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SPECTRAL PROPERTIES OF THE ADJOINT OPERATOR AND APPLICATIONS

(submitted by M.A. Malakhaltsev)

ABSTRACT. We present some spectral properties of the adjoint operator corresponding to an admissible dilatation vector field and its perturbations. Next, we apply these results via the Nash-Moser function inverse theorem to show that the group G of diffeomorphisms on the Euclidean space R^n which are 1-time flat, close to the identity and of small support acts transitively on the affine space of appropriate perturbations of the dilation vector field X_o.

1. INTRODUCTION

Let R^n be the euclidean space endowed with the norm $\|.\|$ and F be the Schwartz space of all functions on R^n which are fast falling together with all derivatives. The convergence is defined by the seminorms

$$\|f\|_k = \max_{m+|\alpha| \le k} \sup_{x \in R^n} \left\{ (1 + \|x\|^2)^{\frac{m}{2}} \|D^\alpha f(x)\| \right\}$$

where $k \in N$ and $\alpha = (\alpha_1, ..., \alpha_n) \in N^n$ is a multi-indices with length $|\alpha| = \alpha_1 + ... + \alpha_n$ and the sup runs over all partial derivatives D^α of degree $|\alpha|$ at most k. Denote by $j_0^1 F$ the closed subspace of F of 1-time flat functions at the origin 0 with the induced gradings.

Knowing from the fundamental work of A.Zajtz [5] that the right invertibility of the derivative of the orbital projection and that of the exponential map reduce to the invertibility of the adjoint operator, we give in the first part of this work some spectral properties related to the adjoint operator corresponding to an admissible dilatation vector field and some of its perturbations.

In the second part of our paper we give some applications of the study done in the first part of this work via the Nash-Moser theorem, essentially we establish that the group G of diffeomorphisms f of R^n close to the identity and with small support such that $f - id \in (j^1 F)^n$(which is tame Lie Fréchet group) acts transitively on the affine space of the perturbed vector fields of the admissible dilatation vector field X_o. As a second application, we state that any diffeomorphism from the shifts $G o \exp(X_o)$ or $\exp(X_o)oG$ in a neighborhood of $\exp(X_o)$ are the value at time1 of a smooth flow. This latter result gives an affirmative answer to the question: *what diffeomorphism on a manifold is imbeddible in a smooth flow in contrast with many negative answer on compact manifold.*

We quote as examples:

Negative results:

(N. Kopell 1970) Let $D^\infty([0,1])$ denote the group of smooth diffeomorphisms f of $[0,1]$ such that $f - id$ has all derivatives globally bounded. There are $f \in D^\infty([0,1])$ arbitrary close to the identity in the C^∞ topology such that f does not embed in a C^1- flow and has no fixed point.

(M.I. Bryn 1974) The subset of all C^2 diffeomorphisms of a smooth compact manifold which embed in a C^1 -flow is nowhere dense in the set of all Morse-Smale diffeomorphisms.

(A. Zajtz 1996) No Anosov diffeomorphism embeds into a flow.

Positive results:

(S. Steinberg 1958) Let f be a local C^∞ volume preserving diffeomorphism of R^n, $n \geq 2$, keeping the origin fixed, with eigenvalues satisfying if $\lambda_i = \lambda_1^{m_1}...\lambda_n^{m_n}$ then $m_i - 1 = m_j$ for all $j \neq i$, lies in one parameter group.

(J. Palis, S. Palais 1969) in D^1 there exists an open set of Morse-Smale diffeomorphisms imbeddible in a topological flow.

(J.Grabovski 1988) Every orientation preserving homeomorphism of the interval $[0,1]$ embeds in a topological flow.

2. Spectral properties of the adjoint operators

Let $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ be a linear vector field on R^n(the dilatation vector field), we assume that:

(i) all the coefficients α_i are nonzero and of the same sign

(ii) for any multi-indices $m = (m_1,...,m_n) \in N^n$ with length $|m| = m_1 + ... + m_n \geq 2$, and

$$a = \min\{|\alpha_i| : i = 1,...,n\}, \quad b = \max\{|\alpha_i| i = 1,...,n\}$$

we have $|m| a - b \geq \rho > 0$.

X_o satisfying the conditions (i) and (ii) will be called *admissible* infinitesimal dilatation. X_o induces a global flow of dilatations (ϕ_t) given by

$$\phi_t(x) = \exp(tX_o)(x) = (x_1 \exp \alpha_1 t, ..., x_n \exp \alpha_n t).$$

Let $\Upsilon_1(t) = (\phi_t)_*$ and $\Upsilon^1(t) = (\phi_t)^*$ be the adjoint diffeomorphisms associated to the infinitesimal generators $ad(-X_o)$ and $ad(X_o)$ defined by

$$(\phi_t)_* Y(x) = (D\phi_t.Y)o\phi_{-t}(x)$$

and

$$(\phi_t)^* Y(x) = (D\phi_{-t}.Y)o\phi_t(x).$$

Denote by F the graded Fréchet space of C^∞−functions on R^n with gradings given in the introduction. Let $j_0^1 F$ denote the closed subspace of F of functions f which are 1-time flat at the origin 0 that is to say $f(0) = Df(0) = 0$, with the induced gradings. Denote by $j_0^1 E$ the graded Fréchet space of vector fields defined on R^n and with components in $j_0^1 F$.

For any vector field $Y = \sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i}$, we have

$$\Upsilon_1(t)Y = (\phi_t)_* Y = \sum_{j=1}^n e^{\alpha_j t} u_j(\phi_{-t}(x)) \frac{\partial}{\partial x_j}$$

and

$$\Upsilon^1(t)Y = (\phi_t)^* Y = \sum_{j=1}^n e^{-\alpha_j t} u_j(\phi_t(x)) \frac{\partial}{\partial x_j}.$$

Then, for any multi-indices $\zeta = (\zeta_1, ..., \zeta_n) \in N^n$ and any unit vector $v = (v_1, ..., v_n) \in R^n$, we put $v^\zeta = (v_1^{\zeta_1}, ..., v_n^{\zeta_n})$.

By derivation, we get

$$D_x^\zeta \Upsilon_1(t)Y.v^\zeta = \sum_{i=1}^n e^{\alpha_i t} D_y^\zeta u_i(y)(D_x \phi_{-t}(x)v)^\zeta \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^n \exp t(\alpha_i - \sum_{j=1}^n \zeta_j \alpha_j) D_y^\zeta u_i(y) v^\zeta \frac{\partial}{\partial x_i},$$

where $y = \phi_t(x)$.

