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UNIVERSAL SEMIGROUPS

Abstract. In this paper we introduce the notion of a universal semigroup and its dual, the universal cosemigroup. We show that the class of universal semigroups include the class of monoids and is included in the class of semigroups with a product that is an epimorphism. Both inclusions are proper. Semigroups in the category of Banach spaces are Banach algebras and we show that all Banach algebras with an approximate unit are universal and construct a finite dimensional Banach algebra that has no unit but is universal. The property of being universal is thus a generalized unit property.

Contents

1. Introduction

The notions of monoids and semigroups are well known and appear naturally in many contexts. In this paper we introduce a new algebraic structure, *the universal semigroup*. Its definition is inspired by the categorical generalization of monoids and semigroups [1]. In a monoidal category a semigroup is a pair $\langle A, \mu\rangle$ where A is an object in the category and $\mu : A \otimes A \to A$ is a morphism such that the following diagram commute

$$\begin{tikzcd}
A\otimes(A\otimes A) \arrow[r, "1_A\otimes\mu_A"] \arrow[d, "\alpha_{A,A,A}"'] & A\otimes A \arrow[r, "\mu_A"] & A \\
(A\otimes A)\otimes A \arrow[ur, "\mu_A\otimes 1_A"'] & &
\end{tikzcd}$$

where α is the associativity constraint in the category. This definition does not only include the usual algebraic notion of semigroups and associative algebras, but also many other algebraic structures. What algebraic structure it represents depends on the choice of monoidal category. Let S_A be the diagram we get by removing the right part of the previous diagram. Thus S_A is the diagram

$$\begin{tikzcd}
A\otimes(A\otimes A) \arrow[r, "1_A\otimes\mu_A"] \arrow[d, "\alpha_{A,A,A}"'] & A\otimes A \\
(A\otimes A)\otimes A \arrow[ur, "\mu_A\otimes 1_A"'] &
\end{tikzcd}$$

Observe that from a categorical point of view $\langle A,\mu\rangle$ is a semigroup if and only if it is a cocone[1] on the diagram S_A. In general $\langle A,\mu\rangle$ is not a universal cocone on S_A. Recall that $\langle B,g\rangle$ is a universal cocone on D_A if it is a cocone and if for any cocone $\langle C,f\rangle$, there exists a unique map $\varphi : B\longrightarrow C$ such that $\varphi\circ g=f$

$$\begin{tikzcd}
A\otimes(A\otimes A) \arrow[r, "1_A\otimes\mu_A"] \arrow[d, "\alpha_{A,A,A}"'] & A\otimes A \arrow[r, "g"] \arrow[dr, "\forall f"] & B \arrow[d, "\exists!\phi"] \\
(A\otimes A)\otimes A \arrow[ur, "\mu_A\otimes 1_A"'] & & C
\end{tikzcd}$$

The universal cocone could be though of as the "smallest" cocone on the diagram, the one that gives the best "commutative fit" to the diagram. In this paper we define $\langle A,\mu\rangle$ to be a universal semigroup if it is a universal cocone on the diagram S_A and show that the class of universal semigroups includes the class of monoids, but that there are universal semigroups that are not monoids. For any monoid the product is an epimorphism. We show that this also holds for any universal semigroup but that there are semigroups that are not universal and where the product is an epimorphism. Thus the class of universal semigroups includes all monoids and is included in the class of semigroups with products that are epimorphisms.

The category of Banach spaces is a monoidal category where the monoidal structure is determined by the projective tensor product of Banach spaces. In this category semigroups are Banach algebras. We show that all Banach algebras with an approximate unit are universal and that there are universal Banach algebras that does not have an approximate unit. All C^*-algebras admit an approximate unit[2] and are thus universal as are L_1 group algebras of locally compact topological groups since they all admit an approximate unit[3].

These results leads to the idea that the universal cocone property for semigroups is a generalized unit condition. Universal semigroups could be expected to share properties with monoids that general semigroups does not.

Note that the property of being a universal semigroup is a typical categorical property in that it depends on which category the semigroup is placed in. One can thus not state that a semigroup is universal without specifying which category we refer to. This may appear a little strange if one is not familiar with the categorical point of view. It is however a fact that even such well known properties as associativity and commutativity of algebraic structures actually depends on the categorical context. It has for instance been shown that both quaternions and Cayley numbers are commutative, associative algebras[6] if placed in the right categories.

2. Universal semigroups

In order to give a categorical definition of universal semigroups or even of semigroups we need the notion of a category with an associative product, or a *semimonoidal category.*

Definition 1. *A semimonoidal category is a triple* $\langle\, \mathcal{C}, \otimes, \alpha\rangle$*, where* $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *is a bifunctor on* $\mathcal{C}$ *and* α *is a natural isomorphism*

$$\alpha : \otimes \circ (1_{\mathcal{C}} \times \otimes) \longrightarrow \otimes \circ (\otimes \times 1_{\mathcal{C}}),$$

such that the following MacLane coherence conditions

$$\alpha_{A\otimes B,C,D} \circ \alpha_{A,B,C\otimes D} = (\alpha_{A,B,C} \otimes 1_D) \circ \alpha_{A,B\otimes C,D} \circ (1_A \otimes \alpha_{B,C,D}),$$

are satisfied for all objects A, B, C, D

Recall that naturality means that the following identities

$$((f \otimes g) \otimes h) \circ \alpha_{A,B,C} = \alpha_{A',B',C'} \circ (f \otimes (g \otimes h)),$$

holds for all choices of arrows $f : A \longrightarrow A'$, $g : B \longrightarrow B'$, $h : C \longrightarrow C'$.

The product $\otimes$ is thus not strictly associative but associative up to isomorphism. A semimonoidal category contains the structures we need to define semigroups[1].

Definition 2. *A semigroup $\langle A, \mu_A \rangle$ in a semimonoidal category is an object A and a morphism $\mu_A : A \otimes A \longrightarrow A$ such that the following diagram commute*

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[r, "1_A \otimes \mu_A"] \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \arrow[r, "\mu_A"] & A \\
(A \otimes A) \otimes A \arrow[ur, "\mu_A \otimes 1_A"'] & &
\end{tikzcd}$$

.

