Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
Vol. 20, 2005, 59–75

V. Gorbunov, A. Gorobetz, and V. Sviridov

THE METHOD OF NORMAL SPLINES FOR LINEAR IMPLICIT DIFFERENTIAL EQUATIONS OF SECOND ORDER

(submitted by A. M. Elizarov)

Abstract. The method of normal splines is specified for the initial and boundary-value problems for systems of linear ordinary differential equations of second order, possible being stiff or unresolved with respect to derivatives (differential-algebraic equations), without their reduction to first order ones. The algorithm of nonuniform collocation grid creation for stiff problems is described. Results of numerical solution to test problems, including linear mathematical physics boundary-value problem of the second order are given. Numerical schemes for the last case are based on the method of lines.

1. Introduction

Many of mechanical and physical problems are initially modeled as systems of ordinary differential equations (ODEs) of second order that can be stiff or unresolved with respect to derivatives. The last class of ODEs is named implicit or differential-algebraic equations (DAEs). Systems of second order can be reduced to ones of first order but such transfer leads to increasing of the system dimension. Specifics of systems

---

2000 Mathematical Subject Classification. 65L05, 65L10, 65L60, 65L80.
Key words and phrases. Normal splines, Singular differential-algebraic equations, Adaptive grids, Partial differential equations, Method of lines.
Supported by Russian Foundation for Basic Research, Grant 01-01-00731.

of higher order can be used for creation of more effective special numerical methods (see, for example, [18]).

The problem of numerical methods' construction for singular DAEs (with arbitrary degenerate main part) is considered in literature as open [13]. We know only theoretical investigations of such DAEs [16]. However, we note, there exists the parameterization method [5], [8], [9] that can be applied for rather wide class of arbitrary degenerate nonlinear ODEs and optimal control problems. The method of normal spline-collocation (NS) presented below should be more effective in linear cases.

The NS method for linear ODEs and integro-differential equations (IDEs) of first order, including implicit systems, has been created by V.Gorbunov in [6], [7]. The theoretical basis of the NS method is the classical functional analysis results: the embedding theorem of Sobolev spaces in the Chebyshev ones [19], and Riesz's theorem [2] of canonical representation of linear continuous functionals in Hilbert spaces as inner products. The last problem is the key one for effective NS algorithms construction.

The NS method consists of minimization of some Hilbert-Sobolev (HS) norm on the set of collocation system solutions. This set is a finite dimensional subset of the used solution space. Differently from the classical collocation methods [15] here the basis system is not entered a priori, but it is constructed according to the chosen norm and to coefficients of the solving problem. The base functions are canonical images of point-wise linear continuous functionals in the HS space (presented as inner product). To find this images it is necessary to construct the corresponding reproducing kernel [1] defined by the norm.

The NS method on the base of creation of adaptive nonuniform grids had appeared effective for stiff problems [6], and for linear DAEs [10], [11]. Significant examples of its application to stiff and singular problems for ODEs and IDEs, including DAEs with variable degeneration of the main part, were presented in these works. Also the NS yields natural way for solving ODEs of arbitrary order provided that the problem is posed in a Sobolev space with norm differentiation factor more than the order of derivatives in the resolving equations.

The main purpose of this work is to specify the computational scheme of the NS method for linear ODEs of second order. It particularly allows applying this method for linear (or linearized) partial differential equations (PDEs) of second order (possible singular) on the base of some partial discretization.

In the second section the initial/boundary-values problem for the common second order linear ODEs is posed in the HS space of sufficiently smooth functions. The matrix under second derivatives can be arbitrary degenerate. The third section is devoted to clarification of the problem of canonical representation of linear point-wise functionals in the used space. Such a problem arises at the NS's algorithm developing. In the forth section the NS scheme for second order ODEs is presented. The fifth section is devoted to presentation of the algorithm of adaptive grids creation for stiff systems. Some test numerical examples are given in the last two sections. In the sixth one two singular and stiff problems for scalar ODEs are resolved by the NS, and in the seventh our method is applied for a boundary-value problem (BVP) for a singular parabolic type equation in combination with the method of lines. Presented results were partially published in [20], [12].

2. Problem statement

Let us consider the system of implicit linear differential equations of second order

$$A(t)\ddot{x}(t) + B(t)\dot{x}(t) + C(t)x(t) = f(t), \quad 0 \le t \le 1, \tag{1}$$

with conditions

$$D^0 x(0) + E^0 \dot{x}(0) = g^0, \quad D^1 x(1) + E^1 \dot{x}(1) = g^1. \tag{2}$$