2.1. **Spectrum of the operator** $\lambda I - \Upsilon(t)$. First, let $K \subset R^n$ be a compact neighborhood of the origin $0 \in R^n$. Then we establish

Lemma 1. *Let* $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ *be an admissible infinitesimal dilatation and* $Y = \sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i}$ *be any vector field in the Fréchet space* $j_0^1 E$ *and* $\Upsilon_1(t)$ *denotes* $(\phi_t)_*$ *(resp.* $\Upsilon(t)$ *denotes* $(\phi_t)^*$*). Then*

(i) If all α_i are positive $(\alpha_i > 0)$, the series $\sum_{m \geq 0} (\Upsilon(t))^m Y$ converges uniformly in $x \in K$ with respect to $t \in R^{*+}$ on the Fréchet space $j_0^1 E$.

(ii) In case all α_i are negative $(\alpha_i < 0)$, the series $\sum_{m \geq 0} (\Upsilon^1(t))^m Y$ converges uniformly in $x \in K$ with respect to $t \in R^{*+}$ on $j_0^1 E$.

Proof. (i) Let $K \subset R^n$ be a compact neighborhood of the origin $0 \in R^n$ and $D = K \times R^{*+}$.   For any $(x,t) \in D$ and any vector field $Y = \sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i} \in j^1 E$, we consider the series

$$\sum_{m \geq 0} (\phi_{t*})^m Y = \sum_{m \geq 0} (\phi_{mt\,*}) Y = \sum_{j=1}^n \left(\sum_{m \geq 0} e^{mt\alpha_j} u_j(\phi_{-mt}(x)) \right) \frac{\partial}{\partial x_j}.$$

For any integer $r \geq 0$ and any multi-indices $\beta = (\beta_1, ..., \beta_n) \in N^n$ with length $|\beta| = \beta_1 + ... + \beta_n$, we obtain by derivation

$$\left\| D_x^\beta (\Upsilon_1(t))^m Y \right\|_r = \left\| D_x^\beta (\phi_{t*})^m Y \right\|_r$$
$$= \max_{i=1,...,n} \sup_{x \in R^n} \left\{ (1 + \|x\|^2)^{\frac{m}{2}} \left| D_x^{\zeta + \beta} u_i(\phi_{-mt}(x)) \right| \right.$$
$$\left. m + |\zeta + \beta| \leq r + |\beta|, \; x \in K \right\}$$

Putting $\xi = \zeta + \beta$, $y = \phi_{-mt}(x)$, we get

$$D_x^\xi u_i(\phi_{-mt}(x)) = \exp(-mt \sum_{i=1}^n \alpha_i \xi_i) D^\xi u_i(y)$$

so

$$\left\| D_x^\beta (\Upsilon_1(t))^m Y \right\|_r$$
$$\leq \max_{i=1,...,n} \sup \left\{ (1 + \|x\|^2)^{\frac{m}{2}} \left| D_y^\xi u_i(\phi_{-mt}(x)) \right| : m + |\xi| \leq r + |\beta|, \; x \in R^n \right\}$$
$$\times \exp \left(-mt(\sum_{j=1}^n \xi_j \alpha_j - \alpha_i) \right)$$
$$\leq \|Y\|_{r+|\beta|} \exp \left(-mt(\sum_{j=1}^n \xi_j \alpha_j - \alpha_i) \right).$$

So if $|\xi| \geq 2$, by the admissibility of the infinitesimal dilatation X_o, we get

$$\left\| D_x^\beta (\Upsilon_1(t))^m Y \right\|_r \leq \|Y\|_{r+|\beta|} \exp(-\rho m t)$$

and

$$\sum_{m \geq 0} \left\| D_x^\beta (\Upsilon_1(t))^m Y \right\|_r \leq \|Y\|_{r+|\beta|} \frac{1}{1 - e^{-\rho t}} \tag{1}$$

for any $(x,t) \in K \times R^{*+}$.

In the case $|\xi| \leq 1$, we use the 1-time flatness of the vector field Y: that is there is a constant $M > 0$ such that $|u_i(x)| \leq \|x\|^2$ and $|Du_i(x)| \leq M \|x\|$ for any $x \in K$ and $i = 1, ..., n$. Then

$$\left| e^{\alpha_i mt} D_x^{\xi} u_i(\phi_{-mt}(x)) \right| \leq M e^{(\alpha_i - 2\alpha_o)mt}$$

where $\alpha_o = \inf \{\alpha_i : i = 1, ..., n\}$. And by the assumption on the infinitesimal dilatation, we get the estimation 1.

(ii) It remains to check that the time depending vector field $X(t) = (\phi_t)_* Y$ belongs to the Fréchet space $j_0^1 E$. It is obvious that if the vector field Y is 1-flat so do the vector field X and the same computations as part (i) of the proof able us to conclude. $\qquad \square$

As a Corollary of Lemma1, we have

Lemma 2. *Let $X_o = \sum_{i=1}^{n} \alpha_i x_i \frac{\partial}{\partial x_i}$ be an admissible infinitesimal dilatation. Then for any $t \in R^{*+}$, the linear operator $(I - \phi_{t*})$ with $\alpha_i > 0$, $i = 1, ..., n$, (resp. $(I - \phi_t^*)$ in case $\alpha_i < 0$, $i = 1, ..., n$) is invertible in the Fréchet space $j_0^1 E$, and its inverse is given by*

$$(I - \phi_{t*})^{-1} Y = \sum_{m \geq 0} (\phi_{t*})^m Y = \sum_{i=1}^{n} \left(\sum_{m \geq 1} e^{mt\alpha_i} u_i(\phi_{-mt}(x)) \right) \frac{\partial}{\partial x_i}$$

$$(resp. \ (I - \phi_t^*)^{-1} Y = \sum_{m \geq 0} (\phi_t^*)^m Y = \sum_{i=1}^{n} \left(\sum_{m \geq 0} e^{-mt\alpha_i} u_i(\phi_{mt}(x)) \right) \frac{\partial}{\partial x_i})$$