Let S_A be the previous diagram with the right-hand node removed

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[r, "1_A \otimes \mu_A"] \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \\
(A \otimes A) \otimes A \arrow[ur, "\mu_A \otimes 1_A"'] &
\end{tikzcd}$$

.

Definition 3. *A universal semigroup in a semimonoidal category $\langle \mathcal{C}, \otimes, \alpha \rangle$ is a universal cocone $\langle A, \mu_A \rangle$ on the diagram S_A.*

2.1. **Examples from *Sets* and general monoidal categories.** Let us consider a few examples from the category *Sets* with the usual monoidal structure. A monoid $\langle A, \mu, e \rangle$ in *Sets* is a semigroup $\langle A, \mu \rangle$ with a neutral element e such that $\mu(e, x) = \mu(x, e) = x$ for all x in A. Let $\langle C, f \rangle$ be any cocone on S_A. This means that $f \circ (1 \times \mu) = f \circ (\mu \times 1)$ or $f(x, yz) = f(xy, z)$ for all x, y, z in A. Here we write $xy = \mu(x, y)$.The condition for $\langle A, \mu \rangle$ to be a universal cocone on S_A is therefore that the equation

$$\varphi \circ \mu = f,$$

has a unique solution for all f such that $f(x, yz) = f(xy, z)$. Let φ and ψ be two solutions. Then we have $\varphi(x) = \varphi(xe) = f(x, e) = \psi(xe) = \psi(x)$ for all x in A. So there can be at most one solution. Define a map $\varphi : A \longrightarrow C$ by $\varphi(x) = f(x, e)$. Then we have

$$(\varphi \circ \mu)(x, y) = \varphi(xy) = f(xy, e) = f(x, ye) = f(x, y)$$

The map φ is therefore a solution and we have proved the following proposition

Proposition 4. *Let $\langle A, \mu, e\rangle$ be a monoid in Sets. Then $\langle A, \mu\rangle$ is a universal semigroup.*

The class of Universal semigroups in *Sets* therefore includes all monoids. This is a proper inclusion, there are universal semigroups that are not monoids as the following two examples show.

Let $A = \{a, b\}$ be a semigroup with two elements and table of multiplication given by

μ	a	b
a	a	b
b	a	b

The conditions on f are for this semigroup given by $f(a, a) = f(b, a)$, $f(a, b) = f(b, b)$. Let φ_1 and φ_2 be two solutions of $\varphi \circ \mu = f$. Then we must have $\varphi_1(a) = \varphi_1(\mu(a, a)) = f(a, a) = \varphi_2(\mu(a, a)) = \varphi_2(a)$ and $\varphi_1(b) = \varphi_1(\mu(b, b)) = f(b, b) = \varphi_2(\mu(b, b)) = \varphi_2(b)$. So $\varphi_1 = \varphi_2$ and there is at most one solution. Define a map $\varphi : A \longrightarrow B$ by $\varphi(a) = f(a, a), \varphi(b) = f(b, b)$. Then we have

$$\begin{aligned}
\varphi(\mu(a, a)) &= \varphi(a) = f(a, a) \\
\varphi(\mu(a, b)) &= \varphi(b) = f(b, b) = f(a, b) \\
\varphi(\mu(b, a)) &= \varphi(a) = f(a, a) = f(b, a) \\
\varphi(\mu(b, b)) &= \varphi(b) = f(b, b)
\end{aligned}$$

This proves that φ is the unique solution to the equation $\varphi \circ \mu = f$ and as a consequence $\langle A, \mu\rangle$ is a universal semigroup. On the other hand $\langle A, \mu\rangle$ does not have a neutral element and is therefore not a monoid.

As a second example of a universal semigroup in *Sets* that is not a monoid let us consider a set A and the projection on the first factor $\mu : A \times A \to A$. Thus $\mu(x, y) = x$. Then we evidently have

$$[\mu \circ (\mu \times 1)](x, y, z) = \mu(x, z) = x$$

$$[\mu \circ (1 \times \mu)](x, y, z) = \mu(x, y) = x$$

so $\langle A, \mu\rangle$ is a semigroup.The multiplication clearly does not have a unit, so $\langle A, \mu\rangle$ is not a monoid. Let now $\langle C, f\rangle$ be a cocone on S_A. Then $f : A \times A \to C$ and we have the condition $f \circ (1 \times \mu) = f \circ (\mu \times 1)$. This means that $f(x, y) = f(x, z)$ for all x, y, z. If φ is any solution of $\varphi \circ \mu = f$ we must have $\varphi(x) = \varphi(\mu(x, y)) = f(x, y)$, so there can be at most one solution. Let x_0 be any element in A. Define the map $\varphi : A \longrightarrow C$ by $\varphi(x) = f(x, x_0)$. Then we have

$$\varphi(\mu(x, y)) = \varphi(x) = f(x, x_0) = f(x, y)$$

so φ is a solution. Therefore $\langle A, \mu\rangle$ is a universal semigroup that is not a monoid. Projection on the second factor would in a similar way produce a universal semigroup.

We will now prove a few simple results that holds for universal semigroups in any semimonoidal category.

Proposition 5. *Let $\langle A, \mu_A\rangle$ be a universal semigroup. Then the arrow μ_A is epi.*

Proof. Let B be any object in the category and let $f, g : A \longrightarrow B$ be two arrows and assume that $f \circ \mu_A = g \circ \mu_A$. Let $h = f \circ \mu_A$. Then the equation

$$\varphi \circ \mu_A = h$$

has both f and g as solutions. Since $\langle A, \mu_A\rangle$ is a universal semigroup we must have $f = g$ and this proves that μ_A is epi. □

Any semigroup in *Sets* with a nonsurjective product is therefore not a universal semigroup in *Sets*.The semigroup $A = \{a, b, c\}$ with table of multiplication given by

μ	a	b	c
a	b	c	c
b	c	c	c
c	c	c	c

does not have a surjective multiplication and is therefore not universal.