Here $x, f, g^0, g^1 \in R^n$, $A(t), B(t), C(t), D^0, E^0, D^1, E^1$ are square n-order matrices. The function $f(t)$ and the matrix coefficients are continuous and have so many derivatives as it is necessary to guarantee appropriate smoothness of the solution $x(t)$ that exists in assumption and belongs to the HS space $W^l_{2,n}[0,1]$ with norm

$$\|x\|_{l,n} = \left\{ \sum_{i=1}^{n} \left[\sum_{r=0}^{l-1} \left(x_i^{(r)}(0) \right)^2 + \int_0^1 \left(x_i^{(l)}(s) \right)^2 ds \right] \right\}^{1/2}, \tag{3}$$

where $x_i^{(r)}(t)$ are derivatives of the order r, and the highest index of derivatives $l \ge 3$. Also notations $\dot{x}_i = x_i^{(1)}$, $\ddot{x}_i = x_i^{(2)}$ will be used below. The inner product corresponding to this norm is

$$\langle x, y \rangle_{(n)} = \sum_{i=1}^{n} \left[\sum_{r=0}^{l-1} \left(x_i^{(r)}(0), y_i^{(r)}(0) \right) + \int_0^1 \left(x_i^{(l)}(s), y_i^{(l)}(s) \right) \right]. \tag{4}$$

These norm and inner product were introduced in [6].

The matrices of the system (1) may be arbitrary degenerate. In general case nontrivial degeneracy of the matrix $A(t)$ generates obstacles for applications of classical numerical methods to solving initial/boundary-value problems for system (1). Such systems are named DAEs.

The complexity of a DAE is determined by a possibility of its transformation to the regular normal form with the help of differentiation and algebraic transformations. The minimal number of required differentiations in such a transformation is called *the differentiation index* (DI) of the DAE [13]. There are special methods [13], [3] for solving DAEs having a finite DI. The matrix at the main part of such a system should have a constant rank.

However, not any DAE can be transformed to the normal form of ODE. Correspondingly, not any DAE has a finite DI. The simplest example such a DAE is

$$t\dot{x}(t) + x(t) = f(t), \quad 0 \le t \le 1.$$

The known numerical methods for solving singular ODEs and DAEs [3], [4], [13], [17], [18] cannot be applied for the initial value problem for this equation with condition in $t = 0$. The assumption of arbitrary degeneracy of the main part of system (1) covers DAEs of any DI, and singular DAEs not having a finite DI.

3. The problem of normal splines for second order ODEs

The NS method is a collocation type one. Introduce some grid

$$0 \le t_1 < t_2 < ... < t_m \le 1, \tag{5}$$

and consider the collocation system

$$A(t_k)\ddot{x}(t_k) + B(t_k)\dot{x}(t_k) + C(t_k)x(t_k) = f(t_k), \quad k = 1, \dots, m. \tag{6}$$

For this system we pose the problem of the normal solution in norm (3). In the case of the compatible system (1), (2) the solution exists and is unique. It is clarified below, that this fact is the consequence of the embedding theorem of Sobolev type mentioned above.

Denote the left parts of the system (6), (2) as

$$l_{ik}(x) = \begin{cases} \sum_{j=1}^{n} d_{ij}^{0} x_j(0) + e_{ij}^{0} \dot{x}_j(0), \quad k = 0; \\ \sum_{j=1}^{n} a_{ij}(t_k)\ddot{x}_j(t_k) + b_{ij}(t_k)\dot{x}_j(t_k) + c_{ij}(t_k)x_j(t_k), 1 \le k \le m; \\ \sum_{j=1}^{n} d_{ij}^{1} x_j(1) + e_{ij}^{1} \dot{x}_j(1), \quad k = m+1; \end{cases} \tag{7}$$

where $1 \le i \le n$. Respectively, the system (6), (2) takes the form

$$l_{i0}(x) = g_i^0, \ l_{ik}(x) = f_i(t_k), \ l_{i(m+1)}(x) = g_i^1, \ 1 \le i \le n, \ 1 \le k \le m. \tag{8}$$

In [6] it has been shown that the functions (7) may be considered as composite linear continuous functionals in the vector-function space $W_{2,n}^l[0,1]$, composed as linear combinations of simple point-wise functionals (values of coordinates $x_i(t)$, and their first and second derivatives in collocation and boundary points $t = t_k$) in the space $W_{2,1}^l[0,1]$. Respectively, each equation of the system (8) defines a hyperplane and the system solution set (an intersection of the hyperplanes) will be nonempty, convex and closed. The minimal norm element x^m of this intersection exists and unique [2]. It has been named [7] *the normal spline.*

Also in [6] it has been shown that the sequence of the normal splines x^m converges to a normal solution x^0 in the norm (3) when the maximal step of the grid (5) tends to zero. It also provides the approximation of the solution derivatives up to the $(l-1)$-th order. We note that the providing of the last property is a special nontrivial problem in the theory of difference schemes for ODEs [17].