Let C be the complex field for any $t > 0$ and $\rho > 0$ the real number of the condition (ii) of the admissibility of the dilatation X_o that is to say: for any multi-indices $m = (m_1, ..., m_n) \in N^n$ with length $|m|$ the constants $a = \min \{|\alpha_i| \ i = 1, ..., n\}$ and $b = \max \{|\alpha_i| \ i = 1, ..., n\}$ fulfill the relation $|m| a - b \geq \rho > 0$. Letting $\rho(\Upsilon(t)) = \{\lambda \in C : |\lambda| > e^{-\rho t}\}$ and K be any compact neighborhood of the origin $0 \in R^n$, we have

Theorem 1. *Let $X_o = \sum_{i=1}^{n} \alpha_i x_i \frac{\partial}{\partial x_i}$ be an admissible infinitesimal dilatation. For any $\lambda \in \rho(\Upsilon(t))$ and any $(x, t) \in D = K \times R^{*+}$, the linear operator $(\lambda I - \phi_{t*})$ with $\alpha_i > 0$, $i = 1, ..., n$ (resp. $(\lambda I - \phi_t^*)$ in case $\alpha_i < 0$) is invertible on the Fréchet space $j_0^1 E$. The inverse operator on $j_0^1 E$ is given by $R(\lambda, \Upsilon(t)) Y = (\lambda I - \Upsilon(t))^{-1} Y = \sum_{m \geq 0} \frac{1}{\lambda^{m+1}} (\Upsilon(t))^m Y$ where $\Upsilon(t)$ denotes ϕ_{t*} (resp. ϕ_t^*).*

Proof. Let $R(\lambda, \Upsilon_1(t))$ denotes the resolvent of the adjoint operator $\Upsilon_1(t) = \phi_{t*}$, we have

$$R(\lambda, \Upsilon_1(t)) Y = (\lambda I - \phi_{t*})^{-1} Y = \lambda^{-1} (I - \frac{1}{\lambda} \phi_{t*})^{-1} Y$$

$$= \sum_{m \geq 0} \frac{1}{\lambda^{m+1}} \left(\phi_{t*}\right)^m Y = \sum_{i=1}^n \left(\sum_{m \geq 0} \frac{1}{\lambda^{m+1}} e^{m\alpha_i t} u_i\big(\phi_{-mt}(x)\big)\right) \frac{\partial}{\partial x_i}.$$

The rest of the proof is similar to that of Lemma1. $\qquad\square$

Corollary 1. *For every $t > 0$, the spectrum $\sigma\left(\Upsilon(t)\right)$ of the adjoint operator $\Upsilon(t) = (\phi_t)_*$ defined on the Fréchet space $j^1 E$ (resp. $\Upsilon(t) = (\phi_t)^*$) is contained in the closed ball centered at the origin and of radius $e^{-\rho t}$.*

2.2. **Right invertibility of the differential operator $\lambda I - ad(X_o)$.** Let $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ be an admissible infinitesimal dilatation and $j_0^1 E$ be the graded Fréchet space defined in the previous section.

Theorem 2. *If all the coefficients are positive $(\alpha_i > 0)$, then for any complex λ with nonpositive real part, the differential operator $\lambda I - ad(X_o)$ is surjective on the Fréchet space $j_0^1 E$. In the case where all α_i are negative $(\alpha_i < 0)$ and the complex λ with nonnegative real part, the differential operator $\lambda I - ad(X_o)$ is surjective on $j_0^1 E$.*

Proof. Letting $Y = \sum_{i=1}^n f_i(x) \frac{\partial}{\partial x_i} \in j_0^1 E$, we look for a vector field $Z = \sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i}$ in the Fréchet space $j_0^1 E$ such that

$$Y = (\lambda I - ad(X_o)) Z = \lambda Z - [X_0, Z]. \qquad (2)$$

1) Consider the case $\alpha_i < 0$, $i = 1, ..., n$ and the real part of the complex λ, $\text{Re}(\lambda) \geq 0$, and let a and b be real constants fulfilling $0 < a \leq -\alpha_i \leq b$.

We claim that a solution of the equation2 is given

$$Z = \int_0^{+\infty} e^{-\lambda t} \left(\phi_t\right)^* Y \, dt \qquad (3)$$

In fact, equation(2.3) writes in coordinates

$$-\sum_{j=1}^n (\alpha_j x_j) \frac{\partial u_i(x)}{\partial x_j} + (\lambda + \alpha_i) u_i(x) = f_i(x) \qquad (4)$$

and we have to check that the function

$$u_i(x) = \int_0^{+\infty} e^{-t(\lambda + \alpha_i)} f_i(e^{t\alpha_1} x_1, ..., e^{t\alpha_n} x_n) \, dt \qquad (5)$$

is well defined and constitutes a solution of equation 4. We do it in the following steps

(i) First we show that the integral u_i converges uniformly. Letting r be nonnegative integer and $\nu = (\nu_1, ..., \nu_n)$ any multi-indices with length $|\nu| = \nu_1 + ... + \nu_n \leq r$, we get by taking the derivative

$$D_x^\nu e^{-t(\lambda+\alpha_i)} f_i(e^{t\alpha_1} x_1, \ldots, e^{t\alpha_n} x_n) = \exp t(-\alpha_i - \lambda + \sum_{j=1}^{n} \nu_j \alpha_j) D_y^\nu f_i(y),$$

where $y_i = e^{\alpha_i t} x_i$.

Then

$$\int_0^{+\infty} \left| D_x^\nu \left(e^{-t(\lambda+\alpha_i)} f_i(e^{t\alpha_1} x_1, ..., e^{t\alpha_n} x_n) \right) \right| dt$$

$$\leq \|Y\|_r \int_0^{+\infty} \exp t(-\alpha_i - \operatorname{Re}(\lambda) + \sum_{j=1}^{n} \nu_j \alpha_j) dt$$

$$\leq \|Y\|_r \int_0^{+\infty} \exp t(-\operatorname{Re}(\lambda) + b - |\nu| a) dt.$$

So if $|\nu| \geq 2$, by the admissibility of the infinitesimal dilatation we get

$$\int_0^{+\infty} \exp t(-\operatorname{Re}(\lambda) + b - |\nu| a) dt \leq \int_0^{+\infty} \exp t(-\operatorname{Re}(\lambda) - \rho) dt$$

$$= \frac{1}{\rho + \operatorname{Re}(\lambda)}$$

and the integral converges uniformly.

In the case $|\nu| \leq 1$, we use the flatness of the vector field Y at the origin 0.