We will next show that monoids are universal semigroups in any semimonoidal category. In order to define the notion of a monoid in a semimonoidal category we need a neutral object for the product $\otimes$. This leads to the well known notion of a monoidal category.

Definition 6. *Let $\mathcal{C}$ be a category and let e be a object in $\mathcal{C}$. Define two functors $L_e, R_e : \mathcal{C} \longrightarrow \mathcal{C}$ on objects and morphisms by*

$$\begin{aligned} L_e(A) &= e \otimes A, \\ L_e(f) &= 1_e \otimes f, \\ R_e(A) &= A \otimes e, \\ R_e(f) &= f \otimes 1_e. \end{aligned}$$

A monoidal category is a 6 tuple $\langle\, \mathcal{C}, \otimes, e, \alpha, \beta, \gamma\rangle$ where $\langle\, \mathcal{C}, \otimes, \alpha\rangle$ is a semimonoidal category and $\beta : L_e \longrightarrow 1_{\mathcal{C}}$ and $\gamma : R_e \longrightarrow 1_{\mathcal{C}}$ are natural *isomorphisms such that the following identities*

$$\begin{aligned} (\gamma_A \otimes 1_B) \circ \alpha_{A,e,B} &= (1_A \otimes \beta_A), \\ \beta_e &= \gamma_e, \end{aligned}$$

holds for all objects A and B.

Naturality of β, γ means that the following identities

$$\begin{aligned} f \circ \beta_a &= \beta_B \circ (1_e \otimes f), \\ f \circ \gamma_A &= \gamma_B \circ (f \otimes 1_e), \end{aligned}$$

holds for all morphism $f : A \longrightarrow B$.

These are the MacLane coherence conditions for a monoidal category. They ensure that all diagrams generated using the functors $\otimes, L_e, R_e$ and the natural isomorphisms α, β, γ will commute.

Definition 7. *A monoid in a monoidal category is a triple $\langle A, \mu_A, u_A \rangle$, where $\langle A, \mu_A \rangle$ is a semigroup in the category and where $u_A : e \longrightarrow A$ is a morphism such that the following unit condition holds*

$$\begin{aligned} \mu_A \circ (1_A \otimes u_A) &= \gamma_A \\ \mu_A \circ (u_A \otimes 1_A) &= \beta_A \end{aligned}$$

We can now prove that monoids are universal semigroups in any monoidal category.

Proposition 8. *Let $\langle A, \mu_A, u_A \rangle$ be a monoid in a monoidal category $\mathcal{C}$. Then $\langle A, \mu_A \rangle$ is a universal semigroup.*

Proof. We want to show that $\langle A, \mu \rangle$ is a universal cocone on the diagram S_A.

$$\begin{tikzcd} A \otimes (A \otimes A) \arrow[r, "1_A \otimes \mu_A"] \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \\ (A \otimes A) \otimes A \arrow[ur, "\mu_A \otimes 1_A"'] & \end{tikzcd}$$

This means that the equation

$$\varphi \circ \mu_A = f$$

should have a unique solution for all $f : A \otimes A \longrightarrow A$ such that $f \circ (1_A \otimes \mu_A) = f \circ (\mu_A \otimes 1_A) \circ \alpha_{A,A,A}$. Define a map $\delta_A : A \to A \otimes A$ by

$\delta_A = (u_A \otimes 1_A) \circ \beta_A^{-1}$. Then we have

$$\mu_A \circ \delta_A = \mu_A \circ (u_A \otimes 1_A) \circ \beta_A^{-1} = \beta_A \circ \beta_A^{-1} = 1_A$$

This identity show that the equation $\varphi \circ \mu = f$ can have at most one solution and this solution must be $\varphi = f \circ \delta_A$. The proof is complete if we can show that this φ really is a solution. From the naturality of α, β and the MacLane coherence conditions we have the following identities

$$\begin{aligned}
(u_A \otimes (1_A \otimes 1_A)) \circ \alpha_{e,A,A}^{-1} &= \alpha_{A,A,A}^{-1} \circ ((u_A \otimes 1_A) \otimes 1_A), \\
\beta_A^{-1} \circ \mu_A &= (1_e \otimes \mu_A) \circ \beta_{A\otimes A}^{-1}, \\
\beta_{A\otimes A}^{-1} &= \alpha_{e,A,A}^{-1} \circ (\beta_A^{-1} \otimes 1_A).
\end{aligned}$$

But then we have

$$\begin{aligned}
\varphi \circ \mu_A &= f \circ \delta_A \circ \mu_A \\
&= f \circ (u_A \otimes 1_A) \circ \beta_A^{-1} \circ \mu_A \\
&= f \circ (u_A \otimes 1_A) \circ (1_e \otimes \mu_A) \circ \beta_{A\otimes A}^{-1} \\
&= f \circ (u_A \otimes \mu_A) \circ \beta_{A\otimes A}^{-1} \\
&= f \circ (1_A \otimes \mu_A) \circ (u_A \otimes (1_A \otimes 1_A)) \circ \beta_{A\otimes A}^{-1} \\
&= f \circ (1_A \otimes \mu_A) \circ (u_A \otimes (1_A \otimes 1_A)) \circ \alpha_{e,A,A}^{-1} \circ (\beta_A^{-1} \otimes 1_A) \\
&= f \circ (1_A \otimes \mu_A) \circ \alpha_{A,A,A}^{-1} \circ ((u_A \otimes 1_A) \otimes 1_A) \circ (\beta_A^{-1} \otimes 1_A) \\
&= f \circ (\mu_A \otimes 1_A) \circ ((u_A \otimes 1_A) \otimes 1_A) \circ (\beta_A^{-1} \otimes 1_A) \\
&= f \circ (\mu_A \otimes 1_A) \circ (\delta_A \otimes 1_A) \\
&= f,
\end{aligned}$$

so $\varphi = f \circ \delta_A$ is a solution □

Proposition 5 and 8 show that in any monoidal category the class of universal semigroups includes all monoids and is included in the class of semigroups where the product is an epimorphism. The following proposition show that one can construct simple universal semigroups in most monoidal categories.