4. Pointwise functionals and reproducing kernel

As it is mentioned in introduction the key problem in the NS method is the problem of canonical representation of linear continuous functionals in HS spaces $W_{2,n}^l[0,1]$, i.e. as the inner product (4). At implementation of the NS method for ODEs point-wise functionals are defined as the values of a seeking function coordinates $x_i(t)$ and their derivatives $x_i^{(r)}(t)$ in a given point τ

$$l_{i,\tau}(x) = x_i^{(r)}(\tau), \quad r = 0, \ 1, \ \ldots, \tag{9}$$

arise. Such functionals have no canonical form. They define in the NS schemes composite functionals (7) as linear combinations. In the case of IDEs non-canonical integral functionals also arise [6], [11].

It can be shown, like in the case of Banach-Sobolev spaces (see [19]), that the convergence in $W^l_{2,n}[0,1]$ implies the convergence in Chebyshev spaces $C^{l-1}[0,1]$, particularly, uniform convergence if $l \geq 1$. Correspondingly, the defined in (9) functionals $l_{i,\tau}(x)$ are linear continuous ones in $W^l_{2,n}[0,1]$ while $0 \leq r \leq l-1$, but their definition is not canonical. According to Riesz's theorem [2] there exists an element $h_{i,\tau} \in W^l_{2,n}[0,1]$ such that the identity

$$l_{i,\tau}(x) = \langle h_{i,\tau}, x\rangle_{(n)}, \quad \forall x \in W^l_{2,n}, \tag{10}$$

holds. Here the inner product $\langle \cdot, \cdot \rangle_{(n)}$ defined in (4).

In [6] (see also [7]) it has been shown that the problem of canonical representation of point-wise functionals (10) (as well as integral functionals) could be reduced to finding a Green function of some BVP. There this problem has been resolved for IDEs of first order under $l \in \{1, 2\}$.

In terms of functional analysis the canonical representation of linear continuous functionals in HS spaces are equivalent to construction of the reproducing kernel corresponding to the space norm. Remind [1], that the *reproducing kernel* (RK) is a function $G(s,t)$ such that

1) $G(\cdot,t) \in W^l_{2,1}[0,1]$ *for any* $t \in [0,1]$;

2) $x_i(t) = \langle G(\cdot,t), x_i\rangle_{(1)}$ *for any* $x_i(\cdot) \in W^l_{2,1}[0,1]$ *and any* $t \in [0,1]$.

In [10] a general RK in HS spaces $W^l_{2,n}[0,1]$ with arbitrary integer l has been created. There it has been shown (V.Petrischev) that the RK for the norm (3) was defined by the formula

$$G(s,t) = \begin{cases} \displaystyle\sum_{i=0}^{l-1} \frac{t^i}{i!}\left(\frac{s^i}{i!} - (-1)^{l+i}\frac{s^{2l-i-1}}{(2l-i-1)!}\right), & 0 \leq s \leq t \leq 1; \\ G(t,s), \quad 0 \leq t < s \leq 1. & \end{cases} \tag{11}$$

The resolution of the RK construction problem for spaces of high smoothness allows to solve a high-order systems without their reduction to the first order ones.

5. The scheme of normal splines for second order ODEs

As it is mentioned above, according to the Riesz theorem a linear continuous functional can be represented as an inner product (4), i.e.

$$l_{ik}(x) = \left\langle h^{\mu(i,k)}, x\right\rangle_{(n)} = \sum_{j=1}^{n}\left\langle h_j^{\mu(i,k)}, x_j\right\rangle_{(1)}. \tag{12}$$

Here $\mu(i,k) = nk + i,\ 1 \leq i \leq n,\ 0 \leq k \leq m+1$.

Elements h^μ_j of the representation (12) may be found with the help of RK $G(s,t)$ of the space of scalar functions $W^l_{2,1}[0,1]$. According to the presented above definition of RK the equalities $x_j(t_k) = \langle G(\cdot, t_k), x_j\rangle_{(1)}$, $\dot{x}_j(t_k) = \langle G'_t(\cdot, t_k), x_j\rangle_{(1)}$, and $\ddot{x}_j(t_k) = \langle G''_{tt}(\cdot, t_k), x_j\rangle_{(1)}$ hold, hence

$$h_j^{\mu(i,k)}(s) = \begin{cases} d^0_{ij} G(s,0) + e^0_{ij} G'_t(s,0), \quad k = 0; \\ a_{ij}(t_k) G''_{tt}(s,t_k) + b_{ij}(t_k) G'_t(s,t_k) + c_{ij}(t_k) G(s,t_k), \\ 1 \le k \le m; \\ d^1_{ij} G(s,1) + e^1_{ij} G'_t(s,1), \quad k = m+1; \end{cases} \tag{13}$$

By virtue of (7) and (12) system (8) is represented as

$$\langle h^\mu, x\rangle_{(n)} = \bar{f}_\mu, \quad \bar{f}_\mu = \begin{cases} g^0_i, \quad k = 0; \\ f_i(t_k), \quad 1 \le k \le m; \\ g^1_i, \quad k = m+1; \end{cases} \tag{14}$$