(ii) We have to verify that the vector field $Z = \sum_{i=1}^{n} u_i(x) \frac{\partial}{\partial x_i}$ belongs to the graded Fréchet space $j_0^1 E$, but this will be easily checked by the same calculation as in step(i).

(iii) It remains to show that u_i is a solution of equation 4. By direct computations we get

$$\sum_{j=1}^{n} \alpha_j x_j \frac{\partial u_i(x)}{\partial x_j} = \int_0^{+\infty} e^{-t(\lambda+\alpha_i)} \frac{d}{dt} f_i(e^{t\alpha_1} x_1, ..., e^{t\alpha_n} x_n) dt$$

$$= \left[e^{-t(\lambda+\alpha_i)} f_i(e^{t\alpha_1} x_1, ..., e^{t\alpha_n} x_n) \right]_0^{+\infty} + (\lambda + \alpha_i) u_i(x)$$

so

$$\sum_{j=1}^{n} \alpha_j x_j \frac{\partial u_i(x)}{\partial x_j} = -f_i(x) + (\lambda + \alpha_i) u_i(x).$$

2) In the case all $\alpha_i > 0$, a solution of 4 is given by

$$Z = \int_0^{+\infty} e^{\lambda t} (\phi_t)_* Y dt$$

with the real part of the complex λ, $\operatorname{Re}(\lambda) \leq 0$.   □

Remark 1. *By doing the change of coordinates $t = e^{-\tau}$, 5 writes*

$$u_i(x) = -\int_0^1 t^{-1+\lambda+\alpha_i} f_i(t^{\alpha_1} x_1, ..., t^{\alpha_n} x_n) dt.$$

2.3. Surjectivity of the perturbed adjoint operator. Let $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ be the infinitesimal dilatation on R^n and $Y_o = X_o + Z$ a perturbed vector field of X_o.

Assume that

(i) all the coefficients $\alpha_i < 0$

(ii) the perturbation Z satisfies the conditions

$$\|D^l Z(x)\| \leq \begin{cases} \begin{cases} c_l \|x\|^k & \text{if } \|x\| < 1 \\ c_l & \text{if } \|x\| \geq 1 \end{cases} & \begin{array}{l} \text{for } l = 0, 1 \\ \text{and any integer} \\ k \geq 1 \end{array} \\ \begin{cases} c_l \|x\|^{k-l} & \text{if } \|x\| < 1 \\ c_l & \text{if } \|x\| \geq 1 \end{cases} & \text{for } 2 \leq l < k \\ c_l & \text{for any } x \in R^n. \end{cases} \tag{P}$$

where c_l is a constant.

Letting $a = \min_{i=1,...,n}(|\alpha_i|)$, $b = \max_{i=1,...,n}(|\alpha_i|)$, put

$$\begin{aligned} a_l &= a - c_l > 0 \quad \text{for } l = 0, 1. \\ a_l &= c_l \qquad l \geq 2 \\ b_l &= b + c_l \\ a_1' &= \min\{a_o, a_1\} \\ b_1' &= \min\{b_o, b_1\}. \end{aligned}$$

With the above notations, the perturbed vector field $Y_o = X_o + Z$ will be said *admissible* if

(i) all the coefficients α_i are nonzero and of the same sign

(ii) For any multi-indices $m = (m_1, ..., m_n) \in N^n$ with length $|m| \geq 2$, $|m| a_1' - b_1' \geq \rho > 0$

where ρ is a positive number.

(iii) the perturbation Z satisfies the condition (P).

Let ψ_t be the flow induced by the vector field $Y_o = X_o + Z$.

The second author stated in [4] a result of global stability. Before quoting this result, we remind :

Definition 1. *An equilibrium point $a \in R^n$ of a flow (ϕ_t) is said globally asymptotically stable if*

(i) *a is an asymptotically stable equilibrium of the flow (ϕ_t)*

(ii) *For any compact set $K \subset R^n$ and any $\varepsilon > 0$ there exists $T_K > 0$ such that for each $t \geq T_K$ we have $\|\phi_t(x) - a\| \leq \varepsilon$ for all $x \in K$.*

Lemma 3. ([4]) *Let K be a compact neighborhood of the origin $0 \in R^n$. Suppose that all coefficients of the dilatation X_o are $\alpha_i < 0$ and the perturbed vector field $Y_o = X_o + Z$ is admissible. Then the origin 0 is globally asymptotically stable and (under the above notations) there are constants $M_l > 0$ such that*

(a) $\|x\| e^{-b_o t} \leq \|\psi_t(x)\| \leq \|x\| e^{-a_o t}$ *for any $t > 0$.*

(b) $M_1 e^{a_1' t} \leq \|D\psi_{-t}(x)\| \leq M_1 e^{b_1' t}$ *for any $(x,t) \in K \times R^{*+}$*

$\left\|D^l \psi_t(x)\right\| \leq M_l e^{-a_1' t}$ *for any $(x,t) \in K \times R^{*+}$*

and any integer $l \geq 2$.

Having Lemma3 in mind, we state

Theorem 3. *Assume that the perturbed vector field $Y_o = X_o + Z$ is admissible , then for any complex λ with nonpositive (resp. nonnegative) real part the differential operator $\lambda I - ad(Y_o)$ is surjective on the Fréchet space $j_0^1 E$ provided that all the coefficients α_i of the linear part X_o of Y_o are positive (resp. all $\alpha_i < 0$).*

Proof. Let $Y = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i} \in j_0^1 E$, we look for a vector field $X = \sum_{i=1}^n u_i(x)\frac{\partial}{\partial x_i}$ from the Fréchet space $j_0^1 E$ such that

$$Y = (\lambda I - ad(Y_o)) X = \lambda X - [Y_o, X]. \tag{6}$$

If the integral

$$X = \int_0^{+\infty} e^{\lambda t} (\psi_t)_* Y \, dt \text{ (resp. } X = \int_0^{+\infty} e^{-\lambda t} (\psi_t)^* Y \, dt \text{)} \tag{7}$$

converges uniformly in the Fréchet space $j_0^1 E$, then the operator $\lambda I - ad(Y)$ is right invertible in $j_0^1 E$ and the equation (6) admits as a solution the vector field given by (7).

We have to show that the integral (7) converges.

