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HARDY TYPE INEQUALITIES IN HIGHER DIMENSIONS WITH EXPLICIT ESTIMATE OF CONSTANTS

Abstract. Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \neq \mathbb{R}^n$. For $1 \leq p < \infty$, $1 < s < \infty$ and $\delta = \operatorname{dist}(x, \partial\Omega)$ we estimate the Hardy constant

$c_p(s,\Omega) = \sup\{\|f/\delta^{s/p}\|_{L^p(\Omega)} : \ f \in C_0^\infty(\Omega), \ \|(\nabla f)/\delta^{s/p-1}\|_{L^p(\Omega)} = 1\}$

and some related quantities.

For open sets $\Omega \subset \mathbb{R}^2$ we prove the following bilateral estimates

$$\min\{2, p\} \ M_0(\Omega) \leq c_p(2, \Omega) \leq 2p \ (\pi M_0(\Omega) + a_0)^2, \ a_0 = 4.38,$$

where $M_0(\Omega)$ is the geometrical parameter defined as the maximum modulus of ring domains in Ω with center on $\partial\Omega$. Since the condition $M_0(\Omega) < \infty$ means the uniformly perfectness of $\partial\Omega$, these estimates give a direct proof of the following Ancona-Pommerenke theorem: $c_2(2, \Omega)$ is finite if and only if the boundary set $\partial\Omega$ is uniformly perfect (see [2], [22] and [40]).

Moreover, we obtain the following direct extension of the one dimensional Hardy inequality to the case $n \geq 2$: if $s > n$, then for arbitrary open sets $\Omega \subset \mathbb{R}^n$ ($\Omega \neq \mathbb{R}^n$) and any $p \in [1, \infty)$ the sharp inequality $c_p(s, \Omega) \leq p/(s-n)$ is valid. This gives a solution of a known problem due to J.L.Lewis [31] and A.Wannebo [44].

Estimates of constants in certain other Hardy and Rellich type inequalities are also considered. In particular, we obtain an improved version of a Hardy type inequality by H.Brezis and M.Marcus [13] for convex domains and give its generalizations.

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Contents.

1. Introduction.

Hardy type inequalities in Sobolev spaces have many applications in Mathematical Physics.

The original Hardy theorem (see [25], Theorem 330) gives that

$$\int_0^{+\infty} \frac{|u(t)|^p}{t^s} dt \le \left(\frac{p}{|s-1|} \right)^p \int_0^{+\infty} \frac{|u'(t)|^p}{t^{s-p}} dt$$

for $p \ge 1$, $s \in \mathbb{R}$, $s \ne 1$ and any absolutely continuous function $u : [0,\infty) \to \mathbb{R}$, $u'/t^{s/p-1} \in L^p[0,\infty)$ such that $u(0)=0$ in the case $s>1$ and $u(+\infty)=0$ in the case $s<1$.

If $p=1$ then equality in the Hardy inequality is valid for any monotone function u; if $p>1$ and $u \not\equiv 0$ then equality is not attained, but the constant $(p/|s-1|)^p$ is still sharp.

The Hardy inequality has been generalized in many ways. Our aim is to consider its direct generalizations when the domain of integration Ω is an open and proper subset of $\mathbb{R}^n$, u and u' are replaced by functions $f \in C_0^\infty(\Omega)$ and ∇f, the gradient of f, and powers of t are replaced by powers of

$$\delta = \delta(x) = \operatorname{dist}(x, \partial\Omega).$$

Let Ω be an open and proper subset of $\mathbb{R}^n$. We first consider the following Hardy constant

$$c_p(s,\Omega) := \sup \left\{ \left\| \frac{f}{\delta^{s/p}} \right\|_{L^p(\Omega)} : \ f \in C_0^\infty(\Omega), \ \left\| \frac{\nabla f}{\delta^{s/p-1}} \right\|_{L^p(\Omega)} = 1 \right\}.$$

The classical examples which are simple consequences of the one dimensional Hardy inequalities are given by the equations : for $p \in [1,\infty)$

and $s \in \mathbb{R}$

$$c_p(s, \mathbb{R}^n \setminus \{0\}) = \frac{p}{|s-n|}, \qquad c_p(s, H) = \frac{p}{|s-1|},$$

where H is a half space in $\mathbb{R}^n$ (see [10], [34], [38]). For $p > 1$ and any open convex set $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ it is known that

$$c_p(s, \Omega) = \frac{p}{p-1}$$

(see [17], [32], [33]). Explicit estimates of $c_p(s, \Omega)$ are also known in some particular cases when Ω is not a convex domain. Namely,

1) if Ω is a simply connected plane domain, then $c_2(2, \Omega) \leq 4$ (see [2], [3], [11], [16]);

2) if Ω is a domain in $\mathbb{R}^n$ with smooth boundary, then $c_p(p, \Omega) \geq p/(p-1)$ (see [16] and [32]).

For $p \geq 1$ and $s > 1$, it is a classical fact that there exists a finite constant $c_p(s, \Omega)$ for any domain Ω with Lipschitz boundary (see, for instance, [10], [16], [38]). It is known that the Lipschitz condition is not a necessary one and can be replaced by more general conditions on the boundary of Ω. In this direction there are several deep results due to A.Ancona [2], H. Brezis and M. Marcus [13], E.B.Davies [16], [17], P. Koskela and X. Zhong [30], J.L.Lewis [31], V.G. Maz'ya [34], V.M.Miklyukov and M.K.Vuorinen [35], and A.Wannebo [44].

The main aim of the present paper is to obtain explicit estimates of $c_p(s, \Omega)$ in the case when $p \in [1, \infty)$, $s \geq n$ and to estimate some related quantities.

In Sections 2 and 3 we examine the quantity $c_p(s, \Omega)$ in the case, when $p \in [1, \infty)$ and $s = n$. In Section 2 the case $p \in [1, \infty)$ and $n = s = 2$ is considered. For plane domains $\Omega \subset \mathbb{R}^2$ we prove the following bilateral estimates

$$\min\{2, p\}\ M_0(\Omega) \leq c_p(2, \Omega) \leq 2p\ (\pi M_0(\Omega) + a_0)^2,\ a_0 = 4.38,$$

where $M_0(\Omega)$ is the geometrical parameter defined as the maximum modulus of genuine annuli in Ω with center on $\partial\Omega$. Note that, by results of Ch. Pommerenke [40] and A. Ancona [2], $c_2(2, \Omega)$ for a plane domain $\Omega \subset \mathbb{R}^2$ is finite if and only if the boundary set $\partial\Omega$ is uniformly perfect. Clearly, our estimates give L^p - version and a direct proof of the Ancona - Pommerenke theorem, since the condition $M_0(\Omega) < \infty$ means the uniformly perfectness of $\partial\Omega$.

In Section 3 we extend our estimates to the quantities

$$\kappa_1(\Omega)^2 := \sup\left\{\|f/\delta\|_{L^2(\Omega)} : f \in C_0^\infty(\Omega), \|\delta\Delta f\|_{L^2(\Omega)} = 1\right\}$$

and

$$\kappa_2(\Omega) := \sup\left\{\|\nabla f\|_{L^2(\Omega)} : f \in C_0^\infty(\Omega), \|\delta\Delta f\|_{L^2(\Omega)} = 1\right\}$$

related to Rellich's constant of $\Omega \subset \mathbb{R}^2$ and discuss a generalization of results to the quantity $c_p(n, \Omega)$ for space domains. In particular, we prove that

$$c_p(n, \Omega) \geq 2 \min\{1, p/n\} M_0(\Omega),$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

One of the main results of the present paper is a direct extension of the original Hardy inequality to the case $n \geq 2$. More precisely, in Section 4 the following assertion is proved: if $s > n$, then for arbitrary open sets $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ and $p \geq 1$ the sharp inequality

$$c_p(s, \Omega) \leq \frac{p}{s-n}$$

is valid. This completes the following known facts: J.L. Lewis [31] discovered that there is $c_p(p, \Omega) < +\infty$ for any open set $\Omega \subset \mathbb{R}^n$, whenever $p > n$. A.Wannebo [44] proved a generalization of this assertion: if $p > n$ and $s > p - \varepsilon(p, n)$ for a convenient $\varepsilon(p, n) > 0$, then $c_p(s, \Omega) < +\infty$ for any open set $\Omega \subset \mathbb{R}^n$.