According to the generalized Lagrange method, the normal solution of the system of linear equations (14) can be written in the form

$$x^m(s) = \sum_{\mu=1}^{(m+2)n} u_\mu h^\mu(s). \tag{15}$$

Coefficients u_μ are defined by the system

$$\sum_{\nu=1}^{(m+2)n} g_{\mu\nu} u_\nu = \bar{f}_\mu, \quad 1 \le \mu \le (m+2)n, \tag{16}$$

where $g_{\mu\nu}$ are the coefficients of the Gram matrix of the system $\{h^\mu\}$, i.e.

$$g_{\mu\nu} = \langle h^\mu, h^\nu\rangle_{(n)} = \sum_{i=1}^{n} \langle h^\mu_i, h^\nu_i\rangle_{(1)}. \tag{17}$$

Thus the realization of the NS method with given partitioning (5) is reduced to the Gram matrix coefficients formation according to (17), to solving the system of linear equation (16) with symmetric, positive defined (as a rule) matrix $\{g_{\mu\nu}\}$, and to the creation of the solution $x^m(s)$ at arbitrary point $s \in [0,1]$ according to (13), (15). The different strategies of the collocation grids improvement are based on the theoretical estimate of the NS method precision [7] as it is shown below.

Effectiveness of the described scheme of the NS method can be essentially increased in case of the initial problem for the equation (1) when the condition (2) has the form $x(0) = g^0$, $\dot{x}(0) = g^1$. In this case the problem can be solved on a sequence of partial adjoining subintervals with a small number of nodes (up to two) on each of them. By such a way total number of calculations ensuring the required precision will be essentially reduced. Corresponding algorithm is described below.

6. The adaptive grid construction

Consider a discrepancy of the equation (1) on the function x^m:

$$\varphi(t; t_1, \dots, t_m) = A(t)\ddot{x}^m(t) + B(t)\dot{x}^m(t) + C(t)x^m(t) - f(t). \tag{18}$$

In the book [7] the following evaluation of deviation of the spline x^m from the exact solution x^0 has been obtained:

$$\left\| x^m - x^0 \right\|_{l,n} \leq \hat{c} \left\| \varphi \right\|_{(l-2),n}. \tag{19}$$

Here $\hat{c}$ is a constant that is depended only on the coefficients of the system (1), (2). The index $l-2$ corresponds to the inclusion $\varphi \in W_{2,n}^{l-2}[0,1]$.

The evaluation (19) opens a way for creation of the optimal nonuniform grids under given number of nodes m, that is very important for solution of stiff problems. Denote

$$\psi_{l-2}(t_1, \dots, t_m) = \left\| \varphi(\cdot; t_1, \dots, t_m) \right\|^2_{(l-2),n}. \tag{20}$$

The grid (5) providing the minimum of the function $\psi_{l-2}(t_1, \dots, t_m)$ is named *optimal*.

Accordingly, a strategy of creation of the optimal grid with fixed nodes number is based on minimization of the function $\psi_{l-2}(t_1, \dots, t_m)$ with the constrains (5). If the system (1),(2) has a finite differentiation index, it is equivalent to some normal system [13]. Such a system is regular, its solution is a smooth function of nodes t_k, and smoothness of function ψ_{l-2} is determined by the order of the norm derivative l and by properties of the function $G(s,t)$. The last function has continuous derivatives up to the order $2l-2$. As it has been shown in [20] (for first order systems) it provides for $l \geq 3$ the differentiability of the function ψ_{l-2} with respect to t_k. In this case the appropriate analytic formulas for partial derivatives have been obtained. We omit them because of their complexity. In general case one may use some direct method (e.g. Hooke-Jeeves [14]) for minimization the function ψ_{l-2}.

Another, more simpler and more effective scheme of the grid improvement is the next. The convergence in the norm $\|\cdot\|_{l,n}$ implies the uniform convergence, therefore we should pass to better grids decreasing the value $\|\varphi\|_{(l-2),n}$ during calculations.

In the collocation nodes t_k the equalities $\varphi(t_k; t_1, \dots, t_m)) = 0$ hold, hence an approximate minimization of the discrepancy norm $\|\varphi\|$ may be achieved by adding nodes into subintervals with the greatest values of $|\varphi(t; t_1, \dots, t_m))|$ between nodes. The detailed algorithm of the step-by-step concentration of the grids has been offered in [7]. The process of creation of such an adaptive condensing grid has two parameters: the initial number of nodes and the number of adding nodes on each step.

Consider an initial problem for the equation (1), when condition (2) has the form $x(0) = g^0$, $\dot{x}(0) = g^1$. In this case an effective scheme of the NS method is the sequential normal spline creation on partial adjoining subintervals with a small number of nodes (up to two). The effectiveness is achieved as a result of reducing of the Gram matrix dimension since the calculation of its elements and solving the linear equation system are the most time-consuming parts of the NS construction.

