ in Theorem 1.2:

$$
\begin{aligned}
K_n^{s'}(K_0) &= (-t^{-10n-15} + t^{-2n-11})\kappa_{n+1}(K_0) \\
&+ (-t^{10n+7} - t^{-10n-13} + t^{2n+3} + t^{-2n-1})\kappa_n(K_0) \\
&+ (-t^{10n+5} + t^{2n-7})\kappa_{n-1}(K_0) = 0
\end{aligned}
$$

As seen in these examples, the formula in Theorem 1.2 gives us a simpler recursive relation than that in Theorem 1.1 gives us. In fact, this phenomenon always holds. More concretely, $\widetilde{K_n^s}(K_0) = K_n^s(K_0) + K_{n-2}^s(K_0)$ always holds for any knot K in S^3 and any element s in $\mathrm{Ker}(\pi_t)$ for K. (Hence the recursive relations given by the formula in Theorem 1.1 are reducible.) In this sense, the formula in Theorem 1.2 is more efficient than that in Theorem 1.1.

ACKNOWLEDGEMENTS

REFERENCES

[B] D. Bullock: *The $(2,\infty)$-skein module of the complement of a $(2,2p+1)$-torus knot*, J. Knot Theory Ramifications **4** (1995), 619–632.

[BL] D. Bullock and W. LoFaro: *The Kauffman bracket skein module of a twist knot exterior*, preprint.

[FG] C. Frohman and R. Gelca: *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. **352** (2000), 4877–4888.

[FGL] C. Frohman, R. Gelca and W. LoFaro: *The A-polynomial from the noncommutative viewpoint*, Trans. Amer. Math. Soc. **354** (2001), 735–747.

[GL] S. Garoufalidis and T.T.Q. Le: *The colored Jones function is q-holonomic*, preprint.

[G1] R. Gelca: *Noncommutative trigonometry and the A-polynomial of the trefoil*, Math. Proc. Cambridge Phil. Soc., **133** (2002), 311–323

[G2] R. Gelca: *On the relation between the A-polynomial and the Jones polynomial*, Proc. Amer. Math. Soc. **130** (2001), 1235–1241.

[HP] J. Hoste and J. Przytycki: *The $(2,\infty)$-skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications **2** (1993), 321–333.

[L] W.B.R. Lickorish: *The skein method for three-manifold invariants*, J. Knot Theory Ramifications **2** (1993), 171–194.

[N] F. Nagasato: *Computing the A-polynomial using noncommutative methods*, to appear.

[P1] J. Przytycki: *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci. **39** (1991), 91–100.

[P2] J. Przytycki: *Fundamentals of Kauffman bracket skein module*, Kobe J. Math. **16** (1999), no. 1, 45–66.

[S] N. Saveliev: *Lectures on the topology of 3-manifolds: an introduction to the Casson invariant*, de Gruyter textbook, Walter de Gruyter, Berlin, 1999.

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