(i) Suppose all the $\alpha_i < 0$ and the real part of the complex λ, $\mathrm{Re}(\lambda) \geq 0$. Let $r \in N$ and $\eta = (\eta_1, ..., \eta_n) \in N^n$ be any multi-indices with length $|\eta| = \eta_1 + ... + \eta_n \leq r$; we show that the following integral

$$I = \int_0^{+\infty} e^{-Re(\lambda)t} D_x^v \left(D\psi_{-t}(\psi_t(x)).Y \circ \psi_t(x)\right) dt$$

converges uniformly for any $Y \in j_0^1 E$. $\qquad\qquad\square$

In coordinates, we have

$$D_x^\eta \left(D\psi_{-t}(\psi_t(x)).Y \circ \psi_t(x) \right)_i = D_x^\eta \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_i(-t, \psi_t(x)).f_j \circ \psi_t(x) \right).$$

Letting, for any multi-indices $k = (k_1, .., k_n) \in N^n$,

$$D_x^k \psi_{-t}(x) = \frac{\partial^{|k|}}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \psi_{-t}(x) = \prod_{i=1}^n \frac{\partial^{k_i}}{\partial x_i^{k_i}} \psi_{-t}(x)$$

we get

$$D_x^\eta \left(D\psi_{-t}(\psi_t(x)).Y \circ \psi_t(x) \right)_i = \sum_{j=1}^n \prod_{\iota=1}^n \frac{\partial^{\eta_\iota}}{\partial x_\iota^{\eta_\iota}} \left(\frac{\partial \psi_i}{\partial x_j}(-t, \psi_t(x)).f_j \circ \psi_t(x) \right)$$

$$= \sum_{j=1}^n \prod_{\iota=1}^n \sum_{k_\iota=0}^{\eta_\iota} C_{\eta_\iota}^{k_\iota} \frac{\partial^{\eta_\iota - k_\iota}}{\partial x_\iota^{\eta_\iota - k_\iota} \partial x_j} \psi_i(-t, \psi_t(x)) . \frac{\partial^{k_\iota}}{\partial x_\iota^{k_\iota}} f_j \circ \psi_t(x)$$

$$= \sum_{j=1}^n \sum_{\substack{k_1=0,...,\eta_1 \\ \dotfill \\ k_n=0,...,\eta_n}} C_\eta^k D_x^{\eta-k} \psi_i(-t, \psi_t(x)) . D_x^k f_j \circ \psi_t(x)$$

where

$$C_\eta^k = \frac{\eta!}{k!(\eta - k)!} \quad \text{and} \quad k! = k_1!...k_n!.$$

Putting $\nu = \eta - k$ i.e. $\nu_i = v_i - k_i$ and letting $v = (v_1, ..., v_n) \in R^n$ be any unit vector, we obtain

$$D_x^{v-k} \psi_i(-t, \psi_t(x)) v^\nu = D_x^\nu \psi_i(-t, y) v^\nu$$

$$= \sum_{l_1=1}^{\nu_1} ... \sum_{l_n=1}^{\nu_n} D_y^l \psi_i(-t, y) \prod_{i=1}^n \sum_{\zeta_1^i + ... + \zeta_{l_i}^i = \nu_i} D_x^{\zeta_1^i} \psi_t(x) v^{\zeta_1^i} ... D_x^{\zeta_{l_i}^i} \psi_t(x) v^{\zeta_{l_i}^i}$$

and

$$D_x^k f_j \circ \psi_t(x) = D_x^k f_j(y)$$

$$= \sum_{l_1=1}^{k_1} ... \sum_{l_n=1}^{k_n} D_y^l f_j(y) \prod_{i=1}^n \sum_{\zeta_1^i + ... + \zeta_{l_i}^i = k_i} D_x^{\zeta_1^i} \psi_t(x) v^{\zeta_1^i} ... D_x^{\zeta_{l_i}^i} \psi_t(x) v^{\zeta_{l_i}^i}.$$

By Lemma3 there are positive constants $M_l > 0$ fulfilling for any $(x, t) \in K \times R^{*+}$

$$M_1 e^{a_1' t} \le \| D \psi_{-t}(x) \| \le M_1 e^{b_1' t}$$

$$\left\| D^l \psi_t(x) \right\| \le M_l e^{-a_1' t} \quad \text{with} \quad l \ge 2$$

so there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\left\| D_x^{v-k} \psi_i \left(-t, \psi_t(x)\right) v^v \right\| \leq C_1 \exp\left(-(|v| - |k|)a_1' + b_1'\right)t$$

and

$$\left\| D_x^k f_j \circ \psi_t(x) \right\| \leq C_2 \left\| D_y^k f_j(y) \right\| \exp(-|k|\, a_1' t).$$

Consequently

$$I \leq \max_{1 \leq i \leq n} \int_0^{+\infty} e^{-Re(\lambda)t} \left| D_x^\eta \left(D\psi_{-t}(\psi_t(x)).Y \circ \psi_t(x)\right)_i \right| dt$$

$$\leq \|Y\|_r \int_0^{+\infty} e^{-t\left(Re(\lambda) + |v|a_1' - b_1'\right)} dt$$

so if $|\eta| \geq 2$, by the admissibility of the perturbed vector field Y_o, we get

$$I \leq \|Y\|_r \int_0^{+\infty} e^{-t(Re(\lambda) + \rho)} dt = \frac{1}{\rho + Re(\lambda)}.$$

In the case $|\eta| \leq 1$, we use the flatness of the vector field Y at the origin to conclude.

To check that

$$X = \int_0^{+\infty} e^{-\lambda t} \left(\psi_t\right)^* Y dt$$

is a solution of the Equation 6, it suffices to remark that

$$[Y_o, X] = \frac{d}{ds}\big|_{s=0} \left(\psi_s\right)^* X = \frac{d}{ds}\big|_{s=0} \left(\psi_s\right)^* \int_0^{+\infty} e^{-\lambda t} \left(\psi_t\right)^* Y dt$$

$$= \frac{d}{ds}\big|_{s=0} \int_0^{+\infty} e^{-\lambda t} \left(\psi_{t+s}\right)^* Y dt$$

$$= \int_0^{+\infty} e^{-\lambda t} \frac{d}{dt} \left(\psi_t\right)^* Y dt$$

$$= e^{-\lambda t} \left(\psi_t\right)^* Y \big|_0^\infty + \lambda \int_0^{+\infty} e^{-\lambda t} \left(\psi_t\right)^* Y dt$$

$$= -Y + \lambda X.$$

(ii) In the case all $\alpha_i > 0$, and $Re(\lambda) \leq 0$, the arguments are as in part (i) .