We find that some Hardy type inequalities are connected with isoperimetric properties of open sets $\Omega \subset \mathbb{R}^n$. Some theorems in this direction are given in Sections 5 and 6. For instance, in Section 5 the constant

$$c(p, \Omega) = sup\left\{\|f/\delta\|_{L^n(\Omega)} : \ f \in C_0^\infty(\Omega), \ \|\nabla f\|_{L^p(\Omega)} = 1\right\}, \ p \in (n, \infty),$$

is considered. For open sets with finite volume $|\Omega| = mes\,\Omega$ we prove the inequalities

$$|\Omega|^{1/n-1/p} \leq c(p, \Omega) \leq \frac{p}{p-n}\ |\Omega|^{1/n-1/p}.$$

We extend the above results on $c_p(s, \Omega)$ and $c(p, \Omega)$ to some other Hardy type inequalities with explicit estimates of all constants in function of parameters and simple geometric quantities of Ω. In particular, we obtain an improved version of a Hardy type inequality by H.Brezis and M.Marcus [13] in convex domains and give its generalizations (see Sections 4, 5 and 6).

For instance, in Section 6 we prove that

$$c_p(s, \Omega) \geq \frac{p}{s-1}, \quad (p \geq 1, s > 1)$$

for any bounded open set $\Omega \subset \mathbb{R}^n$ with finite boundary surface area in the sense of Minkowski. On the other hand, for parameters $p \geq 1$, $s > 1$

and convex open sets $\Omega \subset \mathbb{R}^n$ we prove that any function $f \in C_0^\infty(\Omega)$ satisfies the inequality

$$\int_\Omega \frac{|f|^p}{\delta^s}dx + \frac{1}{(s-1)\delta_0^s}\int_\Omega |f|^p dx \leq \left(\frac{p}{s-1}\right)^p \int_\Omega \frac{|\nabla f|^p}{\delta^{s-p}}dx,$$

where $\delta = \mathrm{dist}(x, \partial\Omega)$, $\delta_0 = \sup\{\delta(x) : x \in \Omega\}$.

The main results of the present paper were announced in our talks in the International Conference "Geometric Analysis and its Applications", Volgograd State University, (2004) (see [5]), in the International Conference and workshop dedicated to the centennial of Sergei Mikhailovich Nikolskii, Russian Academy of Sciences, Moscow (2005) (see [6]) and in the 13-th Saratov winter school on the function theory and its applications, Saratov State University, (2006) (see [7]).

2. Bilateral estimates of Hardy's constant for plane open sets with uniformly perfect boundary

In [19] Fernández observed that Pommerenke's capacity density condition [40] is equivalent to Ancona's condition [2] on domains with strong barrier. This leads to the following excellent fact.

Theorem 1. *If Ω is a plane domain then the Hardy constant*

$$c_2(2,\Omega) = \sup\left\{\left\|\frac{f}{\mathrm{dist}(.,\partial\Omega)}\right\|_{L^2(\Omega)} : f \in C_0^\infty(\Omega),\ \|\nabla f\|_{L^2(\Omega)} = 1\right\}$$

is finite if and only if the boundary set $\partial\Omega$ is uniformly perfect.

One can find this result and many important characterizations of uniformly perfect sets in the recent book by Garnett and Marshall [22], see Page 119 and Pages 343-345. Also, it is known that $c_2(2,\Omega) \geq 2$ for any domain with smooth boundary and $c_2(2,\Omega) = 2$ for convex domains Ω (see [16]). Moreover, if Ω is a simply connected domain in $\mathbb{C}$ then $c_2(2,\Omega) \leq 4$ (see [2], [3], [11] and [16]). In the general case, for instance, in the case when Ω is not a finitely connected domain, explicit estimates of $c_2(2,\Omega)$ are unknown.

In this section, we shall prove L^p-version ($1 \leq p < \infty$) of Theorem 1 with bilateral explicit estimates of the Hardy constant using a simple geometrical parameter of Ω. In particular, we give a direct proof of Theorem 1.

Let Ω be an open set in the complex plane $\mathbb{C}$ such that $\Omega \neq \mathbb{C}$. For any fixed $p \in [1, \infty)$ we consider Hardy's inequality

$$\iint_\Omega \frac{|f|^p}{\operatorname{dist}(z,\partial\Omega)^2} dxdy \le c_p(2,\Omega)^p \iint_\Omega \frac{|\nabla f|^p}{\operatorname{dist}(z,\partial\Omega)^{2-p}} dxdy, \quad \forall f \in C_0^\infty(\Omega),$$

where $z = x + iy$ and $c_p(2, \Omega)$ is the minimum possible constant that generalizes $c_2(2, \Omega)$.

Following to [12] and [40], we characterize the open set Ω by moduli of ring domains that separate $\partial\Omega$. More precisely, we define the maximum modulus

$$M_0(\Omega) := \sup \frac{1}{2\pi} \log \frac{R(A)}{r(A)},$$

where the supremum is taken over all annuli A such that

$$A = \{z \in \mathbb{C} : r(A) < |z - z_0| < R(A)\} \subset \Omega \quad \textit{and} \quad z_0 \in \partial\Omega.$$

We take $M_0(\Omega) = 0$ by definition, when there is no circle in Ω with center on $\partial\Omega$. We say that $\partial\Omega$ is uniformly perfect if $M_0(\Omega) < \infty$.

In the sequel, we need the constant

$$a_0 = \frac{\Gamma(1/4)^{1/4}}{4\pi^2} \approx 4.38 \tag{1}$$

from the sharp form of Landau's theorem (see [26] and [29]).

The main result of this section is the following assertion.

Theorem 2. *If $1 \le p < \infty$ and Ω is an open and proper subset of $\mathbb{C}$, then*

$$\min\{2, p\}\ M_0(\Omega) \le c_p(2, \Omega) \le 2p\ (\pi M_0(\Omega) + a_0)^2. \tag{2}$$

In particular, the Hardy constant $c_p(2, \Omega)$ is finite if and only if $\partial\Omega$ is uniformly perfect.

Proof of Theorem 2. First we prove the lower estimate for $c_p(2, \Omega)$. Clearly, it is sufficient to consider the case, when $0 < M_0(\Omega) \le \infty$ and $0 < c_p(2, \Omega) < \infty$.

We shall examine the cases $p \ge 2$ and $p < 2$ separately. Suppose first that $2 \le p < \infty$ and $c_p(2, \Omega) < 2M_0(\Omega)$ for an open and proper subset of $\mathbb{C}$. From the definition of $M_0(\Omega)$ it follows that there is an annulus $A = \{z \in \mathbb{C} : r(A) < |z - z_0| < R(A)\} \subset \Omega$ such that $z_0 \in \partial\Omega$ and

$$\infty > \log \frac{R(A)}{r(A)} > \pi c_p(2, \Omega).$$

Without loss of generality we can suppose that $z_0 = 0, R(A) = 1$ and $r(A) = \varepsilon \in (0, 1)$, since $c_p(2, \Omega)$ is invariant under linear transformations

of Ω. We have
$$M_0 := \frac{1}{2\pi}\log\frac{1}{\varepsilon} > c_p(2,\Omega)/2$$
and
$$\iint_A \frac{|f|^p}{\operatorname{dist}(z,\partial\Omega)^2}dxdy \le c_p(2,\Omega)^p \iint_A \frac{|\nabla f|^p}{\operatorname{dist}(z,\partial\Omega)^{2-p}}dxdy, \quad \forall\, f \in C_0^\infty(A).$$
Using the polar coordinates, functions $f(r,\theta) = v(r)$ with $v \in C_0^\infty(\varepsilon, 1)$ and the estimate $(z,\partial\Omega) \le |z|$, we obtain
$$\int_\varepsilon^1 \frac{|v(r)|^p r dr}{r^2} \le c_p(2,\Omega)^p \int_\varepsilon^1 \frac{|v'(r)|^p r dr}{r^{2-p}}, \quad \forall\, v \in C_0^\infty(\varepsilon, 1).$$
By the change $r = \varepsilon\exp(2M_0 t)$ and $v(r) = g(t)$ of variables this is equivalent to the Wirtinger type inequality (see [25])
$$\int_0^\pi |g(t)|^p\, dt \le \frac{c_p(2,\Omega)^p}{2^p M_0^p}\int_0^\pi |g'(t)|^p dt, \quad \forall\, g \in C_0^\infty(0,\pi).$$
Approximating $g_0(t) = \sin t$ by functions $g \in C_0^\infty(0,\pi)$, we get
$$c_p(2,\Omega)^p \ge 2^p M_0^p \int_0^\pi |\sin t|^p dt \Big/ \int_0^\pi |\cos t|^p dt = 2^p M_0^p,$$
which contradicts to the assumption $c_p(2,\Omega) < 2M_0$. Hence, $2M_0(\Omega) \le 2M_0 \le c_p(2,\Omega)$ in the case $2 \le p < \infty$.

