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# A NEW SUBCLASS OF QUASI-CONVEX FUNCTIONS WITH RESPECT TO $K$-SYMMETRIC POINTS

(submitted by M. A. Malakhaltsev)

Abstract. In the present paper, we introduce a new subclass $\mathcal{C}_s^{(k)}(\alpha, \beta)$ of quasi-convex functions with respect to $k$-symmetric points. The integral representation and several coefficient inequalities of functions belonging to this class are obtained.

## 1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in C \text{ and } |z| < 1\}.$$

Also let $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathcal{U}$.

We denote by $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Thus, by definition, we have (see, for details, [1] and

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[2]; see also [3] and [4])

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{C} = \left\{ f : f \in \mathcal{A},\ g \in \mathcal{S}^*,\ \text{and } \Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0 \quad (z \in \mathcal{U}) \right\},$$

and

$$\mathcal{C}^* = \left\{ f : f \in \mathcal{A},\ g \in \mathcal{K},\ \text{and } \Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0 \quad (z \in \mathcal{U}) \right\}.$$

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [5]).

Sakaguchi [6] once introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

Following him, many authors discussed this class and its subclasses (see [7]-[14]). Motivated by $\mathcal{S}_s^*$, we can easily obtain the following class $\mathcal{C}_s^*$ of functions convex with respect to symmetric points.

**Definition 1.** Let $\mathcal{C}_s^*$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\Re\left\{\frac{(zf'(z))'}{f'(z) + f'(-z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

In the present paper, we introduce the following class of analytic functions with respect to $k$-symmetric points, and obtain some interesting results.

**Definition 2.** Let $\mathcal{C}_s^{(k)}(\alpha, \beta)$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\left|\frac{(zf'(z))'}{f_k'(z)} - 1\right| < \beta \left|\frac{\alpha(zf'(z))'}{f_k'(z)} + 1\right| \quad (z \in \mathcal{U}), \tag{1.1}$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z) \quad (\varepsilon^k = 1). \tag{1.2}$$

It is easy to know that $\mathcal{C}_s^{(2)}(1,1) = \mathcal{C}_s^*$, so $\mathcal{C}_s^{(k)}(\alpha,\beta)$ is a generalization of $\mathcal{C}_s^*$.

In the present paper, we will discuss the integral representation and coefficient inequalities of functions belonging to the class $\mathcal{C}_s^{(k)}(\alpha,\beta)$.

## 2. Coefficient Estimate of Functions in the Class $\mathcal{C}_s^{(k)}(\alpha,\beta)$

First we give two meaningful conclusions about the class $\mathcal{C}_s^{(k)}(\alpha,\beta)$.

**Theorem 1.** *The function $f(z) \in \mathcal{C}_s^{(k)}(\alpha,\beta)$ if and only if*

$$\frac{(zf'(z))'}{f_k'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z} \quad (z \in \mathcal{U}), \tag{2.1}$$

*where "$\prec$" stands for the subordination.*

*Proof.* Suppose that $f(z) \in \mathcal{C}_s^{(k)}(\alpha,\beta)$, then from (1.1) we have

$$\left|\frac{(zf'(z))'}{f_k'(z)} - 1\right|^2 < \beta^2 \left|\frac{\alpha(zf'(z))'}{f_k'(z)} + 1\right|^2.$$

Expanding it we get

$$(1-\alpha^2\beta^2)\left|\frac{(zf'(z))'}{f_k'(z)}\right|^2 - 2(1+\alpha\beta^2)\Re\left\{\frac{(zf'(z))'}{f_k'(z)}\right\} < \beta^2 - 1.$$

If $\alpha \neq 1$ or $\beta \neq 1$, we have

$$\left|\frac{(zf'(z))'}{f_k'(z)}\right|^2 - \frac{2(1+\alpha\beta^2)}{1-\alpha^2\beta^2}\Re\left\{\frac{(zf'(z))'}{f_k'(z)}\right\} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2 < \frac{\beta^2-1}{1-\alpha^2\beta^2} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2,$$

that is,

$$\left|\frac{(zf'(z))'}{f_k'(z)} - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|^2 < \frac{\beta^2(1+\alpha)^2}{(1-\alpha^2\beta^2)^2},$$

or equivalently,

$$\left|\frac{(zf'(z))'}{f_k'(z)} - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right| < \frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}.$$

This tells us that the value region of $G(z) = (zf'(z))'/f_k'(z)$ is contained in the disk whose center is $(1+\alpha\beta^2)/(1-\alpha^2\beta^2)$ and radius is $[\beta(1+\alpha)]/(1-\alpha^2\beta^2)$. And we know that the function $\omega = p(z) = (1+\beta z)/(1-\alpha\beta z)$ maps the unit disk to the disk:

$$\left|\omega - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right| < \frac{\beta(1+\alpha)}{1-\alpha^2\beta^2}.$$