If partial interval is sufficiently small we can create the spline on two boundary nodes of segments $\{t_{k-1}, t_k\}$ on each step. The initial values on the second and following subintervals are defined as final values of the created spline on the previous subinterval. The right nodes t_k in the created sequence of segments can be chosen so that to provide required precision of the solution.

It is natural to expect, that sequential suppression of a discrepancy (18) in the norms of $W^l_{2,n}(t_{k-1}, t_k)$ can be obtained by passing to the simpler norms of $L_{2,n}(t_{k-1}, t_k)$. It means the suppression of the functions

$$\psi_0^k(t_{k-1}, t_k) = \sum_{i=1}^{n} \int_{t_{k-1}}^{t_k} \left(\varphi_i(s; t_{k-1}, t_k)\right)^2 ds. \tag{21}$$

Introduce an admissible level ε_0 of the value $\psi_0^k(t_1, \dots, t_m)$ and define values

$$E(t_{k-1}, t_k) = \sqrt{\psi_0^k(t_{k-1}, t_k)/(t_k - t_{k-1})}.$$

This values can be estimated by usage of some quadrature formula for integrals in (21). It is easy to see that the condition

$$\psi_0(t_1, \dots, t_m) \le \varepsilon_0 \tag{22}$$

will hold provided

$$E(t_{k-1}, t_k) \le \varepsilon_0 \tag{23}$$

for all $k = 1, \dots, m$.

The algorithm of creation of adaptive condensed grid is the next. Define the maximum level of the grid step h.

(1) $k := 1$, $t_0 := 0$.
(2) $t_k := \min\{t_{k-1} + h, 1\}$.
(3) To construct the NS for the Cauchy problem (1) on (t_{k-1}, t_k).
(4) If (23) holds then go to 6.
(5) $t_k := (t_k + t_{k-1})/2$, go to 3.
(6) If $t_k = 1$, then end, else $t_{k-1} := t_k$.

(7) $k := k + 1$, go to 2.

If the estimation (22) will not be obtained for acceptable number of t_k diminution, then one can pass to the method of the grid condensation described above.

7. Numerical solution to stiff equations

In this section we demonstrate the problem of solving stiff equations of second order by the NS method on two test examples.

Example 1. Consider the initial problem of second order

$$a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) = f(t), \quad 0 \leq t \leq 1,$$

$$x(0) = g_0, \quad \dot{x}(0) = g_1,$$

where $a(t) = sin(10t), b(t) = 10cos(10t), c(s) = -1$. The function $f(t)$ corresponds to the solution $x(t) = te^{-t} + e^{-kt}$, $k > 1$.

The coefficients $a(t)$, $b(t)$ have zero values on the interval $[0, 1]$ and the solution is stiff under large k. The variable degeneracy of the coefficients makes difficulties for application of the known methods of solution to stiff and differential-algebraic equations [3], [4], [13], [18], except for the parameterization method [5], [8]. The last one can be applied for arbitrary degenerate nonlinear problems, however, the NS method should be more effective in linear cases.

The NS method allows different numerical schemes. In [11] this problem had been solved by reducing the initial second order equation to the equivalent integro-differential equation of first order. Such a way in some cases is more effective with respect to the transformation of the initial equation to the normal system of two equations of first order [6].

Table 1 presents some results for two variants of the NS method: the sequential scheme with eight nodes on each subinterval, and the scheme with uniform grid. The problem has been solved in the space W_2^3 with $\varepsilon_0 = 0.01$ in the sequential scheme. The last two column presents deviations of obtained normal splines from the exact solution on the doubled grids.

Table 1

k	m	sequential	uniform
10	8*4=32	3.43e-03	1.16e-02
100	8*14=112	1.38e-03	2.10-01
1000	8*42=336	8.58e-04	4.47e+00

Comparison of these results with ones of [11] shows approximate equivalence of the new scheme and the integro-differential reduction, however the problem of canonical transformation and Gram matrix formation in the approach presented here is easier.

Example 2. Consider the BVP arising at the solution to steady-state heat conductivity equations with the property of central spherical symmetry [17]:

$$\left(\frac{\varepsilon}{t}\right)^2 \frac{d}{dt}\left(t^2 \frac{dx}{dt}\right) + c(t)x(t) = f(t), \quad t \in (0,1),$$

$$\dot{x}(0) = 0, \quad d_1 x(1) + e_1 \varepsilon \dot{x}(1) = g_1, \quad \varepsilon \in (0,1].$$

Parameters of this singular problem with degenerate coefficient $(\varepsilon/t)^2$ at the higher derivative are the next: $c(t) = (1 + 0.1t^2)$, $d_1 = 5$, $g_1 = 1$, $e_1 = 0$. Function $f(t)$ corresponds to the solution

$$x(t) = -1 + 6/5sh(t/\varepsilon)/(tsh(1/\epsilon)) + 10t^2(1-t)^2.$$

This solution has initial layers with increasing sharpness when parameter ε vanishes.