Proposition 9. *Let A be a object and assume that there is an arrow $\epsilon_A : A \longrightarrow e$ from A to the neutral object e in the category. Define an arrow $\mu_A : A \otimes A \longrightarrow A$ by*

$$\mu_A = \beta_A \circ (\epsilon_A \otimes 1_A)$$

Then $\langle A, \mu_A \rangle$ is a semigroup. It is a universal semigroup if μ_A has a right inverse

Proof. It is easy to show, using the monoidal structure and naturality, that the following set of identities holds

$$
\begin{aligned}
(\beta_A \otimes 1_A) \circ ((1_e \otimes \epsilon_A) \otimes 1_A) &= (\epsilon_A \otimes 1_A) \circ (\beta_A \otimes 1_A), \\
((\epsilon_A \otimes \epsilon_A) \otimes 1_A) \circ \alpha_{A,A,A} &= \alpha_{e,e,A} \circ (\epsilon_A \otimes (\epsilon_A \otimes 1_A)), \\
(\beta_A \otimes 1_A) \circ \alpha_{e,e,A} &= (1_e \otimes \beta_A).
\end{aligned}
$$

Using these identities we have

$$
\begin{aligned}
& \mu^l_A \circ (\mu^l_A \otimes 1_A) \circ \alpha_{A,A,A} \\
= \; & \beta_A \circ (\epsilon_A \otimes 1_A) \circ (\beta_A \circ (\epsilon_A \otimes 1_A) \otimes 1_A) \circ \alpha_{A,A,A} \\
= \; & \beta_A \circ (\epsilon_A \otimes 1_A) \circ (\beta_A \otimes 1_A) \circ ((\epsilon_A \otimes 1_A) \otimes 1_A) \circ \alpha_{A,A,A} \\
= \; & \alpha_A \circ (\beta_e \otimes 1_A) \circ ((1_e \otimes \epsilon_A) \otimes 1_A) \circ ((\epsilon_A \otimes 1_A) \otimes 1_A) \circ \alpha_{A,A,A} \\
= \; & \beta_A \circ (\beta_e \otimes 1_A) \circ ((\epsilon_A \otimes \epsilon_A) \otimes 1_A) \circ \alpha_{A,A,A} \\
= \; & \beta_A \circ (\beta_e \otimes 1_A) \circ \alpha_{e,e,A} \circ (\epsilon_A \otimes (\epsilon_A \otimes 1_A)) \\
= \; & \beta_A \circ (1_e \otimes \beta_A) \circ (\epsilon_A \otimes (\epsilon_A \otimes 1_A)) \\
= \; & \beta_A \circ (\epsilon_A \otimes \beta_A \circ (\epsilon_A \otimes 1_A)) \\
= \; & \beta_A \circ (\epsilon_A \otimes 1_A) \circ (1_A \otimes \beta_A \circ (\epsilon_A \otimes 1_A)) \\
= \; & \mu^l_A \circ (1_A \otimes \mu^l_A),
\end{aligned}
$$

and this proves that $\langle A, \mu_A \rangle$ is a semigroup. In order to prove that it is universal when μ_A has a right inverse we have to prove that the equation $\varphi \circ \mu_A = f$ has a unique solution $\varphi : A \longrightarrow B$ for all $f : A \otimes A \longrightarrow B$ such that $f \circ (1_A \otimes \mu_A) = f \circ (\mu_A \otimes 1_A) \circ \alpha_{A,A,A}$. Let the right inverse for μ_A be δ_A. We thus have the identity $\mu_A \circ \delta_A = 1_A$. But then the only possible solution of the equation $\varphi \circ \mu_A = f$ is $\varphi = f \circ \delta_A$. We must now show that this in fact is a solution. The following identities follow from unit coherence and naturality

$$
\begin{aligned}
\delta_A \circ \beta_A &= \beta_{A \otimes A} \circ (1_e \otimes \delta_A), \\
\alpha_{e,A,A} \circ (\epsilon_A \otimes (1_A \otimes 1_A)) &= ((\epsilon_A \otimes 1_A) \otimes 1_A) \circ \alpha_{A,A,A}, \\
(\beta_A \otimes 1_A) \circ \alpha_{e,A,A} &= \beta_{A \otimes A}.
\end{aligned}
$$

But then we have

$$\begin{aligned}
& f\circ\delta^l_A\circ\mu^l_A\\
= \;& f\circ\delta^l_A\circ\beta_A\circ(\epsilon_A\otimes 1_A)\\
= \;& f\circ\beta_{A\otimes A}\circ(1_e\otimes\delta^l_A)\circ(\epsilon_A\otimes 1_A)\\
= \;& f\circ\beta_{A\otimes A}\circ(\epsilon_A\otimes\delta^l_A)\\
= \;& f\circ\beta_{A\otimes A}\circ(\epsilon_A\otimes(1_A\otimes 1_A))\circ(1_A\otimes\delta^l_A)\\
= \;& f\circ(\beta_A\otimes 1_A)\circ\alpha_{e,A,A}\circ(\epsilon_A\otimes(1_A\otimes 1_A))\circ(1_A\otimes\delta^l_A)\\
= \;& f\circ(\beta_A\otimes 1_A)\circ((\epsilon_A\otimes 1_A)\otimes 1_A)\circ\alpha_{A,A,A}\circ(1_A\otimes\delta^l_A)\\
= \;& f\circ(\mu^l_A\otimes 1_A)\circ\alpha_{A,A,A}\circ(1_A\otimes\delta^l_A)\\
= \;& f\circ(1_A\otimes\mu^l_A)\circ(1_A\otimes\delta^l_A)\\
= \;& f.
\end{aligned}$$

□

A special case of the previous proposition is

Corollary 10. *Assume that there exists a map $u_A : e \longrightarrow A$ such that $\epsilon_A\circ u_A = 1_e$. Then $\langle A,\mu_A\rangle$ is a universal semigroup.*

Proof. Define an arrows $\delta_A : A \longrightarrow A\otimes A$ by

$$\delta_A = (u_A\otimes 1_A)\circ\beta_A^{-1}$$

Then we have

$$\begin{aligned}
\mu_A\circ\delta_A &= \beta_A\circ(\epsilon_A\otimes 1_A)\circ(u_A\otimes 1_A)\circ\beta_A\\
&= \beta_A\circ(\epsilon_A\circ u_A\otimes 1_A)\circ\beta_A^{-1}\\
&= \beta_A\circ\beta_A^{-1} = 1_A.
\end{aligned}$$

But this show that δ_A is the right inverse of μ_A and the result follows from the previous proposition. □

In a similar way we find that $\langle A,\mu_A\rangle$ is a universal semigroup if we define

$$\mu_A = \beta_A\circ(1_A\otimes\epsilon_A)$$

and assume that μ_A has a right inverse. Note that in general the universal semigroups described in these propositions are not monoids.