Notice that $G(0) = p(0)$, $G(\mathcal{U}) \subset p(\mathcal{U})$, and $p(z)$ is univalent in $\mathcal{U}$, we obtain the following conclusion

$$\frac{(zf'(z))'}{f_k'(z)} \prec p(z) = \frac{1+\beta z}{1-\alpha\beta z}.$$

Conversely, let

$$\frac{(zf'(z))'}{f_k'(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

then

$$\frac{(zf'(z))'}{f_k'(z)} = \frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}, \tag{2.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$, and $\omega(0) = 0$, $|\omega(z)| < 1$. By calculation we can easily obtain from (2.2) that

$$\left|\frac{(zf'(z))'}{f_k'(z)} - 1\right| < \beta\left|\frac{\alpha(zf'(z))'}{f_k'(z)} + 1\right|,$$

that is $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$.

If $\alpha = \beta = 1$, inequality (1.1) becomes

$$\left|\frac{(zf'(z))'}{f_k'(z)} - 1\right| < \left|\frac{(zf'(z))'}{f_k'(z)} + 1\right|.$$

It is obvious that

$$\frac{(zf'(z))'}{f_k'(z)} \prec \frac{1+z}{1-z}.$$

Therefore, the proof of Theorem 1 is complete.

**Remark 1.** From Theorem 1 we know that

$$\Re\left\{\frac{(zf'(z))'}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.3}$$

Because of

$$\Re\left\{\frac{1+\beta z}{1-\alpha\beta z}\right\} > 0 \quad (z \in \mathcal{U}).$$

**Theorem 2.** *Let $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, then $f_k(z) \in \mathcal{K} \subset \mathcal{S}$.*

*Proof.* For $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, we can obtain inequality (2.3) from Theorem 1. Substituting $z$ by $\varepsilon^\mu z$ in (2.3) respectively ($\mu = 0, 1, 2, \ldots, k-1$; $\varepsilon^k = 1$), then (2.3) is also true, that is,

$$\Re\left\{\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu z f''(\varepsilon^\mu z)}{f_k'(\varepsilon^\mu z)}\right\} > 0 \quad (z \in \mathcal{U};\ \mu = 0, 1, 2, \ldots, k-1). \tag{2.4}$$

According to the definition of $f_k(z)$ and $\varepsilon^k = 1$, we know $f_k'(\varepsilon^\mu z) = f_k'(z)$. Then inequality (2.4) becomes

$$\Re\left\{\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu z f''(\varepsilon^\mu z)}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.5}$$

Let $\mu = 0, 1, 2, \ldots, k-1$ in (2.5) respectively, and sum them we can get

$$\Re\left\{\frac{\sum_{\mu=0}^{k-1} f'(\varepsilon^\mu z) + z\sum_{\mu=0}^{k-1} \varepsilon^\mu f''(\varepsilon^\mu z)}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

or equivalently,

$$\Re\left\{\frac{f_k'(z) + z f_k''(z)}{f_k'(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

that is $f_k(z) \in \mathcal{K} \subset \mathcal{S}$.

**Remark 2.** From Theorem 2 and inequality (2.3), we know that if $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, then $f(z)$ is a quasi-convex function. So $\mathcal{C}_s^{(k)}(\alpha, \beta)$ is a subclass of the class $\mathcal{C}^*$ of quasi-convex functions.

In order to give the coefficient estimate of functions in the class $\mathcal{C}_s^{(k)}(\alpha, \beta)$, we need the following lemma.

**Lemma 1** ([14])**.** *Let $f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^\infty b_n z^n \in \mathcal{S}$, and satisfy the inequality*

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < \beta\left|\frac{\alpha z f'(z)}{g(z)} + 1\right| \quad (z \in \mathcal{U}),$$

*where $0 \le \alpha \le 1$, $0 < \beta \le 1$. Then for $n \ge 2$, we have*

$$|na_n - b_n|^2 \le 2(1 + \alpha\beta^2)\sum_{k=1}^{n-1} k\,|a_k|\,|b_k| \quad (|a_1| = |b_1| = 1). \tag{2.6}$$

Now we give the following theorem.