It is accepted to characterize quality of numerical methods for such problems by notions of classical and uniform convergence of approximations with respect to small parameter ε [4]. Following to [17] we use here the next convergence measures. Let x_ε^0 is the precise solution to initial problem, x_ε^m is the normal spline considered on uniform grid with step $h \in H \equiv \{h_0/2^j | j = 0, 1, \ldots, k\}$, and $\varepsilon \in E \equiv \{\varepsilon_0/2^j | j = 0, 1, \ldots, 8\}$.

Denote

$$\delta(h, \varepsilon) = max|x_\varepsilon^0 - x_\varepsilon^m|, \quad \Delta(h) \equiv max\{\delta(h, \varepsilon) : \varepsilon \in E\}.$$

Here the difference is calculated on the condensed grid (in ten times). The experimental orders p of uniform convergence and p_0 of classical one are defined by the formulas

$$p = \frac{1}{ln(2)} ln \left\{ \frac{1}{k} \sum_{j=0}^{k-1} [\Delta(h_0/2^j)/\Delta(h_0/2^{j+1})] \right\},$$

$$p_0 = \frac{1}{ln(2)} ln \left\{ \frac{1}{k} \sum_{j=0}^{k-1} [\delta(h_0/2^j, \varepsilon_0)/\delta(h_0/2^{j+1}, \varepsilon_0)] \right\}$$

for $h_0 = 1/8$, $\varepsilon_0 = 1/2$, $k = 7$.

These characteristics of the NS method as well as ones from [17] are presented in table 2.

TABLE 2

Method	$x(t)$		$\dot{x}(t)$	
	p	p_0	p	p_0
[17]	1,04	2	0,61	1,97
NS	2,33	1,29	1,55	0,92

One can note that NS method has advantage with respect to method [17] in uniform convergence and conversely in classical one.

This problem also has been solved on adaptive condensed grid. The start grid had 8 points and on each step 6 points has been added on the interval with maximum discrepancy. Corresponding deviation of the normal spline x_ε^m from the precise solution x_ε^0 is presented in table 3. The last row marked by (*) corresponds to the uniform grid.

TABLE 3

m\ ε	0.015625	0.0078125	0.00390625	0.001953125
8	3.54e+00	1.55e+01	6.36e+01	2.56e+02
14	4.06e-02	2.43e-01	1.37e+00	6.63e+00
20	9.81e-03	2.59e-02	1.58e-02	3.88e-02
26	8.89e-03	4.60e-03	3.41e-03	3.81e-02
32	8.01e-03	4.02e-03	2.92e-03	2.89e-02
128*	2.66e-03	1.39e-02	8.82e-02	5,38E-01

8. The NS method for a BVP of a parabolic type equation

In this section we demonstrate the application of the NS method for solving BVP for partial differential equations (PDE) with two variables by the combination of the lines method (LM) with the NS. The LM consists of discretization of one of variables and the transition from the PDE to the system of ODEs. The last can be solved by the NS.

Consider the parabolic type differential equation with respect to function $u(x,t)$:

$$x\frac{\partial u}{\partial t} = a^2\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + f(x,t), \tag{24}$$

in the unit square $\Omega = \{x, t : 0 \le x, t \le 1\}$ with initial and boundary conditions:

$$u(x,0) = \varphi(x), \quad u(0,t) = \psi_0(t), \quad u(1,t) = \psi_1(t). \tag{25}$$

Note, that the equation (24) is singular. Let us assume, the functions $f(x,t)$, $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are sufficiently smooth, and the singular BVP (24), (25) is solvable in the space $W_2^3(\Omega)$.

There are two version of the LM application to PDEs with two variables (x,t) which are defined by the choice of the discretized variable. Since the presented above NS method is directly applicable to equations of second order then it is seemed efficient to discretize the equation (24) on t - variable.

Introduce a uniform grid on t - variable with the step $h = 1/n$ and nodes $t_i = ih, \;\; i = \overline{0,n}$. Define the x-variable functions $v_i(x) = u(x,t_i)$. In view of the first condition (25),

$$v_0(x) = \varphi_0(x) \tag{26}$$

the approximating problem can be stated with respect to n-vector function

$$v(x) = (v_1(x), ..., v_n(x)). \tag{27}$$

Taking into consideration the initial condition (26), we approximate derivatives $\partial u(x,t_i)/\partial t$ by finite differences:

$$\frac{\partial u(x,t_i)}{\partial t} \approx \begin{cases} \dfrac{v_1(x) - \varphi_0(x)}{h}, & i = 0, \\ \dfrac{v_{i+1}(x) - v_{i-1}(x)}{2h}, & i = \overline{1, n-1}, \\ \dfrac{v_n(x) - v_{n-1}(x)}{h}, & i = n. \end{cases}$$