Let us consider a few examples of the previous construction. Let us first consider the case of *Sets* with the cartesian product as monoidal structure and neutral object given by the set $T = \{*\}$. Since T is the terminal object in *Sets* there exists a unique map $\epsilon_A : A \longrightarrow T$ from any set A to the terminal T. This map is obviously defined by $\epsilon_A(x) = *$ for all $x \in A$. In $Sets$ the natural isomorphism $\beta_A : T \times A \longrightarrow A$ is given by $\beta_A(*, x) = x$. Let x_0 be any element in A and define a map $u_A : T \longrightarrow A$ by $u_A(*) = x_0$. Then $\epsilon_A \circ u_A = 1_e$ and we can conclude that $\langle A, \mu_A \rangle$ is a universal semigroups. The product is explicitly given by

$$\mu_A(x, y) = [\beta_A \circ (\epsilon_A \times 1_A)](x, y) = \beta_A(*, y) = y$$

so μ_A is the projection on the second factor. We have previously shown directly that these maps gives universal semigroups.

Let $Vect_{\mathbb{R}}$be the category of real vector spaces with linear maps as arrows.In this category tensor product of vector spaces $\otimes = \otimes_{\mathbb{R}}$ is a monoidal structure. The neutral object is $\mathbb{R}$ and the arrows β_A and its inverse $\beta_A^{-1}$are given by

$$\begin{aligned} \beta_A(r \otimes a) &= ra \\ \beta_A^{-1}(a) &= 1 \otimes a \end{aligned}$$

where r is any real number, a a element in A and 1 is the unit in $\mathbb{R}$. Note that a semigroup in this category is a real associative algebra.

Let A be a vector space with a positive definite inner product $I_A : A \otimes A \longrightarrow \mathbb{R}$ and let $a \in A$. Define a map $\epsilon_A : A \longrightarrow \mathbb{R}$ by

$$\epsilon_A(b) = I_A(a \otimes b)$$

This is clearly a linear map. Define now the map $\mu_A : A \otimes A \longrightarrow A$ by $\mu_A = \beta_A \circ (\epsilon_A \otimes 1_A)$. Explicitly we have

$$\mu_A(x \otimes y) = \beta_A(I_A(a \otimes x) \otimes y) = I_A(a \otimes x)y$$

Then the general theory show that $\langle A, \mu_A \rangle$ is a semigroup in $Vect_{\mathbb{R}}$.. Now define a map $u_A : \mathbb{R} \longrightarrow A$ by

$$u_A(r) = \frac{ra}{I_A(a, a)}$$

Then u_A is a linear map and

$$(\epsilon_A \circ u_A)(r) = \epsilon_A(\frac{ra}{I_A(a, a)}) = \frac{r}{I_A(a, a)}\epsilon_A(a) = r$$

so we have $\epsilon_A \circ u_A = 1$ and the semigroup $\langle A, \mu_A \rangle$ is in fact a universal semigroup. This example be generalized in the following way. We consider a subcategory of topological vector spaces over a field $\mathcal{F}$ that is closed with respect to tensor product. Such subcategories certainly exists. Let α be a continuous linear functional $\alpha : A \longrightarrow \mathbb{R}$. The morphism $\epsilon_A : A \longrightarrow \mathcal{F}$ is now defined by

$$\epsilon_A(a) = \alpha(a)$$

and the product in the semigroup $\langle A, \mu_A \rangle$ is

$$\mu_A(a \otimes a') = \alpha(a)b$$

The resulting semigroup is a universal semigroup because the continuous linear map $u_A : \mathcal{F} \longrightarrow A$ defined by $u_A(r) = r(\alpha(b))^{-1}b$,where $b \in A$ and $\alpha(b) \neq 0$, satisfy $\epsilon_A \circ u_A = 1_A$.

We have seen that in general the class of universal semigroups is strictly larger than the class of monoids. We also know that the class of universal semigroups is included in the class of semigroups with a product rule that is an epimorphisms. In general this last inclusion is also proper as the following example show.

Let A be a real vector space of dimension three with basis $\{i, j, k\}$. Define a product, μ on A by the following multiplication table

μ	i	j	k
i	i	0	k
j	j	0	0
k	0	0	0

By direct calculation one can show that $\langle A, \mu \rangle$ is a semigroup in $Vect_{\mathbb{R}}$,or in other words, a real associative algebra. The product is clearly an epimorphism. The chosen basis for A induce in the usual way a basis for $A \otimes A$. Let now V be a one dimensional vector space and let $f : A \otimes A \to V$ be a linear map that is zero on all basis vectors except on the vector $j \otimes k$ where it has a nonzero value. It is a straight forward calculation to show that $\langle V, f \rangle$ is a cocone on the diagram S_A. If φ is a solution of the equation $\varphi \circ \mu = f$ we must have

$$\begin{aligned} \varphi(i) &= \varphi(\mu(i \otimes i)) = f(i \otimes i) = 0, \\ \varphi(j) &= \varphi(\mu(j \otimes i)) = f(j \otimes i) = 0, \\ \varphi(k) &= \varphi(\mu(i \otimes k)) = f(i \otimes k) = 0, \end{aligned}$$

so $\varphi = 0$. But $\varphi = 0$ is not a solution of the equation $\varphi \circ \mu = f$ because $f \neq 0$. The equation $\varphi \circ \mu = f$ therefore have no solution for the cocone $\langle V, f \rangle$ and then by definition $\langle A, \mu \rangle$ is not a universal algebra.