In the case $1 \le p < 2$ and $c_p(2,\Omega) < \infty$, we combine the Hardy and Hölder inequalities in the following way
$$\begin{aligned}\iint_\Omega \frac{|f|^2}{\operatorname{dist}(z,\partial\Omega)^2}dxdy &= \iint_\Omega \frac{(|f|^{2/p})^p}{\operatorname{dist}(z,\partial\Omega)^2}dxdy \\ &\le \left(\frac{2}{p}\right)^p c_p(2,\Omega)^p \iint_\Omega \frac{|f|^{2-p}|\nabla f|^p}{\operatorname{dist}(z,\partial\Omega)^{2-p}}dxdy \\ \le \left(\frac{2}{p}\right)^p c_p(2,\Omega)^p &\left(\iint_\Omega \frac{|f|^2}{\operatorname{dist}(z,\partial\Omega)^2}dxdy\right)^{1-p/2}\left(\iint_\Omega |\nabla f|^2\, dxdy\right)^{p/2}.\end{aligned}$$
It follows that $c_2(2,\Omega) \le \frac{2}{p}c_p(2,\Omega)$. As is proved that $c_2(2,\Omega) \ge 2M_0(\Omega)$, we get
$$c_p(2,\Omega) \ge pM_0(\Omega),$$
when $p \in [1,2)$.

Now we suppose that $M_0(\Omega) < \infty$ and we prove the upper estimate. Clearly, the condition $M_0(\Omega) < \infty$ assure that $\partial\Omega$ has no isolated point. Also, it is sufficient to obtain the upper inequality in (2) for connected components of Ω.

Since $\mathbb{C}\backslash\Omega \ne \emptyset$ and $\partial\Omega$ has no isolated point, any connected component of Ω is a hyperbolic domain in $\mathbb{C}$, i.e. its boundary has more than one point in $\mathbb{C}$. Without loss of generality we can suppose that Ω itself is a hyperbolic domain in $\mathbb{C}$. Let λ_Ω be the density of the Poincaré metric on Ω with curvature -4 (see [1], [9]).

Let $p > 1$ and let $f \in C_0^\infty(\Omega)$. Since $p > 1$, we have that $|f|^p \in C_0^1(\Omega)$ and $|\nabla |f|^p| = p|f|^{p-1}|\nabla f|$. Using the Liouville equation in the form

$$\frac{\Delta \log \lambda_\Omega(z)^{-1}}{\lambda_\Omega(z)^2} = -4, \ \ z = x + iy \in \Omega,$$

and the Green formula

$$\iint_\Omega [u\Delta v + (\nabla u, \nabla v)] dxdy = 0$$

for $v = \log \lambda_\Omega^{-1}$ and $u = |f|^p, f \in C_0^\infty(\Omega)$, we obtain

$$4 \iint_\Omega |f|^p \lambda_\Omega(z)^2 \ dxdy = p \iint_\Omega |f|^{p-1} \lambda_\Omega(z)(\nabla |f|, \nabla \lambda_\Omega(z)^{-1}) \ dxdy.$$

Combining this with the Hölder inequality

$$\iint_\Omega |f|^{p-1} \lambda_\Omega |(\nabla |f|, \nabla \lambda_\Omega^{-1})| \ dxdy$$

$$\leq \left(\iint_\Omega |f|^p \lambda_\Omega^2 \ dxdy \right)^{1-1/p} \ \left(\iint_\Omega \lambda_\Omega^{2-p} |(\nabla f, \nabla \lambda_\Omega^{-1})|^p \ dxdy \right)^{1/p},$$

we immediately get

$$\iint_\Omega |f|^p \lambda_\Omega^2 \ dxdy \leq \left(\frac{p}{4}\right)^p \iint_\Omega \lambda_\Omega^{2-p} |(\nabla f, \nabla \lambda_\Omega^{-1})|^p \ dxdy \qquad (3)$$

for any $f \in C_0^\infty(\Omega)$. Since this inequality is proved for any $p \in (1, \infty)$, letting $p \to 1$ for a fixed $f \in C_0^\infty(\Omega)$ we obtain that (3) is true in the case $p = 1$, too.

Using (3) and Osgood's inequality [39]

$$\lambda_\Omega(z) |\nabla \lambda_\Omega(z)^{-1}| \leq \frac{2}{\operatorname{dist}(z, \partial\Omega)}, \ \ z = x + iy \in \Omega,$$

one gets

$$\iint_\Omega |f|^p \lambda_\Omega(z)^2 \ dxdy \leq \left(\frac{p}{2}\right)^p \iint_\Omega \frac{\lambda_\Omega(z)^{2-2p} |\nabla f|^p}{\operatorname{dist}(z, \partial\Omega)^p} \ dxdy$$

Hence, for any $p \in [1, \infty)$ and any $f \in C_0^\infty(\Omega)$,

$$\alpha(\Omega)^2 \iint_\Omega \frac{|f|^p}{\operatorname{dist}(z, \partial\Omega)^2} \ dxdy \leq \frac{(p/2)^p}{\alpha(\Omega)^{2p-2}} \iint_\Omega \frac{|\nabla f|^p}{\operatorname{dist}(z, \partial\Omega)^{2-p}} \ dxdy, \quad (4)$$

where

$$\alpha(\Omega) := \inf\{\lambda_\Omega(z) \operatorname{dist}(z, \partial\Omega) : z \in \Omega\}.$$

Beardon and Pommerenke [12] proved that the condition $M_0(\Omega) < \infty$ holds if and only if $\alpha(\Omega) > 0$ and, in particular,

$$\alpha(\Omega)^{-1} \leq 2\pi M_0(\Omega) + 2a_0, \tag{5}$$

where a_0 is the constant from Landau's theorem. By [26] and [29] it is known that the sharp value of a_0 is given by formula (1).

From (4) it follows that

$$c_p(2,\Omega) \leq \frac{p}{2\ \alpha(\Omega)^2}. \tag{6}$$

Evidently, inequalities (5) and (6) imply the upper bound in (2).

The proof of Theorem 2 is complete.

Consider the upper bound corresponding to the case $M_0(\Omega) = 0$ in (2).

Corollary 2.1. *Let Ω be an open and proper subset of $\mathbb{C}$. If there is no circle in Ω with center on $\partial\Omega$, then $c_p(2,\Omega) < 38.4\ p$.*

Let us mention that $M_0(\Omega) = 0$ for any simply connected domain, but the converse is not true. The family $\{\Omega : M_0(\Omega) = 0\}$ is a large collection of open sets $\Omega \subset \mathbb{C}$ and it contains domains of arbitrary connectivity. For example, *the equality $M_0(\Omega_0 \setminus \overline{K}) = 0$ holds for domains satisfying the following conditions:*

(1) Ω_0 *is an open set in* $\mathbb{C}$ *such that* $\sup\{\mathrm{dist}(z, \partial\Omega_0) : z \in \Omega_0\} = 1$, *in particular,* Ω_0 *is a stripe with width* 2*;*

(2) $K = \bigcup_{m=1}^{\infty} C_m$, *where* C_m *are continuums (connected compact sets) such that* $\mathrm{diam}\ C_m \geq 2$ *for all* $m \geq 1$*;*

(3) $K \subset \Omega_0$ *and* $\Omega_0 \setminus \overline{K}$ *is nonempty.*

The Pommerenke condition on the capacity density for $\mathbb{C} \setminus \Omega$ has the form

$$C(\Omega) := \inf\left\{\frac{cap\ (\{|z - z_o| \leq r\} \bigcap (\mathbb{C} \setminus \Omega))}{r} : z_0 \in \partial\Omega,\ 0 < r < \infty\right\} > 0\,,$$

where *cap* E is the logarithmic capacity of E. In our notations, Pommerenke's estimates of the capacity density (see the proof of Theorem 1 in [40]) can be written as

$$M_0(\Omega) \leq \frac{1}{2\pi}\ \log\frac{1}{C(\Omega)} \leq 2M_0(\Omega) + \frac{4\ \log 2}{\pi}\ . \tag{7}$$

From (7) and Theorem 2, one immediately obtains the following assertion.

Corollary 2.2. *If $1 \le p < \infty$ and Ω is an open and proper subset of $\mathbb{C}$, then*

$$\frac{1}{4\pi}\ \log\frac{1}{C(\Omega)} - \frac{\log 4}{\pi} \le c_p(2,\Omega) \le \frac{p}{2}\ \left(\ \log\frac{1}{C(\Omega)} + 2a_0\right)^2.$$

We complete this section by three examples considering domains of the form $B(0,3)\backslash E_j = \{z \in \mathbb{C} : |z| < 3\}\backslash E_j$ with perfect boundaries.