3. Applications to dynamic systems and to the exponential map

3.1. Category of tame Fréchet manifolds.

Definition 2. *Let E and F be Fréchet spaces with graded countable collections of seminorms. We say that a non linear map $P : U \subset E \to F$ satisfies a tame estimate of degree r and base b if*

$$\|P(f)\|_n \leq C(1 + \|f\|_{n+r})$$

for all $f \in U$ and all $n \geq b$, and a constant C which may depend on n, P is a tame map if it is defined on an open set, is continuous, and satisfies a tame estimate in a neighborhood of each point.

3.2. General estimates.
Applying the product and the chain rules of the differentiation and using the interpolation formulae (see [3]) one gets the estimates on any compact set K of R^n.

$$\|fg\|_k \leq C \left(\|f\|_k \|g\|_0 + \|f\|_0 \|g\|_k\right) \tag{8}$$

$$\left\|D^k \left(f \circ g\right)\right\| \leq C \|g\|_1^{k-1} \left(\|f\|_k \|g\|_1 + \|f\|_1 \|g\|_k\right) \tag{9}$$

$$\left\|f^{-1}\right\|_k \leq C_k \left(1 + \|f\|_k\right) \tag{10}$$

for all $k \geq 1$ the latter estimate holds if

$$\|f - id\| \leq \varepsilon \text{ where } \varepsilon \text{ is small.}$$

Example 1. (*of tame map*) *If f is a diffeomorphism on a compact manifold M close to the identity then the adjoint operators $f_* = Ad(f)$ and its inverse $f^* = Ad(f^{-1})$ are tame of degree 0 and base 0 on the space of vector fields on M*

In fact from the general above estimates, we get, for any vector field on M,

$$\|f_* X\|_n = \left\|(Df.X)of^{-1}\right\|_n$$
$$\leq \left\|f^{-1}\right\|_1^{n-1} \left(\|Df.X\|_n \left\|f^{-1}\right\|_1 + \|Df.X\|_1 \left\|f^{-1}\right\|_n\right)$$
$$\leq C_1 \left(1 + \|f\|_n\right)^n \|f\|_{n+1} \|X\|_n$$

so

$$\|f_* X\|_n \leq C \|X\|_n$$

where C is a constant.

Example 2. *In particular, any continuous map from E to a Banach space and any continuous map of finite dimension space into F are tame.*

Definition 3. *P is a smooth tame map if it is smooth and all its derivatives are tame. A linear map $L : E \to F$ is tame if it satisfies a tame estimate*

$$\|L(f)\|_n \leq C \|f\|_{n+r}$$

for some r and all $n \geq b$.

Definition 4. *A graded Fréchet space E is tame if it is direct summand of a space $\sum(B)$ of exponentially decreasing sequences in a Banach space B, so that we have*

$$E \xrightarrow{L} \sum(B) \xrightarrow{M} E$$

with $MoL = id_E$ and M, L are tame.

Example 3. *By a tame manifold we mean a smooth manifold modeled on a tame Fréchet space; in this category we have also the notion of tame Lie group (see [3]).*

We quote the Nash-Moser function theorem

Theorem 4. *(Nash, Moser) Let F and G be tame space and $P : F \to G$ a smooth tame map. Suppose that the equation for the derivative $DP(f)h = k$ has a unique solution $h = VP(f)k$ for all f in U and all k, and that the family of inverses $VP : U \times G \to F$ is a smooth tame map. Then P is locally invertible, and each local inverse P^{-1} is smooth tame map.*

Theorem 5. *Suppose DP is surjective with smooth tame family of right inverse VP. Then P is locally surjective. Moreover in a neighborhood of any point, P has a smooth tame right inverse.*

From the Nash-Moser inverse function theorem, Hamilton deduced the following (cf.[3])

Theorem 6. *(Nash-Moser-Hamilton) Let G be a tame Fréchet Lie group acting tamely on a tame Fréchet manifold F with $A : G \times F \to F$ the action. For any $f \in F$ there is a linear map*

$$A'_f = D_G A(e, f) : T_e G \to T_f F.$$

Suppose that F is connected, and for each $f \in F$ the map A'_f is surjective with a tame linear right inverse. Then G acts transitively on F.

Let (as in the previous sections) R^n be the Euclidean space endowed with the usual norm $\|.\|$; let F denote the Schwartz space of all functions on R^n which are fast falling together with all derivatives. F is a tame Fréchet space (see [3]). Denote by E the graded Fréchet space of vectors fields with components in F and by $j_0^1 E$ the closed subspace of E of vector fields which are 1-time flat at the origin 0 with the induced gradings. $j_0^1 E$ is a tame Fréchet space. Denote by G the group of diffeomorphisms f on R^n such that $f - id \in j_0^1 E$ (1-flat diffeomorphisms). The group G modeled on the tame Fréchet space $j_0^1 E$, is a tame Lie Fréchet group.

Canonically G acts on $j_0^1 E$ by the adjoint action

$$A : G \times j_0^1 E \rightarrow j_0^1 E \ \ A(f, X) = f_* X = (Df.X) o f^{-1}.$$

Let X_o be the infinitesimal dilatation, the affine space

$$F = \left\{ X_o + Z : Z \in j_0^1 E \right\}$$

is a tame Fréchet space and obviously F is invariant by the Fréchet Lie group G. The tangent space $T_{X_o} F$ at X_o of the space affine F is identified to the Fréchet space $j_0^1 E$. And also the tangent space $T_{id} G$ to the group at the identity is identified to $j_0^1 E$.

For any $X_0 \in j_0^1 E$ the derivative of the orbital projection $f \rightarrow f_* X_0$ writes as

$$A'_{X_0} = L(X_0) : j_0^1 E \rightarrow j_0^1 E \text{ with } L(X_0) Y = ad_Y X_0 = [Y, X_0].$$

We are going to apply the Nash-Moser-Hamilton theorem to the above action; to reach this aim, we state.