2.2. **Examples from the category of Banach spaces.** Let $\mathcal{B}$ be the category where objects are Banach spaces and where morphisms are bounded linear maps. The Banach spaces are vector spaces over a field F where F is $\mathbb{R}$ or $\mathbb{C}$. We introduce a monoidal structure in this category by defining $X \otimes Y$ to be the projective tensor product[4] of Banach spaces. The unit object for this product is the Banach space F and the natural isomorphisms α,β and γ are the standard ones. The standard interpretation of the projective tensor product in terms of properties of bilinear maps[5] shows that $\langle A, \mu \rangle$ is a semigroups in $\mathcal{B}$ iff it is a Banach algebra. Recall that an approximate unit[3] in a Banach algebra A is a net $\{u_\lambda\}_{\lambda \in \Lambda}$ such that $\|u_\lambda\| \leq 1$ for all $\lambda \in \Lambda$ and such that for all elements $a \in A$ we have

$$\begin{aligned} \lim_\lambda u_\lambda a &= a, \\ \lim_\lambda a u_\lambda &= a. \end{aligned}$$

Recall that a morphism $f : X \to Y$ in any category is a epimorphism iff for all morphisms $g, h : Y \to Z$ with $f \circ g = f \circ h$ we have $g = h$. In the category of *Sets* epimorphisms are exactly the surjective maps. In the category of Banach spaces it is easy to show that any morphism with a dense image is a epimorphism. Using this observation we first prove the following result

Proposition 11. *Let $\langle A, \mu \rangle$ be a Banach algebra with an approximate unit. Then $\mu : A \otimes A \to A$ is a epimorphism.*

Proof. Let $m = \mu(A \otimes A) \subset A$ be the image of μ. We must show that the image is dense in A. Let $a \in A$ and let $\{u_\lambda\}$ be an approximate unit in A. Define a net $a_\lambda = u_\lambda a$ in A. Then we have by the defining property of an approximate unit that

$$\lim_\lambda a_\lambda = a.$$

But this means that the closure of m is A and thus that m is dense in A. □

Therefore Banach algebras with an approximate unit is included in the class of algebras where the product is an epimorphism. We now use this result to prove our main result in this section.

Theorem 12. *Let $\langle A, \mu \rangle$ be a Banach algebra with an approximate unit. Then $\langle A, \mu \rangle$ is a universal semigroup in the category of Banach spaces.*

Proof. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be the approximate unit. For each $\lambda \in \Lambda$ define a linear map $\delta_\lambda : A \to A \otimes A$ by

$$\delta_\lambda(a) = u_\lambda \otimes a.$$

Then since the projective norm is a cross norm we have

$$\|\delta_\lambda(a)\| = \|u_\lambda \otimes a\| = \|u_\lambda\| \|a\| \leq \|a\|,$$

and therefore each δ_λ is bounded with $\|\delta_\lambda\| \leq 1$. Let now B be any other Banach space and let $f : A \otimes A \to B$ be a bounded linear map that satisfy

$$f \circ (\mu \otimes 1_A) = f \circ (1_A \otimes \mu).$$

Note that because of linearity and continuity of f this condition holds iff

$$f(ab \otimes c) = f(a \otimes bc),$$

holds for all elements a, b and c in A.

We must show that the equation $\varphi \circ \mu = f$ has one and only one solution. We know from proposition 11 that μ is a epimorphism so there can be at most one solution. For each $\lambda \in \Lambda$ define $\varphi_\lambda : A \to B$ by

$$\varphi_\lambda = f \circ \delta_\lambda.$$

This map is continuous and

$$\|\varphi_\lambda\| \leq \|f\|\|\delta_\lambda\| \leq \|f\|,$$

so the family of maps $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded in λ by $\|f\|$. Let

$$m = \{\sum_n a_n b_n \mid a_n, b_n \in A\} \subset \mu(A \otimes A).$$

Then for $a \in m$ we have

$$\varphi_\lambda(a) = \varphi_\lambda(\sum_n a_n b_n) = f(u_\lambda \otimes \sum_n a_n b_n) = f(\sum_n u_\lambda a_n \otimes b_n),$$

and therefore by continuity

$$\lim_\lambda \varphi_\lambda(a) = \lim_\lambda f(\sum_n u_\lambda a_n \otimes b_n) = f(\sum_n \lim_\lambda (u_\lambda a_n) \otimes b_n) = f(\sum_n a_n \otimes b_n).$$

Thus $\lim_\lambda \varphi_\lambda(a)$ exists for all $a \in m$. We know that

$$R = \{\sum_n a_n \otimes b_n \mid a_n, b_n \in A\},$$

is dense in $A \otimes A$. Therefore $m = \mu(R)$ is dense in A and so m is dense in A. Let now $a \in A$. Then there exists a sequence $\{a_n\}$ in m with $a_n \to a$ and if $\lambda, \beta \in \Lambda$ we have

$$\begin{aligned}
\|\varphi_\lambda(a) - \varphi_\beta(a)\| &= \|\varphi_\lambda(a) - \varphi_\lambda(a_n) + \varphi_\lambda(a_n) - \varphi_\beta(a) - \varphi_\beta(a_n) + \varphi_\beta(a_n)\| \\
&\leq \|\varphi_\lambda(a - a_n)\| + \|\varphi_\lambda(a_n) - \varphi_\beta(a_n)\| + \|\varphi_\beta(a) - \varphi_\beta(a_n)\| \\
&\leq (\|\varphi_\lambda\| + \|\varphi_\beta\|)\|a - a_n\| + \|\varphi_\lambda(a_n) - \varphi_\beta(a_n)\| \\
&\leq 2\|f\|\|a - a_n\| + \|\varphi_\lambda(a_n) - \varphi_\beta(a_n)\|.
\end{aligned}$$