Example 1. Suppose that $E_1 = \bigcup_{m=1}^{\infty} K_m \bigcup\{0\}$, where

$$K_m = \{z = x + iy \in \mathbb{C} : y = 0,\ m^{-m} \le x \le 2m^{-m}\}.$$

The domain $\Omega_1 = B(0,3)\backslash E_1$ contains the annuli

$$A_m = \{z \in \mathbb{C} : 2(m+1)^{-m-1} < |z| < m^{-m}\}$$

and

$$\frac{R(A_m)}{r(A_m)} = \frac{m+1}{2}\left(1+\frac{1}{m}\right)^m \to \infty \text{ as } m \to \infty.$$

Consequently, $M_0(\Omega_1) = c_p(2,\Omega_1) = \infty$.

Example 2. Now, we consider $E_2 = \bigcup_{m=1}^{\infty} L_{2m-1} \bigcup\{0\}$, where

$$L_{2m-1} = \{z = x + iy \in \mathbb{C} : y = 0,\ 3^{-2m+1} \le x \le 3^{-2m+2}\}.$$

For any annulus A in $\Omega_2 = B(0,3)\backslash E_2$ with center on $\partial\Omega_2$ we have $R(A)/r(A) \le 3$. It is an easy task to show that $2\pi M_0(\Omega_2) = \log\ 3$. Accordingly, $c_p(2,\Omega_2) \le (8.76 + \log 3)^2 p/2 < 48\ p$.

Example 3. Let E_3 be the classical Cantor set. In [12] estimates for $\alpha(\mathbb{C}\backslash E_3)$ are proved. We consider

$$\Omega_3 = B(0,3)\backslash E_3.$$

It is easy to show that $M_0(\Omega_3) = M_0(\Omega_2) = (2\pi)^{-1}\log 3$. Thus, $c_p(2,\Omega_3) < 48\ p$.

3. Other results connected with uniformly perfect sets, a conjecture in the spatial case

We shall extend Theorem 1 to certain functionals connected with Rellich's constants and discuss a generalization of Theorem 1 to the space domains.

First, we consider two following quantities used in [3] for simply connected domains and related to Rellich's constants (compare [18], [36] and [42]).

Let Ω be an open and proper subset of $\mathbb{C}$. We define

$$\kappa_1(\Omega)^2 := \sup\left\{\left\|\frac{f}{\operatorname{dist}(.,\partial\Omega)}\right\|_{L^2(\Omega)} : f \in C_0^\infty(\Omega), \right.$$
$$\left. \|\operatorname{dist}(.,\partial\Omega)\Delta f\|_{L^2(\Omega)} = 1\right\}$$

and

$$\kappa_2(\Omega) := \sup\left\{\|\nabla f\|_{L^2(\Omega)} : f \in C_0^\infty(\Omega),\ \|\operatorname{dist}(.,\partial\Omega)\Delta f\|_{L^2(\Omega)} = 1\right\},$$

where $z = x + iy$ and Δ is the Laplace operator.

In [3] it is proved that $\kappa_1(\Omega) \le 4$ and $\kappa_2(\Omega) \le 4$ for any simply connected domain Ω. In the next theorem we give bilateral estimates of $\kappa_1(\Omega)$ and $\kappa_2(\Omega)$ for open sets with uniformly perfect boundary. Also, we obtain an improvement of the upper bound of $c_2(2,\Omega)$, $\kappa_1(\Omega)$ and $\kappa_2(\Omega)$ for doubly connected domains.

Theorem 3. *If Ω is an open and proper subset of $\mathbb{C}$, then the quantity $\kappa_j(\Omega)$ $(j = 1, 2)$ is finite if and only if $\partial\Omega$ is a uniformly perfect set. Moreover,*

$$2M_0(\Omega) \le \kappa_1(\Omega) \le 4(\pi M_0(\Omega) + a_0)^2,$$
$$2M_0(\Omega) \le \kappa_2(\Omega) \le 4(\pi M_0(\Omega) + a_0)^2.$$

If Ω is a doubly connected domain in $\mathbb{C}$, then

$$M_0(\Omega) \le c_2(2,\Omega)/2 \le \pi M_0(\Omega) + a_0,$$
$$M_0(\Omega) \le \kappa_1(\Omega)/2 \le \pi M_0(\Omega) + a_0,$$
$$M_0(\Omega) \le \kappa_2(\Omega)/2 \le \pi M_0(\Omega) + a_0.$$

Proof of Theorem 3. Suppose that $\partial\Omega$ is uniformly perfect and $f \in C_0^\infty(\Omega)$. Using the Green formula and the Cauchy - Schwartz inequality one gets

$$\iint_\Omega |\nabla f|^2 dxdy = -\iint_\Omega f\Delta f\ dxdy$$
$$\le \left(\iint_\Omega |f|^2 \operatorname{dist}(z,\partial\Omega)^{-2}\ dxdy\right)^{1/2} \left(\iint_\Omega \operatorname{dist}(z,\partial\Omega)^2 |\Delta f|^2\ dxdy\right)^{1/2}.$$

This inequality and the definitions of $c_2(2,\Omega)$ and $\kappa_1(\Omega)$ imply

$$\iint_\Omega |\nabla f|^2 dxdy \le \kappa_1(\Omega)^2,\ \ \iint_\Omega |f|^2 \operatorname{dist}(z,\partial\Omega)^{-2}\ dxdy \le c_2(2,\Omega)^4$$

for any $f \in C_0^\infty(\Omega)$ satisfying

$$\iint_\Omega \operatorname{dist}(z,\partial\Omega)^2 |\Delta f|^2\ dxdy = 1.$$

Consequently,

$$\kappa_2(\Omega) \leq \kappa_1(\Omega) \leq c_2(2,\Omega). \tag{8}$$

Inequalities (2) and (8) immediately give the upper bounds of $\kappa_1(\Omega)$ and $\kappa_2(\Omega)$ in the general case. Moreover, it is known (see, for instance, [20], Lemma 1.1) that the inequality

$$\iint_\Omega |f|^2 \lambda_\Omega^2 \, dxdy \leq \iint_\Omega |\nabla f|^2 \, dxdy, \quad \forall f \in C_0^\infty(\Omega), \tag{9}$$

is valid for any simply or doubly connected hyperbolic domain Ω. It is obvious that inequalities (5), (8) and (9) imply the upper bounds in Theorem 3 for doubly connected domains.

Thank to (8), we have to prove the lower estimates of Theorem 3 for $\kappa_2(\Omega)$, only. To this end we assume that $\kappa_2(\Omega) < 2M_0(\Omega)$. Without loss of generality we can suppose that $0 \in \Omega$ and there exists an annulus $A = \{z \in \mathbb{C} : \varepsilon < |z| < 1\}$ such that $A \subset \Omega$ and $M_0 := \frac{1}{2\pi} \log \frac{1}{\varepsilon} > \kappa_2(\Omega)/2$. One has

$$\iint_A |\nabla f|^2 \, dxdy \leq \kappa_2(\Omega)^2 \iint_A \operatorname{dist}(z,\partial\Omega)^2 |\Delta f|^2 \, dxdy, \quad \forall \ f \in C_0^\infty(A)$$

and $\operatorname{dist}(z,\partial\Omega) < |z|$ for $z \in A$. Consequently,

$$\int_\varepsilon^1 v'(r)^2 r \, dr \leq \kappa_2(\Omega)^2 \int_\varepsilon^1 (rv''(r) + v'(r))^2 r \, dr$$

for radial functions $v(r) = f(r,\theta), v \in C_0^\infty(\varepsilon, 1)$.

After the changes $r = \varepsilon \exp(2M_0 t)$ and $v(r) = g(t)$, the last inequality can be written as the following Wirtinger type inequality

$$\int_0^\pi |g'(t)|^2 \, dt \leq \frac{\kappa_2(\Omega)^2}{(2M_0)^2} \int_0^\pi |g''(t)|^2 dt, \quad \forall \ g \in C_0^\infty(0,\pi).$$

Consequently, $\kappa_2(\Omega) \geq 2M_0$.

This completes the proof of Theorem 3.

Finally, we consider the spatial case. Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \neq \mathbb{R}^n$. Suppose that $1 \leq p < \infty, 1 < s < \infty$ and consider the Hardy constant

$$c_p(s,\Omega) = \sup \left\{ \left\| \frac{f}{\delta^{s/p}} \right\|_{L^p(\Omega)} : \ f \in C_0^\infty(\Omega), \ \left\| \frac{\nabla f}{\delta^{s/p-1}} \right\|_{L^p(\Omega)} = 1 \right\},$$

where $\delta = \operatorname{dist}(x,\partial\Omega)$.