Substitution of these differences in equalities (24) under $t = t_i$ in view of (26) provides under $i = 0$ the finite equality

$$xv_1(x) = \varphi(x) + ha^2\left(x\varphi^{"}(x) + \varphi^{'}(x)\right) + hf(x,0), \tag{28}$$

and under $i > 0$ the system of ODEs of second order with respect to the seeking function (27):

$$\begin{cases} 2ha^2\left(xv_1^{"}(x) + v_1^{'}(x)\right) - xv_2(x) = -x\varphi(x) - 2hf(x,t_1); \\ 2ha^2\left(xv_i^{"}(x) + v_i^{'}(x)\right) - xv_{i+1}(x) + xv_{i-1}(x) = -2hf(x,t_i), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = \overline{2, n-1}; \\ ha^2\left(xv_n^{"}(x) + v_n^{'}(x)\right) - xv_n(x) + xv_{n-1}(x) = -hf(x,t_n). \end{cases} \tag{29}$$

Thus, the singular PDE (24) on the unit square Ω is changed by the system of one finite equation (28) and n ODEs (29) on the unit segment $\Sigma = \{0 \le x \le 1\}$. All these equations have singularity in the initial point $x = 0$. Matrix of the system (29) main part $A(x) = ha^2 diag\{2x, ..., 2x, x\}$ is the zero-matrix at $x = 0$, and it is regular one under $x > 0$.

The last pare of conditions (25) provides boundary conditions on the seeking functions (27):

$$v_i(0) = \psi_0(t_i), \quad v_i(1) = \psi_1(t_i), \quad i = \overline{1, n}. \tag{30}$$

Formally, the system (28), (29), (30) is the overdetermined one. It contains on one equality more then it is necessary for existence and uniqueness of a solution in a regular case. However, there is not a theory of singular BVP that guarantees correctness of the obtained problem. So we can use heuristically only the system of ODEs (29) with boundary conditions (30) for numerical creation of the seeking solution by the NS method. The last problem for the singular ODEs belongs to the type (1), (2). The variability of its rank encumbers or makes impossible the application of other known numerical methods besides the NS one.

In the case of a possible failure the equation (28) can be used as an additional condition. Particularly, it can be used in the second equation of the system (29) for a substitution of the term $xv_1(x)$ by the known right hand of (28).

In table 4 results of numerical solution to the problem (24), (25) are presented. The free function $f(x,t)$ in (24) and the right-hand functions in (25) were constructed on the solution $u(x,t) = x^2t + 1$.

We have used the classical finite-differences (FD) method realized in the software MATLAB, and the combination of the NS one presented above with the LM. In both cases the uniform grids with equal nodes n along two variables were chosen for simplicity. Note, for the used variant of the combined LM+NS method the t-grid corresponds to the LM, and the x-grid is the NS collocation one. Values Δ present the maximal absolute deviations of the obtained approximations (the grid-function in the FD or the spline in the LM+NS) from exact solution on the grids.

TABLE 4

FD		LM+NS	
n	Δ	n	Δ
10	0.02	5	0.0023
20	0.005	7	0.0013
100	0.0003	10	0.0006

One can note that accuracy of the LM+NS method for this problem on the same net (n=10) overcome the accuracy of classical FD method on two order.

9. Conclusions and future work

Presented above numerical results also as results of [6], [7], [10], [11], [12], [20] demonstrate the ability of the NS method to obtain appropriate approximation of solutions to arbitrary degenerate problems of linear differential and integral equations. Particularly for equations that can not be reduced to the normal Cauchy form, i.e. equations which are not having of a finite differentiation index. Such a kind of nonlinear problems can be resolved also by the parameterization method [5], [8]. We know only theoretical works of another authors devoted to such kind of ODEs problems [16].

We are inclined to explain the success of the presented NS method by passage from an initial singular problem to the approximate variational problem to minimize the Hilbert-Sobolev norm on the set of solutions of a collocation system. Usage of two classical results of functional analysis, namely, of embedding of Sobolev's spaces in Chebyshev's ones, and the Riesz theorem of canonical representation of linear continuous functionals in Hilbert spaces, has allowed to construct effective algorithm for solving this robust variational problem.

In future work in the area of numerical methods for hard problems of differential and integral equations we plan to develop the NS method for singular problems on infinite intervals, particularly, for numerical inversion of the Laplace and the Fourier transformations, for nonlinear equations, and multi-dimensional problems.

References

[1] Aronszajn N. *Theory of reproducing kernels*, in 'Transactions of the AMS', 1950, Vol. 68, pp. 337–404.

[2] Balakrishnan A. *Applied Functional Analysis*, Springer-Verlag, New York, 1976.