**Theorem 3.** *Let $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta) \subset \mathcal{C}^*$, then we have*

(i) *For $l \ge 2$,*

$$\begin{aligned}&\{k(l-1)[(l-1)k+1]\}^2\left|a_{(l-1)k+1}\right|^2\\ &\le 2(1+\alpha\beta^2)\textstyle\sum_{i=1}^{l-1}[(i-1)k+1]^3\left|a_{(i-1)k+1}\right|^2 \qquad (a_1 = 1);\end{aligned} \tag{2.7}$$

(ii) *For* $n \geq 2,\ n \neq (l-1)k+1,$

$$n^4 \left|a_n\right|^2 \leq 2(1+\alpha\beta^2) \sum_{i=1}^{\left[\frac{n-2}{k}+1\right]} [(i-1)k+1]^3 \left|a_{(i-1)k+1}\right|^2 \quad (a_1 = 1), \quad (2.8)$$

*where* $\left[\frac{n-2}{k}+1\right]$ *denotes the biggest integer* $\leq \frac{n-2}{k}+1$.

*Proof.* It is easy to know that the condition (1.1) can write as

$$\left|\frac{z(zf'(z))'}{zf'_k(z)} - 1\right| < \beta \left|\frac{\alpha z(zf'(z))'}{zf'_k(z)} + 1\right| \quad (z \in \mathcal{U}).$$

Now suppose that $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, it is well-know that

$$f(z) \in \mathcal{C}^* \subset \mathcal{S} \Longleftrightarrow zf'(z) \in \mathcal{C} \subset \mathcal{S}.$$

And from theorem 2 we know

$$f_k(z) \in \mathcal{K} \subset \mathcal{S} \Longleftrightarrow zf'_k(z) \in \mathcal{S}^* \subset \mathcal{S}.$$

So $zf'(z)$ and $zf'_k(z)$ satisfy the condition of Lemma 1. At the same time, by the definition of $f_k(z)$ we have

$$\begin{aligned} f_k(z) &= \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z) \\ &= \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} \left[\varepsilon^\nu z + \sum_{n=2}^{\infty} a_n (\varepsilon^\nu z)^n\right] \\ &= z + \sum_{l=2}^{\infty} a_{(l-1)k+1} z^{(l-1)k+1}. \end{aligned}$$

Using Lemma 1, let $n = (l-1)k+1$ in (2.6), we can get (2.7), if $n \neq (l-1)k+1,\ n \geq 2$, from (2.6), we can get (2.8).

## 3. The Integral Representation of Functions in the Class $\mathcal{C}_s^{(k)}(\alpha, \beta)$

In this section, we give the integral representation of functions in the class $\mathcal{C}_s^{(k)}(\alpha, \beta)$.

**Theorem 4.** *Let* $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, *then we have*

$$f_k(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu \zeta} \frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))} dt\right\} d\zeta, \qquad (3.1)$$

*where* $f_k(z)$ *is defined by equality (1.2),* $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0$, $|\omega(z)| < 1$.

*Proof.* Let $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, from Theorem 1 we have

$$\frac{(zf'(z))'}{f_k'(z)} = \frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}, \tag{3.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$. Substituting $z$ by $\varepsilon^\mu z$ in (3.2) respectively ($\mu = 0, 1, 2, \ldots, k-1$; $\varepsilon^k = 1$), we have

$$\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu z f''(\varepsilon^\mu z)}{f_k'(\varepsilon^\mu z)} = \frac{1+\beta\omega(\varepsilon^\mu z)}{1-\alpha\beta\omega(\varepsilon^\mu z)} \quad (\mu = 0, 1, 2, \ldots, k-1). \tag{3.3}$$

It is easy to know that $f_k'(\varepsilon^\mu z) = f_k'(z)$, sum (3.3) we can obtain

$$\frac{(zf_k'(z))'}{f_k'(z)} = \frac{1}{k}\sum_{\mu=0}^{k-1} \frac{1+\beta\omega(\varepsilon^\mu z)}{1-\alpha\beta\omega(\varepsilon^\mu z)}, \tag{3.4}$$

from equality (3.4) we get

$$\frac{(zf_k'(z))'}{zf_k'(z)} - \frac{1}{z} = \frac{1}{k}\sum_{\mu=0}^{k-1} \frac{(1+\alpha)\beta\omega(\varepsilon^\mu z)}{z(1-\alpha\beta\omega(\varepsilon^\mu z))}. \tag{3.5}$$

Integrating equality (3.5), we have

$$\begin{aligned} \log\{f_k'(z)\} &= \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^z \frac{(1+\alpha)\beta\omega(\varepsilon^\mu \zeta)}{\zeta(1-\alpha\beta\omega(\varepsilon^\mu \zeta))} d\zeta \\ &= \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))} dt, \end{aligned}$$

that is,

$$f_k'(z) = \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))} dt\right\}. \tag{3.6}$$

Therefore, integrating equality (3.6) we can obtain equality (3.1).