It is known that $c_p(s,\Omega) < \infty$ for any domain Ω with Lipschitz boundary (see [38]). More general families of domains with finite $c_p(s,\Omega)$ are given in the papers by Ancona [2] (case $p = s = 2$), Lewis [31] (case $p = s > 1$) and Wannebo [44] (case $p \geq 1, s > 1$).

In the case $n \geq 3$, as is indicated in the paper [28] of Järvi and Vuorinen, the concept of uniformly perfect sets is not equivalent to the known density concepts used in the theory of Hardy's inequalities in higher dimensions.

For an open set $\Omega \subset \mathbb{R}^n$ ($\Omega \neq \mathbb{R}^n, n \geq 3$) we consider the quantity $M_0(\Omega)$ defined as in the planar case. More precisely, let

$$M_0(\Omega) := \sup \frac{1}{2\pi} \log \frac{R(A)}{r(A)},$$

where the supremum is taken over all ring domains A such that $A = \{x \in \mathbb{R}^n : r(A) < |x - x_0| < R(A)\} \subset \Omega$ and $x_0 \in \partial\Omega$. If such a ring domain A doesn't exist, then $M_0(\Omega) = 0$. If $M_0(\Omega) < \infty$ then the set $\partial\Omega$ is said to be uniformly perfect (compare [28]).

It is clear that the conditions $M_0(\Omega) < \infty$ and $c_p(s, \Omega) < \infty$ are not equivalent in the case $s \neq n$. For instance, if $s > n$, then $c_p(s, \Omega) \leq p/(n - s)$ for any open set $\Omega \subset \mathbb{R}^n$ ($\Omega \neq \mathbb{R}^n$) (see Theorem 5, below). If $1 < s < n$ and B_0 is a punctured ball, then it is easy to show that $M_0(B_0) = \infty$ but $c_p(s, B_0)$ is finite for any $p \in [1, \infty)$.

We conjecture that a direct comparison of $M_0(\Omega)$ and the Hardy constant is possible in the case $s = n$. At least, if $c_p(n, \Omega)$ is finite, then $\partial\Omega$ is uniformly perfect. Evidently, the last assertion is a consequence of the following theorem.

Theorem 4. *If $p \in [1, \infty)$, $n \geq 3$ and Ω is an open and proper subset of $\mathbb{R}^n$, then*

$$c_p(n, \Omega) \geq 2 \min\{1, p/n\} M_0(\Omega). \tag{10}$$

Proof of Theorem 4. We will follow the proof of lower estimates in Theorem 2 with some necessary changes. In particular, we consider the cases $p \geq n$ and $p < n$ separately.

Assume that $n \leq p < \infty$, $0 < M_0(\Omega) \leq \infty$ and $c_p(n, \Omega) < 2M_0(\Omega)$. Without loss of generality we can conclude that there is a positive constant ε such that

$$c_p(n, \Omega) < 2M_0 := \frac{1}{\pi} \log \frac{1}{\varepsilon} < \infty$$

and

$$A = \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\} \subset \Omega, \quad (0, ..., 0) \in \partial\Omega.$$

Using radial functions, the spherical coordinates and the obvious inequality $\operatorname{dist}(x, \partial\Omega) \leq |x|$ for $x \in A$, we can write (compare the proof of

Theorem 2 for details)

$$\int_\varepsilon^1 \frac{|v(r)|^p r^{n-1} dr}{r^n} \le c_p(n,\Omega)^p \int_\varepsilon^1 \frac{|v'(r)|^p r^{n-1} dr}{r^{n-p}}, \quad \forall\, v \in C_0^\infty(\varepsilon, 1).$$

Using the changes $r = \varepsilon \exp(2M_0 t)$, $u(r) = g(t)$ and straightforward computations, we obtain

$$\int_0^\pi |g(t)|^p \, dt \le \frac{c_p(n,\Omega)^p}{2^p M_0^p} \int_0^\pi |g'(t)|^p \, dt, \quad \forall\, g \in C_0^\infty(0,\pi).$$

Hence, $c_p(n,\Omega)/(2M_0) \ge 1$. This completes the proof of (10) in the case $n \le p$.

In the case $1 \le p < n$ and $c_p(n,\Omega) < \infty$, we again combine the Hardy and Hölder inequalities in the following way

$$\int_\Omega \frac{|f|^n}{\delta^n} dx = \int_\Omega \frac{(|f|^{n/p})^p}{\delta^n} dx$$

$$\le \left(\frac{n}{p}\right)^p c_p(n,\Omega)^p \int_\Omega \frac{|f|^{n-p} |\nabla f|^p}{\delta^{n-p}} dx$$

$$\le \left(\frac{n}{p}\right)^p c_p(n,\Omega)^p \left(\int_\Omega \frac{|f|^n}{\delta^n} dx\right)^{1-p/n} \left(\int_\Omega |\nabla f|^n \, dx\right)^{p/n},$$

where $\delta = \operatorname{dist}(x, \partial\Omega)$. It follows that $c_n(n,\Omega) \le \frac{n}{p} c_p(n,\Omega)$. Since $c_n(n,\Omega) \ge 2M_0(\Omega)$, we get

$$c_p(n,\Omega) \ge \frac{p}{n} M_0(\Omega),$$

when $p \in [1, n)$.

The proof of Theorem 4 is complete.

It seems to be natural the following generalization of Theorem 1.

Conjecture. *The equivalence* $\{c_p(n,\Omega) < \infty \Longleftrightarrow M_0(\Omega) < \infty\}$ *is true in the spatial case, too.*

Remark. In the literature on uniformly perfect sets one can find several definitions of maximum modulus of Ω. To define $M_0(\Omega)$ we have used ring domains in Ω with centers on $\partial\Omega$. This directly is connected with the basic definition of Ch. Pommerenke in [40]. One can consider a slightly different parameter $M(\Omega)$ defined as the maximum modulus of ring domains which are in Ω and separate $\partial\Omega \bigcup \{\infty\}$ (compare [12] and [28]). It is easy to show that

$$M_0(\Omega) \le M(\Omega) \le M_0(\Omega) + \frac{1}{2\pi} \log 3.$$

4. Solution of a problem by J.L. Lewis and A. Wannebo

In [31], J. L. Lewis proved that there is $c_p(p,\Omega) < +\infty$ for any open set $\Omega \subset \mathbb{R}^n$ if $p > n$. A. Wannebo [44] proved a generalization of this assertion: if $p > n$ and $s > p - \varepsilon(p,n)$ for a convenient $\varepsilon(p,n) > 0$, then $c_p(s,\Omega) < +\infty$ for any open set $\Omega \subset \mathbb{R}^n$.

We find that the single condition $s > n$ assure that $c_p(s,\Omega) < +\infty$ for any open set $\Omega \subset \mathbb{R}^n$ and any $p \geq 1$. Surprisingly, the constant has a simple upper bound in this case. More precisely, we obtain the following extension of the one dimensional Hardy inequality.

Theorem 5. *Let Ω be an open and proper subset of $\mathbb{R}^n$. If $p \geq 1$ and $s > n$ then*

$$\int_\Omega \frac{|f|^p}{\delta^s} dx \leq \left(\frac{p}{s-n}\right)^p \int_\Omega \frac{|\nabla f|^p}{\delta^{s-p}} dx, \quad \forall f \in C_0^\infty(\Omega), \tag{11}$$

where $\delta = \delta(x) = \operatorname{dist}(x, \partial\Omega)$.

The constant $p^p(s-n)^{-p}$ in (11) is the best one for many sets Ω. For example, it is sharp for every Ω of the form $\Omega_0 \setminus \{x_0\}$, where Ω_0 is an open set in $\mathbb{R}^n$ and $x_0 \in \Omega_0$.

From Theorem 5 it follows that the basic inequality of Hardy

$$\int_0^{+\infty} \frac{|u(t)|^2}{t^2} dt \leq 4 \int_0^{+\infty} |u'(t)|^2 dt, \quad u' \in L^2, \quad u(0) = 0,$$

has a sharp analog in $\mathbb{R}^n$:

$$\int_\Omega \frac{|f|^{2n}}{\delta^{2n}} dx \leq 4^n \int_\Omega |\nabla f|^{2n} dx, \quad \forall f \in C_0^\infty(\Omega),$$

which is valid for any open set $\Omega \subset \mathbb{R}^n$ ($\Omega \neq \mathbb{R}^n$).