Let $\phi_t = \exp(t X_0)$, $\psi_t = \exp t(X_o + Z)$ with $X_o = \sum_{i=1}^{n} \alpha_i x_i \frac{\partial}{\partial x_i}$ be the dilatation which all $\alpha_i < 0$ and Z be a perturbation of X_o such that the perturbed dilatation $Y_o = X_o + Z$ is admissible. Put $f_t = \phi_{-t} o \psi_t$.

Lemma 4. $f_t - id$ *is globally bounded to infinite order uniformly in* $t > 0$.

Proof. For any multi-indices $\beta = (\beta_1, ..., \beta_n)$ and any unit vector $v = (v_1, ..., v_n)$, we have

$$D^\beta f_t(x) v^\beta = D\phi_{-t}(\psi_t(x)) D^\beta \psi_t(x) v^\beta$$

so

$$D^\beta f_t(x)' v^\beta = -(D\phi_{-t}.X)(\psi_t(x)) D^\beta \psi_t(x) v^\beta$$
$$+ (D\phi_{-t}.D^\beta (X + Z) v^\beta) o \psi_t(x) = (D\phi_{-t}(D^\beta Z v^\beta)) o \psi_t(x).$$

Letting in mind the notation introduced in subsection 2.3, by integrating we get

$$\left\| D^\beta(f_t - id) \right\| \leq \int_0^t \exp(bs) \left\| D^\beta Z(\psi_s(x)) \right\| ds.$$

As in section 1, we obtain

$$D^\beta Z(\psi_s(x))v^\beta = D^\beta Z(y)v^\beta$$

$$= \sum_{l_1=1}^{\beta_1} \cdots \sum_{l_n=1}^{\beta_n} D_y^l Z(y) \prod_{\iota=1}^{n} \sum_{\eta_1^i+\ldots+\eta_{l_i}^i=\beta_i} D^{\eta_1^i}\psi_t(x)v_1^{\eta_1^i}\ldots D^{\eta_{l_i}^i}\psi_t(x)v_{l_i}^{\eta_{l_i}^i}$$

and by Lemma 3, we have with $y = \psi_t(x)$

$$\left\| D_x^\beta Z(y)v^\beta \right\| \leq \|Z\|_r \exp(-na_1' t)$$

where $|\beta| \leq r$.

Consequently for any any positive integer r and $t \in R^{*+}$, we obtain

$$\|f_t - id\|_r \leq \|Z\|_r \int_0^t e^{(b-na_1)t} dt$$

so, if $n \geq 2$, we have the conclusion.

In the case $n = 1$, we use the flatness of Z at the origin 0.   □

3.3. **Right inverse of the operator** $L(X_o)$. Let $X_o = \sum_{i=1}^n \alpha_i x_i$ be an admissible infinitesimal dilatation and consider the following equation

$$Z = L(X_o)Y = [Y, X_o]$$

in the tame space $j_0^1 E$.

Using the general relations

$$(\phi_t)_* ad(X_o) = -\frac{d}{dt}(\phi_t)_* = \frac{d}{dt}(\phi_t)^*,$$

we obtain by integrating

$$(I - \phi_{t*})Y = \int_0^t (\phi_s)_* Z ds.$$

Suppose that all the coefficients $\alpha_i > 0$. Since by Lemma2, $I - \phi_{t*}$ is invertible, we have

$$Y(Z) = (I - \phi_{t*})^{-1} \int_0^t (\phi_s)_* Z ds \quad \text{for any } t \in R^+.$$

Putting $t = 1$, we get

$$Y(Z) = (I - \phi_*)^{-1} \int_0^1 (\phi_s)_* Z ds.$$

Letting

$$A = \int_0^1 (\phi_s)_* Z \, ds$$

be the mean adjoint operator we obtain

$$L(X)^{-1} = (I - \phi_*)^{-1} A.$$

3.4. **Tame property of the inverse $L(X_o)^{-1}$.** Suppose that the infinitesimal dilatation X_o is admissible with positive coefficients.

By Lemma 2, the inverse map is given by

$$(L(X))^{-1} : j_0^1 E \to j_0^1 E , \quad (L(X))^{-1} Z = Y(Z) = (I - \phi_*)^{-1} A$$

and, by the general estimates (8), (9), and (10), we get for any integer $r \geq 1$

$$\left\| (I - \phi_*)^{-1} A \right\|_r \leq C_1 \left(\left\| (I - \phi_*)^{-1} \right\|_r A_0 + \left\| (I - \phi_*)^{-1} \right\|_0 \|A\|_r \right),$$

$$\left\| (I - \phi_*)^{-1} \right\|_r \leq C_2 \left(1 + \|I - \phi_*\|_r \right),$$

$$\left\| \phi_s^{-1} \right\|_r \leq C_r \left(1 + \|\phi_s\|_r \right)$$

and

$$\left\| (\phi_s)_* Z \right\|_r \leq C_3 \left(\|\phi_s\|_1 \left\| \phi_s^{-1} \right\|_r + \left\| \phi_s^{-1} \right\|_0 \|\phi_s\|_r \right) \|Z\|_r^{\psi_s(K)}$$

so

$$\|Y(Z)\|_r \leq C_4 \left(1 + \|Z\|_{r+n} \right).$$

Let K be a compact neighborhood of the origin $0 \in R$.

Theorem 7. *Let $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ be the dilatation vector field, and $Y_o = X_o + Z$ be an admissible perturbed vector field of X_o defined on K and with the perturbation Z having small support. Let F be the affine space of vector fields of the form $X_o + Z$. Let G be the group of diffeomorphisms of K which are close to the identity and have small support. Then G acts transitively on the space F.*

Proof. Since F is a tame Fréchet space and G is a Lie Fréchet group, by the Nash-Moser-Hamilton Theorem, we deduce that the Lie Fréchet group G acts transitively on F, that means that for every vector fields X_1 and X_2 in the space F there is a diffeomorphism $f \in G$ so that $f_* X_1 = X_2$.