**Theorem 5.** *Let $f(z) \in \mathcal{C}_s^{(k)}(\alpha, \beta)$, then we have*

$$\begin{aligned} f(z) = \int_0^z \frac{1}{\xi}\int_0^\xi \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta} \frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))} dt\right\} \\ \cdot \frac{1+\beta\omega(\zeta)}{1-\alpha\beta\omega(\zeta)} d\zeta d\xi, \end{aligned} \tag{3.7}$$

*where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.*

*Proof.* Let $f(z)\in\mathcal{C}_s^{(k)}(\alpha,\beta)$, from equalities (3.2) and (3.6) we have

$$\begin{aligned}(zf'(z))' &= f_k'(z)\cdot\frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}\\ &= \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))}dt\right\}\cdot\frac{1+\beta\omega(z)}{1-\alpha\beta\omega(z)}.\end{aligned}\tag{3.8}$$

Integrating equality (3.8) we can obtain

$$f'(z)=\frac{1}{z}\int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{(1+\alpha)\beta\omega(t)}{t(1-\alpha\beta\omega(t))}dt\right\}\cdot\frac{1+\beta\omega(\zeta)}{1-\alpha\beta\omega(\zeta)}d\zeta.\tag{3.9}$$

Therefore, integrating equality (3.9) we can obtain equality (3.7).

## 4. Sufficient Condition for Functions Belonging to the Class $\mathcal{C}_s^{(k)}(\alpha,\beta)$

At last, we give the sufficient condition for functions belonging to the class $\mathcal{C}_s^{(k)}(\alpha,\beta)$.

**Theorem 6.** *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in $\mathcal{U}$, if for $0\le\alpha\le 1$, $0<\beta\le 1$, we have*

$$\begin{aligned}&\sum_{n=1}^{\infty}(nk+1)[(1+\alpha\beta)(nk+1)+\beta-1]\,|a_{nk+1}|\\ &+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty} n^2(1+\alpha\beta)\,|a_n|\le(1+\alpha)\beta.\end{aligned}\tag{4.1}$$

*Then $f(z)\in\mathcal{C}_s^{(k)}(\alpha,\beta)$.*

*Proof.* Suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, and $f_k(z)$ is defined by equality (1.2). Then for $z\in\mathcal{U}$, we have

$$\begin{aligned}M &= |(zf'(z))'-f_k'(z)|-\beta\,|\alpha(zf'(z))'+f_k'(z)|\\ &= \left|1+\sum_{n=2}^{\infty}n^2a_nz^{n-1}-1-\sum_{n=2}^{\infty}na_nb_nz^{n-1}\right|\\ &\quad -\beta\left|\alpha+\sum_{n=2}^{\infty}\alpha n^2a_nz^{n-1}+1+\sum_{n=2}^{\infty}na_nb_nz^{n-1}\right|,\end{aligned}$$

where

$$b_n=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{(n-1)\nu}\quad(\varepsilon^k=1).$$

Thus, for $|z| = r < 1$, we have

$$\begin{aligned} M &\leq \sum_{n=2}^{\infty} n(n-b_n)\,|a_n|\,r^{n-1} \\ &\quad -\beta\left[(1+\alpha) - \sum_{n=2}^{\infty} n(\alpha n + b_n)\,|a_n|\,r^{n-1}\right] \\ &< \sum_{n=2}^{\infty} n[(n-b_n) + \beta(\alpha n + b_n)]\,|a_n| - (1+\alpha)\beta. \end{aligned} \tag{4.2}$$

From the definition of $b_n$ we know

$$b_n = \begin{cases} 1, & n = lk+1, \\ 0, & n \neq lk+1. \end{cases} \tag{4.3}$$

Substituting (4.3) into inequality (4.2), we get

$$\begin{aligned} M &< \sum_{n=1}^{\infty} (nk+1)[(1+\alpha\beta)(nk+1) + \beta - 1]\,|a_{nk+1}| \\ &\quad + \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty} n^2(1+\alpha\beta)\,|a_n| - (1+\alpha)\beta. \end{aligned}$$

From (4.1) we know $M < 0$. Thus we have

$$\left|\frac{(zf'(z))'}{f_k'(z)} - 1\right| < \beta\left|\frac{\alpha(zf'(z))'}{f_k'(z)} + 1\right| \quad (z \in \mathcal{U}),$$

that is $f(z) \in \mathcal{C}_s^{(k)}(\alpha,\beta)$. Therefore, the proof of theorem 6 is complete.

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