We will prove that equality in (11) is not attained in the corresponding Sobolev space if $f \not\equiv 0$ and

$$\delta_0 = \delta_0(\Omega) := \sup\{\operatorname{dist}(x, \partial\Omega) \, : x \in \Omega\} < +\infty.$$

More precisely, we prove the following refined version of Theorem 5.

Theorem 6. *Let Ω be an open and proper subset of $\mathbb{R}^n$. If $p \geq 1$ and $s > n$, then*

$$\int_\Omega \frac{|f|^p}{\delta^s} dx + \frac{1}{(s-1)\delta_0^s} \int_\Omega |f|^p dx \leq \left(\frac{p}{s-n}\right)^p \int_\Omega \frac{|\nabla f|^p}{\delta^{s-p}} dx, \quad \forall f \in C_0^\infty(\Omega), \tag{12}$$

where $\delta = \operatorname{dist}(x, \partial\Omega)$, $\delta_0 = \sup\{\delta(x) \, : x \in \Omega\}$.

Proof of Theorems 5 *and* 6. Let Ω be an open and proper subset of $\mathbb{R}^n$. For given $f \in C_0^\infty(\Omega)$ we use the following approximation of Ω.

For $h \in (0,1)$ we consider the simplest covering of $\mathbb{R}^n$ by cubes

$$Q_{h,z} = [0,h]^n + hz, \qquad z \in \mathbf{Z}^n,$$

and define the finite set

$$\mathbf{Z}^n(\Omega,h) = \{z \in \mathbf{Z}^n \, : Q_{h,z} \subset \Omega \cap \{x \in \mathbb{R}^n \, : |x| < 1/h\}\}$$

and the following approximation of Ω:

$$\Omega_h = \operatorname{int} \cup_{z \in \mathbf{Z}^n(\Omega,h)} Q_{h,z}.$$

For a given $f \in C_0^\infty(\Omega)$ it is clear that it suffices to prove (11) and (12) with $\Omega = \Omega_h$ and any $h \in (0,1)$. By the change $y = x/h, \;\; x \in \Omega_h$, of variables we also see that (11) and (12) for $\Omega = \Omega_h$ and $\Omega = \Omega_1$ are equivalent. Thus, we have to prove (11) and (12) for a set of the form

$$\Omega_1 = \operatorname{int} \cup_{j=1}^m ([0,1]^n + z_j), \qquad z_j \in \mathbf{Z}^n.$$

Let S be a q-face of Q_{1,z_j}. Suppose that $S \subset \partial\Omega_1$ and define the following subset of Ω_1:

$$Q(S) = \{x \in \mathbb{R}^n \, : x' \in \operatorname{int} S, B(x, |x - x'|) \subset \Omega_1\},$$

where $B(x, |x-x'|)$ is the ball $\{y \in \mathbb{R}^n \, : |y - x| < |x' - x|\}$. We have to note that the interior of S is taken in $\mathbb{R}^q \supset S$ and, by definition, $\operatorname{int} S = S$ if S is a 0-face i.e. a point.

Suppose that $Q(S) \neq \emptyset$, this is always the case if S is a $(n-1)$-face and $S \subset \partial\Omega_1$. The set $Q(S) \neq \emptyset$ is starlike with respect to S, i.e. $x' + t(x - x') \in Q(S)$ for every $x' \in \operatorname{int} S$ and all $t \in (0,1)$ if $|x - x'| = \operatorname{dist}(x, \partial\Omega_1)$ and $x \in Q(S)$. Up to a rotation, $Q(S) \subset S \times \mathbb{R}_+^{n-q}$, and

$$\overline{Q}(S) = S \times \{t \in \mathbb{R}_+^{n-q} \, : 0 \leq |t| \leq \varphi(\frac{t}{|t|}; S, Q)\},$$

where φ is a positive function satisfying the inequality

$$\sup \varphi \leq \sup\{\operatorname{dist}(x, \partial\Omega_1) \, : x \in \Omega_1\}.$$

If S' is a cubic j-face $(j = 0, 1, \ldots, n-1)$ and $S' \subset (\partial\Omega_1) \setminus S$ then the set

$$(S, S') := \{x \in \Omega : \operatorname{dist}(x, S) = \operatorname{dist}(x, S') \leq \operatorname{dist}(x, \partial\Omega_1)\}$$

is a bounded subset of a $(n-1)$-plane or a $(n-1)$-surface of order 2. Since $\operatorname{mes}_n(S, S') = 0$ and

$$(\partial Q(S)) \setminus S \subset \cup_{S'} (S, S'),$$

we obtain that $\mathrm{mes}_n \partial Q(S) = 0$. Consequently, for any $g \in C(\overline{\Omega}_1)$

$$\int_{\Omega_1} g(x)\,dx = \sum_{S \subset \partial\Omega_1} \int_{Q(S)} g(x)\,dx. \tag{13}$$

In the sequel we will need the notations:

$$S_+^{n-q} = \{\omega \in \mathbf{R}_+^{n-q} \, : |\omega| = 1\},$$
$$\varphi_q = \varphi(.; S, Q_1).$$

Let S be a cubic $(n-k)$-face such that $S \subset \partial\Omega_1$ and $Q(S) \neq \emptyset$, where $k = 1, 2, \ldots, n$. By Fubini's theorem, we get the following formulas, depending of the dimension of S:

if $k = 1$, then

$$\int_{Q(S)} g(x)\,dx = \int_S dx' \int_0^{\varphi_{n-1}(x')} g(x' + r\nu(x'))dr; \tag{14}$$

if $2 \leq k \leq n-1$, then

$$\int_{Q(S)} g(x)\,dx = \int_S dx' \int_{S_+^k} d\omega \int_0^{\varphi_{n-k}(\omega)} g(x' + \omega r) r^{k-1} dr; \tag{15}$$

if $k = n$ and $S = \{x'\}$, then

$$\int_{Q(S)} g(x)\,dx = \int_{S_+^n} d\omega \int_0^{\varphi_0(\omega)} g(x' + \omega r) r^{n-1} dr. \tag{16}$$

Suppose that $f \in C_0^\infty(\Omega_1)$, $p \geq 1$, $s > n$, $\delta = \delta(x) = \mathrm{dist}(x, \partial\Omega_1)$, $\delta_0 = \sup\{\delta(x) \, : x \in \Omega_1\}$. We will use (14), (15) and (16) for the function

$$g(x) = |f(x)|^p \left(\frac{1}{\delta^s(x)} + \frac{1}{(s-1)\delta_0^s} \right).$$

Since $\delta(x) = r$, $1 \leq k \leq n$ in (14)–(16), we have

$$\begin{aligned}
&\int_0^{\varphi_{n-k}} |f|^p \left(\frac{1}{r^s} + \frac{1}{(s-1)\delta_0^s} \right) r^{k-1} dr \\
&\leq \int_0^{\varphi_{n-k}} \left(t^{k-s-1} + \frac{t^{k-1}}{(s-1)\delta_0^s} \right) dt \int_0^t |f|^{p-1} |\nabla f| dr \\
&= p \int_0^{\varphi_{n-k}} |f|^{p-1} |\nabla f| A(r, \varphi_{n-k}) dr,
\end{aligned}$$

where

$$\begin{aligned}
A(r, \varphi_{n-k}) &= \frac{1}{s-k} \left(\frac{1}{r^{s-k}} - \frac{1}{\varphi_{n-k}^{s-k}} \right) + \frac{1}{k(s-1)\delta_0^s} (\varphi_{n-k}^k - r^k) \\
&\leq \frac{1}{s-k} \left(\frac{1}{r^{s-k}} - \frac{r^k}{\delta_0^s} \right)
\end{aligned}$$

$$\leq \frac{r^{k-1}}{(s-n)r^{s-1}}.$$

Therefore, one gets

$$\int_{Q(S)} |f|^p \left(\frac{1}{\delta^s(x)} + \frac{1}{(s-1)\delta_0^s} \right) dx \leq \frac{p}{s-n} \int_{Q(S)} \frac{|f|^{p-1}|\nabla f|}{\delta^{s-1}(x)} dx$$

for all $k = 1, 2, \ldots, n$. By using this and formula (13) we obtain

$$\int_{\Omega_1} |f|^p \left[\frac{1}{\delta^s(x)} + \frac{1}{(s-1)\delta_0^s} \right] dx \leq \frac{p}{s-n} \int_{\Omega_1} \frac{|f|^{p-1}|\nabla f|}{\delta^{s-1}(x)} dx.$$

This is the inequality to prove in the case $p = 1$. If $p > 1$ then we apply the Hölder inequality to get (12) for $\Omega = \Omega_1$.

The proof is complete.