Let now n be fixed and so large that $2\|f\|\|a-a_n\| < \frac{1}{2}\epsilon$. Since $a_n \in m$ we have

$$a_n = \sum_{j=1}^{p} a_j^n b_j^n,$$

and therefore

$$\begin{aligned}\varphi_\lambda(a_n) - \varphi_\beta(a_n) &= f(u_\lambda \otimes \sum_{j=1}^{p} a_j^n b_j^n) - f(u_\beta \otimes \sum_{j=1}^{p} a_j^n b_j^n) \\ &= f(\sum_{j=1}^{p}(u_\lambda a_j^n - u_\beta a_j^n) \otimes b_j^n),\end{aligned}$$

and so

$$\|\varphi_\lambda(a_n) - \varphi_\beta(a_n)\| \leq \|f\| M_n \sum_{j=1}^{p} \|u_\lambda a_j^n - u_\beta a_j^n\|,$$

where we have assumed without loss of generality that $\|b_j^n\| \leq M_n$ for $j = 1...p$. But u_λ is an approximate unit and therefore $u_\lambda a_j^n$ converges $\forall j$ and is thus Cauchy since A is a Banach space. But then there exists $\lambda_j \in \Lambda$ $j = 1..p$ such that

$$\|u_\lambda a_j^n - u_\beta a_j^n\| < \frac{1}{2p\epsilon\|f\|M_n}.$$

Since Λ is a directed set there exists a element λ_0 such that all these inequalities holds when $\lambda, \beta > \lambda_0$. But then for such λ and β we have using the previously derived inequalities that

$$\|\varphi_\lambda(a) - \varphi_\beta(a)\| < \epsilon,$$

and this means that the net $\{\varphi_\lambda(a)\}_{\lambda\in\Lambda}$ is Cauchy and therefore converges. We can now define a map $\varphi : A \to B$ by

$$\varphi(a) = \lim_\lambda \varphi_\lambda(a).$$

This map is linear and bounded because

$$\|\varphi(a)\| = \lim_\lambda \|\varphi_\lambda(a)\| \leq \|f\|\|a\|.$$

But for any element $\xi = \sum_j a_j \otimes b_j$ in the dense set $R \subset A \otimes A$ we have

$$\varphi(\mu(\xi)) = \lim_\lambda \varphi_\lambda(\sum_j a_j b_j) = \lim_\lambda f(\sum_j u_\lambda a_j \otimes b_j) = f(\sum_j a_j \otimes b_j) = f(\xi).$$

Therefore the bounded maps $\varphi \circ \mu$ and f agree on a dense set and we can conclude that

$$\varphi \circ \mu = f,$$

and this shows that the algebra A is universal. □

Universal Banach algebras thus include all Banach algebras with an approximate unit. On the other hand it is not hard to construct Banach algebras without unit that are universal, so the inclusion is proper. Let $\langle A, \mu \rangle$ be a three dimensional algebra with basis $\{i, j, k\}$ and table of multiplication given by

μ	i	j	k
i	i	0	0
j	j	0	0
k	0	0	k

By direct calculation one can easily show that this algebra is universal but does not have a unit (or approximate unit).

3. Universal cosemigroups

To any categorical concept described in terms of diagrams there is a dual concept that we get by reversing all arrows. For the notion of a universal semigroup this procedure leads to the notion of a universal cosemigroup. Let $\langle \mathcal{C}, \otimes, \alpha \rangle$ be a semimonoidal category. A cosemigroup in $\mathcal{C}$ is a pair $\langle A, \delta_A \rangle$ where A is a object in $\mathcal{C}$ and $\delta_A : A \longrightarrow A \otimes A$ is a arrow in $\mathcal{C}$ and where the following diagram commute.

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \arrow[l, "1_A \otimes \delta_A"'] \arrow[dl, "\delta_A \otimes 1_A"] & A \arrow[l, "\delta_A"'] \\
(A \otimes A) \otimes A & &
\end{tikzcd}$$

Let CS_A be the diagram we get by removing the right-hand node.

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \arrow[l, "1_A \otimes \delta_A"'] \arrow[dl, "\delta_A \otimes 1_A"] \\
(A \otimes A) \otimes A &
\end{tikzcd}$$

Evidently $\langle A, \delta_A \rangle$ is a cosemigroup iff $\langle A, \delta_A \rangle$ is a cone on the diagram CS_A. In general a cosemigroup does not give a universal cone.

By definition $\langle B, f \rangle$ is a universal cone on the diagram CS_A if it is a cone and if for all cones $\langle C, g \rangle$ there exists a unique arrow $\varphi : C \longrightarrow B$ in $\mathcal{C}$ such that

$$f \circ \varphi = g$$

Definition 13. *A universal cosemigroup in a semimonoidal category* $\langle\, \mathcal{C}, \otimes, \alpha\rangle$ *is a cosemigroup* $\langle A, \delta_A\rangle$ *such that* $\langle A, \delta_A\rangle$ *is a universal cocone on the diagram* CS_A.

3.1. **Examples from** *Sets* **and general monoidal categories.** We have the following result that show that not all cosemigroups are universal.

Proposition 14. *Let* $\langle A, \delta_A\rangle$ *be a universal cosemigroup. Then* δ_A *is a monomorphism.*

Proof. Let B be any object in the category and let $f, g : B \longrightarrow A$ be two arrows and assume that $\delta_A \circ f = \delta_A \circ g$. Let $h = \delta_A \circ f$. Then the equation

$$\delta_A \circ \varphi = h$$

has both f and g as solutions. Since $\langle A, \delta_A\rangle$ is a universal cosemigroup we must have $f = g$ and this proves that δ_A is a monomorphism. □