[3] Bulatov M.V., Chistyakov V.F. *A numerical method for solving differantial-algebraic equations*, in 'Zh. Vichisl. Mat. Mat. Fiz.', 2002, Vol. 42, No 4, pp. 439–449 (in Russian, English transl. in 'Comput. Math. Math. Phys.', 2002, Vol 42, No 4).

[4] Doolan E., Miller J., Schilders W. *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Bool Press, Dublin, 1980.

[5] Gorbunov V.K. *The parameterization method for optimal control problems*, in 'Zh. Vichisl. Mat. Mat. Fiz.', 1979, Vol. 19, No 2, pp. 212-224 (in Russian, English transl. in 'Comput. Math. Math. Phys.', 1979, Vol. 19, No 2).

[6] Gorbunov V.K. *The method of normal spline-collocation*, in 'Zh. Vichisl. Mat. Mat. Fiz.', 1989, Vol 29, No 2, pp. 212-224 (in Russian, English transl. in 'Comput. Math. Math. Phys.', 1989, No 2).

[7] Gorbunov, V.K. *Extremum Problems of Measurements Data Processing*, Ilim, Frunze, 1990 (in Russian).

[8] Gorbunov V.K., Lutoshkin I.V. *Development and experience of applying the parameterization method in degenerate problems of dynamical optimization*, in 'Izvestia RAN. Teoria i Systemy Upravlenia', 2004, No 5 pp.67-84 (in Russian, English transl. in 'J. of Computer and Systems Sciences International', 2004, No 5).

[9] Gorbunov V.K., Lutoshkin I.V. *The parameterization method in optimal control problems and differential-algebraic equations*, in 'J. of Comput. and Applied Math.', 2006, Vol. 185, No 2, pp. 377-390.

[10] Gorbunov V.K., Petrischev V.V. *Development of the method of normal spline collocation for linear differential equations*, in 'Zh. Vichisl. Mat. Mat. Fiz.', 2003, Vol. 43, No 8, pp.1150-1159 (in Russian, English transl. in 'Comput. Math. Math. Phys.', 2003, Vol 43, No 8, pp.1099-1108).

[11] Gorbunov V.K., Petrischev V.V., Sviridov V.Yu. *Development of the normal spline method for linear integro-differential equations*, in 'P. Slot et al, eds., Computational Science - ICCS 2003, LNCS 2658', Springer, Berlin, 2003, pp. 492-499.

[12] Gorobetz A. *Normal spline-collocation's method for linear differential equations of second order*, in 'Differential Equations and Applications, Proceedings of the International Scientific Conference, 26-31 May, 2002', Samara State Architectural - Building Academy, Samara, 2002, pp. 99-104 (in Russian).

[13] Hairer E., Wanner G. *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems*, Springer-Verlag, Berlin, 1996.

[14] Himmelblau D. *Applied Nonlinear Programming*, McGraw-Hill Book Company, Texas, 1972.

[15] Krasnoselsky M.A. et al, *Approximate Solution to Operator Equations*, Nauka, Moskow, 1969 (in Russian).

[16] März R., Weinmüller E. *Solvability of boundary value problems for systems of singular differential-algebraic equations*, in 'J. Math. Anal., SIAM', 1993, Vol. 24, pp. 200–215.

[17] Rafatov I.R., Sclyar S.N. *Difference schemes for singular perturbated boundary-value problems arising at solving elliptic equations with spherical symmetry property*, in 'Zh. Vichisl. Mat. Mat. Fiz.', 2002, Vol. 42, No 9, pp. 1383-1393 (in Russian, English transl. in 'Comput. Math. Math. Phys.', 2002, Vol.42).

[18] Shablitskaja L.N. *Solution of initial-value problems for systems of the second order linear differential equations which are not solved with respect to the highest derivative*, in 'Zh. Vichisl. Mat. Mat. Fiz.', 1996, Vol.36, No 9, pp. 71-82 (in Russian, English transl. in 'Comput. Math. Math. Phys.', 1996, Vol.36).

[19] Sobolev S.L. *Applications of Functional Analysis to Mathematical Physics*, Amer. Math. Soc., Providence RI, 1963.

[20] Sviridov V.Yu. *Grid optimization in normal spline's method for integro-differential equations*, in 'Proceedings of the Middle Volga Mathematical Society', SVMO, Saransk, 2002, Vol. 3-4, No 1, pp. 236–245 (in Russian).

Ulyanovsk State University, L.Tolstoy str. 42, Ulyanovsk, 432970, Russia

E-mail address: vkgorb@vens.ru

Scientific-Research Institute of Atomic Reactors, Dimitrovgrad, 433510, Russia

E-mail address: gas_2001@rambler.ru

Ulyanovsk State University, L.Tolstoy str. 42, Ulyanovsk, 432970, Russia

E-mail address: svu@newmail.ru

Received October 29, 2005