Finally, we consider a simple example to get that the upper bound $(p/(s-n))^p$ in Theorems 5 and 6 is sharp (compare with the example of Hardy [25] for one dimensional case).

Let Ω_0 be an open set in $\mathbb{R}^n$ such that

$$0 \in \partial\Omega_0, \quad \{x \in \mathbb{R}^n : 0 < |x| < 3\} \subset \Omega_0.$$

Let us denote

$$X = \int_{\Omega_0} \frac{|u|^p}{\delta^s} dx, \quad Y = \int_{\Omega_0} \frac{|\nabla u|^p}{\delta^{s-p}} dx, \quad \delta = \mathrm{dist}(x, \partial\Omega_0).$$

If $p \geq 1$, $s > n$, $\varepsilon > 0$ and

$$\begin{aligned} u_\varepsilon(x) &= |x|^{(s-n+\varepsilon)/p}, \ 0 < |x| \leq 1, \\ u_\varepsilon(x) &= 2 - |x| \qquad , \ 1 < |x| \leq 2, \\ u_\varepsilon(x) &= 0 \qquad\qquad , \ 2 < |x| < \infty, \end{aligned}$$

then straightforward computations give ($\omega_{n-1} = |\partial B_1|$)

$$X(u_\varepsilon) = \frac{\omega_{n-1}}{\varepsilon} + O(1), \quad Y(u_\varepsilon) = \frac{\omega_{n-1}}{\varepsilon} \left(\frac{s-n+\varepsilon}{p} \right)^p + O(1).$$

Approximating u_ε by radial functions that belong to $C_0^\infty(B(0,3) \setminus \{0\})$ and letting $\varepsilon \to 0$ we obtain that $c_p(s, \Omega_0) \geq p(s-n)^{-1}$. Consequently, $c_p(s, \Omega_0) = p(s-n)^{-1}$ for any $p \in [1, \infty)$ and any $s \in (n, \infty)$. In particular, the constant from Theorem 5 is sharp for the punctured ball $B(0,3) \setminus \{0\}$. Since the Hardy constant is invariant under linear transformations of Ω, there exist extremal domains with given $\delta_0 \in (0, \infty]$. For instance, if $\Omega_0 = B(0, 2\delta_0) \setminus \{0\}$ and $s \in (n, \infty)$, then $c_p(s, \Omega_0) = p(s-n)^{-1}$ and $\delta_0(\Omega_0) = \delta_0$. Hence, the constant $p^p(s-n)^{-p}$ in Theorem 6 is sharp, too.

5. Boundary moments of an open set in connection with constants in Hardy type inequalities

In [3] and [4], we used α-moment of Ω about its boundary i.e. the quantity

$$\int_\Omega \operatorname{dist}(x, \partial\Omega)^\alpha dx$$

to get bilateral estimates of constants in some inequalities of Mathematical Physics. The aim of this section is to show that these moments are also connected with constants in some Hardy type inequalities.

In the sequel we will use the following consequences of formulas (14), (15) and (16):

$$\int_{Q(S)} \frac{|f|^p}{\delta^s} dx \le p \int_{Q(S)} \frac{|f|^{p-1}}{\delta^{s-1}} |\nabla f| \Phi_s(x, S) dx, \tag{17}$$

where $p \ge 1$, $s \in \mathbb{R}$, S is a cubic $(n-k)$-face and

$$\Phi_s(x, S) = \delta^{s-k} \int_\delta^{\psi_{n-k}} \frac{dt}{t^{s-k+1}} \quad (1 \le k \le n). \tag{18}$$

Evidently, by using (17) and (18) and following the proof of Theorem 5, one can give generalizations of Theorem 5 for admissible values of parameters in the inequality

$$\left(\int_\Omega \frac{|f|^q}{\delta^\alpha} dx \right)^{1/q} \le c \left(\int_\Omega \frac{|\nabla f|^p}{\delta^\beta} dx \right)^{1/p}, \qquad \forall f \in C_0^\infty(\Omega).$$

We illustrate this idea by some particular cases, only. Consider first a case, when the constant in a Hardy type inequality is connected with the volume of Ω, i. e. with the 0-moment of Ω. Let us denote

$$c(p, \Omega) = \sup \left\{ \left\| \frac{f}{\delta} \right\|_{L^n(\Omega)} : \ \|\nabla f\|_{L^p(\Omega)} = 1, \ f \in C_0^\infty(\Omega) \right\}, \ p \in (n, \infty).$$

Theorem 7. *Let Ω be an open set in $\mathbb{R}^n (n \ge 1)$ with finite volume $|\Omega| = \operatorname{mes} \Omega$. If $p > n$, then*

$$|\Omega|^{1/n-1/p} \le c(p, \Omega) \le \frac{p}{p-n} |\Omega|^{1/n-1/p}, \tag{19}$$

i.e. the following inequality

$$\left(\frac{1}{|\Omega|} \int_\Omega \frac{|f|^n}{\delta^n} dx \right)^{1/n} \le \lambda \left(\frac{1}{|\Omega|} \int_\Omega |\nabla f|^p dx \right)^{1/p}, \quad \forall f \in C_0^\infty(\Omega), \tag{20}$$

is valid with a constant λ such that

$$1 \le \lambda \le \frac{p}{p-n} .$$

Proof of Theorem 7. Let $f \in C_0^\infty(\Omega)$. Applying the Hölder inequality with the exponents p/n and $p/(p-n)$ and Theorem 5, we easily obtain

$$\left(\int_\Omega \frac{|f|^n}{\delta^n} dx\right)^{1/n} \le |\Omega|^{1/n-1/p} \left(\int_\Omega \frac{|f|^p}{\delta^p} dx\right)^{1/p}$$
$$\le \frac{p}{p-n} |\Omega|^{1/n-1/p} \left(\int_\Omega |\nabla f|^p \, dx\right)^{1/p}.$$

From this one immediately obtains the upper bounds for $c(p,\Omega)$ and λ.

Now we prove the lower estimate for $c(p,\Omega)$. According to the theorem on regularized distance functions (see V.I. Burenkov [14], P. 78, compare A.P. Calderon and A. Zigmund [15] and L.E. Fraenkel [21]), for any open set $\Omega \subset \mathbb{R}^n$ ($\Omega \ne \mathbb{R}^n$) and for any $\beta \in (0,1)$ there exists a $C^\infty(\Omega)$-function $\delta_\beta(.,\Omega)$ such that

$$\beta\delta(x,\Omega) \le \delta_\beta(x,\Omega) \le \delta(x,\Omega), \qquad |\nabla \delta_\beta(x,\Omega)| \le 1, \qquad x \in \Omega.$$

Consider the functions

$$f_{\alpha\beta\varepsilon}(x) = \begin{cases} (\delta_\beta(x,\Omega) - \varepsilon)^\alpha, & \text{if } \ x \in \Omega(\beta,\varepsilon), \\ 0, & \text{if } \ x \in \Omega \setminus \Omega(\beta,\varepsilon), \end{cases}$$

where $0 < \beta \le 1$, $1 \le \alpha < \infty$, $0 < \varepsilon < \beta\delta_0(\Omega)$ and

$$\Omega(\beta,\varepsilon) = \{x \in \Omega : \delta_\beta(x,\Omega)| > \varepsilon\}, \qquad \Omega(1,\varepsilon) = \{x \in \Omega : \delta(x,\Omega)| > \varepsilon\}.$$

The set $\Omega(\beta,\varepsilon)$ is bounded since the volume of Ω is finite. It is clear that $f_{\alpha\beta\varepsilon} \in C_0^1(\Omega)$ for $\alpha > 1$ and $\beta < 1$. Since $C_0^\infty(\Omega)$ is dense in $C_0^1(\Omega)$ (see, for instance, [14]), one can write

$$c(p,\Omega) \ge \left(\int_{\Omega(\beta,\varepsilon)} \frac{|f_{\alpha\beta\varepsilon}|^n}{\delta^n} dx\right)^{1/n} \left(\int_{\Omega(\beta,\varepsilon)} |\nabla f_{\alpha\beta\varepsilon}|^p \, dx\right)^{-1/p},$$

where $\alpha > 1$ and $\beta < 1$. Letting $\alpha \to 1$ and $\beta \to 1$ and using $|\nabla \delta_\beta(x,\Omega)| \le 1$, we get

$$c(p,\Omega) \ge \left(\int_{\Omega(1,\varepsilon)} \frac{|f_{11\varepsilon}|^n}{\delta^n} dx\right)^{1/n} \left(\int_{\Omega(1,\varepsilon)} dx\right)^{-1/p}$$
$$= \left(\int_{\Omega(1,\varepsilon)} \frac{(\delta-\varepsilon)^n}{\delta^n} dx\right)^{1/n} |\Omega(1,\varepsilon)|^{-1/p},$$

where $\delta = \mathrm{dist}(x,\partial\Omega)$. Lebesgue's theorem on majorized convergence applied to the last inequality as $\varepsilon \to 0$ gives

$$c(p,\Omega) \ge |\Omega|^{1/n-1/p}.$$

This completes the proof of Theorem 7.