We will now illustrate this definition with several examples. Let us first work in the category *Sets*. This is a monoidal category with cartesian product as product bifunctor and with trivial associativity constraint. Let A be any set and define a map $\delta_A : A \longrightarrow A \times A$ by $\delta_A(x) = (x, x)$. This is the diagonal map in *Sets*. We have

$$\begin{aligned} [(1_A \times \delta_A) \circ \delta_A](x) &= (1_A \times \delta_A)(x, x) = (x, x, x) \\ [(\delta_A \times 1_A) \circ \delta_A](x) &= (\delta_A \times 1_A)(x, x) = (x, x, x) \end{aligned}$$

so $\langle A, \delta_A\rangle$ is a cone. Let $\langle B, f\rangle$ be any cone and consider the system

$$\delta_A \circ \varphi = f$$

Since $\langle B, f\rangle$ is a cone we have $(1_A \times \delta_A) \circ f = (\delta_A \times 1_A) \circ f$. We have $f : B \longrightarrow A \times A$ so we can write $f(x) = (f_1(x), f_2(x))$. The cone condition then gives

$$(f_1(x), f_2(x), f_2(x)) = (f_1(x), f_1(x), f_2(x))$$

so we must have $f_1(x) = f_2(x)$for all x in A. Let φ be any solution to $\delta_A \circ \varphi = f$. Then we have $(\varphi(x), \varphi(x)) = (f_1(x), f_2(x))$ or $\varphi = f_1$. So there is at most one solution. Let $\varphi = f_1$ then we find

$$[\delta_A \circ \varphi](x) = \delta_A(f_1(x)) = (f_1(x), f_1(x)) = (f_1(x), f_2(x)) = f(x)$$

so $\varphi = f_1$ is a solution. We have therefore proved

Proposition 15. *Let* $\delta_A : A \longrightarrow A \times A$ *be the diagonal map of sets. Then* $\langle A, \delta_A\rangle$ *is a universal cosemigroup in Sets.*

As our next example let us consider a pointed set. This is a set A with a chosen point $x_0 \in A$. Define a map $\delta_A : A \longrightarrow A\times A$ by $\delta_A(x) = (x_0, x)$. Then we have

$$\begin{aligned}
[(1_A \times \delta_A) \circ \delta_A](x) &= (1_A \times \delta_A)(x_0, x) = (x_0, x_0, x) \\
[(\delta_A \times 1_A) \circ \delta_A](x) &= (\delta_A \times 1_A)(x_0, x) = (x_0, x_0, x)
\end{aligned}$$

so $\langle A, \delta_A\rangle$ is a cone. Let $\langle B, f\rangle$ be any cone. This means that $f : B \longrightarrow A \times A$ and $(1_A \times \delta_A) \circ f = (\delta_A \times 1_A) \circ f$. As above we write $f(x) = (f_1(x), f_2(x))$ and the cone condition becomes

$$(f_1(x), x_0, f_2(x)) = (x_0, f_1(x), f_2(x))$$

so we must have $f_1(x) = x_0$. But then we have

$$[\delta_A \circ \varphi](x) = \delta_A(\varphi(x)) = (x_0, \varphi(x)) = (f_1(x), f_2(x)) = f(x)$$

if we choose $\varphi(x) = f_2(x)$. This is clearly the only solution. We therefore have

Proposition 16. *Let $\langle A, x_0\rangle$ be a pointed set. Define $\delta_A : A \longrightarrow A \times A$ by $\delta_A(x) = (x_0, x)$. Then $\langle A, \delta_A\rangle$ is a universal cosemigroup.*

The map $\delta_A : A \longrightarrow A\times A$ defined by $\delta_A(x) = (x, x_0)$ will similarly give a universal cosemigroup. If $\langle A, \mu_A, e\rangle$ is a monoid we get two universal cosemigroups by choosing $x_0 = e$.

Finite sets offer many examples of universal cosemigroups. Consider the set $A = \{a, b, c\}$. Define $\delta_A : A \longrightarrow A \times A$ by

$$\begin{aligned}
\delta_A(a) &= (a, a) \\
\delta_A(b) &= (b, a) \\
\delta_A(c) &= (c, c)
\end{aligned}$$

Then we have

$$\begin{aligned}
[(\delta_A \times 1_A) \circ \delta_A](a) &= (\delta_A \times 1_A)(a, a) = (a, a, a) \\
[(1_A \times \delta_A) \circ \delta_A](a) &= (1_A \times \delta_A)(a, a) = (a, a, a) \\
[(\delta_A \times 1_A) \circ \delta_A](b) &= (\delta_A \times 1_A)(b, a) = (b, a, a) \\
[(1_A \times \delta_A) \circ \delta_A](b) &= (1_A \times \delta_A)(b, a) = (b, a, a) \\
[(\delta_A \times 1_A) \circ \delta_A](c) &= (\delta_A \times 1_A)(c, c) = (c, c, c) \\
[(1_A \times \delta_A) \circ \delta_A](c) &= (1_A \times \delta_A)(c, c) = (c, c, c)
\end{aligned}$$

so $\langle A, \delta_A\rangle$ is a cone. Next let $\langle B, f\rangle$ be any cone. This implies that $(\delta_A \times 1_A) \circ f = (1_A \times \delta_A) \circ f$. Direct computation show that $(\delta_A \times 1_A)(x, y) = (1_A \times \delta_A)(x, y)$ iff (x, y) is a element in the range of δ_A. We therefore can conclude that $\langle B, f\rangle$ is a cone if the image of f is included in the range of δ_A. Since δ_A is injective there is at most one solution to the equation $\delta_A \circ \varphi = f$. Let b be any element in B. Then there exists a unique element a in A such that $\delta_A(a) = f(b)$. This is true since the range of f is equal to the range of δ_A and δ_A is injective. Define $\varphi(b) = a$, then φ is well defined and clearly a solution to the equation $\delta_A \circ \varphi = f$. This proves that $\langle A, \delta_A\rangle$ is a universal cosemigroup.

We have seen that a necessary condition for a cosemigroup $\langle A, \delta_A\rangle$ to be a universal cosemigroup is that δ_A is a monomorphism. This is not sufficient in general. Let $A = \{a, b, c\}$ and define a map $\delta_A : A \longrightarrow A \times A$ by

$$\begin{aligned} \delta_A(a) &= (a, a), \\ \delta_A(b) &= (b, a), \\ \delta_A(c) &= (a, c). \end{aligned}$$

The map δ_A is a monomorphism and a direct calculation like the one above show that $\langle A, \delta_A\rangle$ is a cosemigroup but that it is not a universal cosemigroup. In general one can prove by dualizing the proof for monoids that comonoid in a monoidal category are universal cosemigroups. Thus universal cosemigroups is a class that contains all comonoids and is included in all cosemigroups with a coproduct that is a monomorphism.

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