In particular there is $f \in G$ such that $f_* X_o = X_o + Z$. $\square$

4. INVERTIBILITY OF THE MAP $X \to \exp X$

4.1. $X-$derivative of the exponential map. Let χ_K be the Lie algebra of smooth vectors fields on a compact neighborhood of the origin $0 \in R^n$. Each vector field X in χ_K induces a global one parameter group $t \to \exp(tX)$, for $t \in R$. Thus we have the map

$$\theta : \chi_K \times R^n \times R \to R^n; (X, x, t) \to \exp(tX)(x).$$

Since X is smooth, the map ϕ is also smooth in (x, t). The X-derivative at X in the direction of Y has been computed in [5] as

$$D \exp(X)Y = \int_0^1 (\exp sX)_* Y ds \circ \exp X. \tag{11}$$

Let $\phi = \exp(X)$, we have the mean adjoint operator A on χ_K

$$AY = \int_0^1 (\phi_s)_* Y ds$$

and (11) writes as

$$D\phi Y = AY o\phi$$

so

$$AY = D\phi Y o\phi^{-1}.$$

4.2. Right inverse to the exponential map. Let as in the previous $j_0^1 E$ be the Fréchet space of smooth vector fields with components in the Schwartz space F and which are 1-time flat at the origin, $j_0^1 E$ is a tame Fréchet space. Denote by G be the group of diffeomorphisms f on R^n such that $f - id \in j_0^1 E$. The group G modeled on the tame space $j_0^1 E$ is a tame Lie Fréchet group. Let $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ be an admissible infinitesimal dilatation. We are going to apply the Hamilton-Nash-Moser theorem to the mapping

$$P : Z \in j_0^1 E \to \exp(X_o + Z) \exp(-X_o) \in G.$$

P is a smooth tame map from the Fréchet space $j_0^1 E$ into the tame manifold G. With Z fixed, set $\phi = \exp(X_o)$, $\psi = \exp(X_o + Z)$ and $g = \psi o\phi^{-1}$.

The derivative of P at Z is

$$D_Z P.Y = D\psi \int_0^1 (\psi_t)_* Y dto(\psi o\phi^{-1}) \in T_g G$$

where $\psi_t = \exp t(X_o + Z)$ and $T_g G$ the tangent space to G at g. Then $(Z, Y) \to D_Z P.Y$ is a smooth tame family of maps from $j_0^1 E \times j_0^1 E$ into

T_gG. Now, we look for a smooth family of inverses, that is solution $Y_{Z,W}$ of the equation

$$D_Z P Y = W \, og$$

with Z and W from $j_0^1 E$.

We have([5])

Proposition 1. *If $I - \psi_*$ or $I - \psi^*$ is invertible on $j_0^1 E$ then the solution is given by*

$$Y = ad(X_o + Z)\left(I - \psi_*\right)^{-1} W$$

or

$$Y = ad(X_o + Z)\left(I - \psi^*\right)^{-1} W$$

respectively.

We are going to show that the series

$$(I - \psi_*)^{-1} Y(x) = \sum_{m \geq 0} (\psi_m)_* Y(x) = \sum_{m \geq 0} (\psi_{-m})^* Y(x)$$

converges.

Lemma 5. *Under the conditions of Lemma2, the series*

$$(I - \psi_*)^{-1} Y(x) = \sum_{m \geq 0} (\psi_m)_* Y(x)$$

(resp. the series $(I - \psi^*)^{-1} Y(x) = \sum_{m \geq 1} (\psi_m)^* Y(x)$) converges on the Fréchet space $j_0^1 E$.

Proof. Putting $Z(t) = \phi_t^* Y$ and using the estimates 8, 9 and 10, we get for any $k \geq 1$,

$$\left\| D^k f_t^* Z(t) \right\| = \left\| D^k (D f_{-t} Z) o f_t \right\|$$
$$\leq C_1 (\|f_t\|_1^{k-1} \|f_t\|_1 \|D f_{-t} Z\|_k + \|D f_{-t} Z\|_1 \|f_t\|_k)$$
$$\leq C_1 \|f_t\|_1^{k-1} \left\{ C_2 \|f_t\|_1 \left(\|f_{-t}\|_{k+1} \|Z\|_0 + \|f_{-t}\|_2 \|Z\|_k \right. \right.$$
$$\left. + C_3 \|f_t\|_k \|f_{-t}\|_2 \|Z\|_0 + \|f_{-t}\|_1 \|Z\|_1 \right\}$$
$$\leq C \|f_t\|_1^{k-1} \left(\|f_t\|_1 \|f_{-t}\|_{k+1} + \|f_t\|_k \|f_{-t}\|_2 \right) \|Z\|_k .$$

By Lemma 4, all $\|f_t\|_j$ are independent of t and, for any $Y \in j_0^1 E$, one obtains the estimate

$$\|(\psi_t)_* Y\|_k = \|f_{t*}(\phi_{t*} Y)\|_k \leq C \|Z(t)\|_k \leq C \|\phi_{t*} Y\|_k .$$

Since, by Lemma 2, the series $\sum_{m \geq 1} \|(\phi_t)_* Y\|_k$ tamely converge, so do the one for ψ_t. □

Lemma 6. *The map $(Z, Y) \in j_0^1 E \times j_0^1 E \to (I - (\psi_t)_*)^{-1} Y \in j_0^1 E$ is smooth and gives a tame family of inverses.*

Proof. Since, by Lemma 5, $(I - \psi_t)_*)$ is tamely invertible on $j_0^1 E$ and the map $Z \to \exp(X_o + Z)$ is smooth and tame so do the map given in Lemma 6. $\square$

We have checked the hypothesis of the Nash-Moser theorem, now we get the main result of this section

Theorem 8. *Let $X_o = \sum_{i=1}^n \alpha_i x_i \frac{\partial}{\partial x_i}$ be the dilatation vector field and $\digamma$ the affine space of admissible vector fields $Y_o = X_o + Z$ of X_o defined on R^n and with Z of small support. Then for every diffeomorphism f on R^n which is 1-time flat, close to the identity and of small support, there exists a vector field Z such that $X_o + Z \in \digamma$, fulfilling*

$$f = \exp(X_o + Z) o \exp(-X_o).$$

In [1], using the Steinberg linearization theorem, the first author obtained the following result concerning germs at the origin 0 similar to Theorem 8.

Theorem 9. *Let f be a germ of diffeomorphism at the origin 0 of R^n which is 1-time flat at 0 and A be a germ of a linear vector field given in coordinates by $Ax = (\alpha_1 x_1, ..., \alpha_n x_n)$ where α_i satisfy the additive Steinberg condition $\alpha_i \neq \sum_{i=1}^n m_i \alpha_i$ for all nonnegative integers m_i satisfying $2 \leq m_1 + ... + m_n$. Then there exists a germ of vector field X on R^n which is 1-time flat at the origin 0 such that $f = \exp(-A) o \exp(A + Z)$.*

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