In the next theorem we consider the following inequality

$$\int_\Omega \frac{|f|}{\delta^s}dx \le c\left(\int_\Omega \frac{|\nabla f|^p}{\delta^s}dx\right)^{1/p}, \qquad \forall f \in C_0^\infty(\Omega). \tag{21}$$

Theorem 8. *Suppose that Ω is an open set in $\mathbb{R}^n$ $(n \ge 1)$ such that*

$$M = \int_\Omega \delta^{\frac{p}{p-1}-s}dx < +\infty.$$

If $p > 1$, $s \ge n$ and $1/p + 1/s > 1$, then the best constant in (21) satisfies the inequalities

$$1 - \frac{1}{p} \le \frac{c}{M^{1-1/p}} \le \Gamma^{1-1/p}\left(\frac{2p}{p-1}\right)\left(1 - \frac{1}{p}\right), \tag{22}$$

where Γ is Euler's gamma function.

Proof of Theorem 8. Following the proof of Theorem 6 with a little change we obtain the upper estimate in (22). Namely, by using (17) and (18) for $s > n$ and $p = 1$, summing over all S with $Q(S) \ne \emptyset$, and applying the Hölder inequality with the exponents p and $q = \frac{p}{p-1}$, we easily obtain

$$\int_{\Omega_1} \frac{|f|}{\delta^s}dx \le \left(\int_{\Omega_1} \frac{|\nabla f|^p}{\delta^s}dx\right)^{1/p}\left(\int_{\Omega_1} \Phi^q dx\right)^{1/q},$$

where

$$\Phi|Q(S) = \frac{\delta^{s/p}}{\delta^{s-1}}\int_\delta^{\psi_{n-k}} \frac{dt}{t^{s-k+1}} \quad (k = 1, 2, \ldots, n)$$

for any cubic $(n-k)$-face S. From this it follows that

$$\int_{\Omega_1} \Phi^q dx \le \frac{1}{q^q}\Gamma(q+1)\cdot M$$

since

$$\begin{aligned}
&\int_0^\varphi \left(\frac{r^{s/p}}{r^{s-1}}\int_r^\varphi \frac{dt}{t^{s-k+1}}\right)^q r^{k-1}dr \\
&= \frac{\varphi^{q-s+k}}{(s-k)^q}\int_0^1 \tau^{q-s}(1-\tau^{s-k})^q \tau^{k-1}d\tau \\
&= \frac{\Gamma(\frac{q}{s-k})\Gamma(q+1)}{(s-k)^q\Gamma(\frac{q}{s-k}+q)}\int_0^\varphi r^{q-s}r^{k-1}dr \\
&\le q^{-q}\Gamma(q+1)\int_0^\varphi r^{q-s}r^{k-1}dr.
\end{aligned}$$

To obtain the last inequality we have used that

$$\frac{\Gamma(\frac{q}{s-k})}{(s-k)^q\Gamma(\frac{q}{s-k}+q)} \le q^{-q}.$$

This inequality is a simple consequence of the identity $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ and the sharp estimates by W. Gautschi for the gamma function (see [23] or the book by D. S. Mitrinovic [37]).

To obtain the lower estimate in (22), we first observe that

$$\int_\Omega \frac{|f_0|}{\delta^s}dx \left(\int_\Omega \frac{|\nabla f_0|^p}{\delta^s}dx\right)^{-1/p} = (1-1/p)M^{1-1/p}$$

for the function $f_0=\delta^{\frac{p}{p-1}}$.

To complete the proof, we apply the Calderon - Zigmund - Burenkov theorem on regularized distance functions ([14], P. 78) as in the proof of Theorem 7.

6. An improved form of the Brezis-Marcus inequality and related results.

We shall obtain the following generalization of the cited equation

$$c_p(p,\Omega)=p/(p-1)$$

for convex domains.

Theorem 9. *Let Ω be an open, convex and proper subset of $\mathbb{R}^n$. If $p\ge 1$ and $s>1$ then*

$$\int_\Omega \frac{|f|^p}{\delta^s}dx \le \left(\frac{p}{s-1}\right)^p \int_\Omega \frac{|\nabla f|^p}{\delta^{s-p}}dx, \quad \forall f\in C_0^\infty(\Omega), \tag{23}$$

where $\delta=\delta(x)=\mathrm{dist}(x,\partial\Omega)$.

Also, we shall prove the following lower estimate (compare the case $\alpha=n-1$ of Theorem 5.1 in [16]). Let Ω be a bounded open set in $\mathbb{R}^n$. Consider its boundary surface area by Minkowski (see [24]):

$$\sigma(\Omega)=\lim_{t\to+0}\sup\frac{A(t)}{t},$$

where $A(t)=\mathrm{mes}\{x\in\Omega : \mathrm{dist}(x,\partial\Omega)<t\}$.

Theorem 10. *If $p\ge 1$ and $s>1$ and Ω is a bounded open set in $\mathbb{R}^n$ with finite surface area $\sigma(\Omega)$, then*

$$c_p(s,\Omega)\ge\frac{p}{s-1}.$$

From Theorems 9 and 10 it follows that $c_p(s,\Omega)=p/(s-1)$ for $p\ge 1$ and $s>1$ and any bounded convex domain $\Omega\subset\mathbb{R}^n$. The main aim of this section is to improve this result using additional terms in the inequality (23). To this end, examine first the following theorem of H.Brezis and

M.Marcus [13]: *if Ω is a bounded open and convex subset of $\mathbb{R}^n$ and $\lambda = 1/\mathrm{diam}(\Omega)^2$ then*

$$\int_\Omega \frac{|f|^2}{\delta^2}dx + \lambda \int_\Omega |f|^2 dx \le 4 \int_\Omega |\nabla f|^2 dx, \quad \forall f \in C_0^\infty(\Omega). \tag{24}$$

In [27] M.Hoffmann–Ostenhof, T.Hoffmann–Ostenhof and A.Laptev proved that λ in (24) can be replaced by $c(n)/|\Omega|^{2/n}$, where $|\Omega| = \mathrm{mes}\,\Omega$.

It is natural to ask whether inequality (24) is valid with some $\lambda > 0$ for an unbounded convex domain. It is clear that the validity of (24) implies

$$\lambda \le 4\lambda_1(\Omega),$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue for the Laplace equation. According to the theory of Isoperimetric Inequalities in Mathematical Physics (see [8], [10], [41]) we have

$$\lambda \le \frac{\text{const.}}{\delta_0^2(\Omega)}.$$

This argument shows that (24) is not true with $\lambda > 0$ for any unbounded convex domain Ω in the case $\delta_0 = \delta_0(\Omega) = +\infty$. It is also clear that there are unbounded convex domains $\Omega \subset \mathbb{R}^n$ with $\delta_0(\Omega) < +\infty$ in the case $n \ge 2$, only.

We extend the Brezis - Marcus inequality to certain unbounded convex domains. More precisely, we prove that (24) is true with $\lambda = 1/\delta_0^2$, and that a similar improved version of (23) is valid.

Theorem 11. *Let Ω be an open, convex and proper subset of $\mathbb{R}^n$. If $p \ge 1$ and $s > 1$, then for any $f \in C_0^\infty(\Omega)$*

$$\int_\Omega \frac{|f|^p}{\delta^s}dx + \frac{1}{(s-1)\delta_0^s}\int_\Omega |f|^p dx \le \left(\frac{p}{s-1}\right)^p \int_\Omega \frac{|\nabla f|^p}{\delta^{s-p}}dx, \tag{25}$$

where $\delta = \mathrm{dist}(x, \partial\Omega)$, $\delta_0 = \sup\{\delta(x) \,:\, x \in \Omega\}$.

Proof of Theorems 9 *and* 11. Let Ω be an open, convex and proper subset of $\mathbb{R}^n$. It is known that for any compact set $K \subset \Omega$ there exists a convex n-dimensional polytope Q such that $K \subset \mathrm{int}Q \subset \Omega$ (see [24]). Hence, for given $f \in C_0^\infty(\Omega)$ it is sufficient to prove inequalities (23) and (25) for every convex, n-dimensional polytope Q such that

$$\mathrm{supp}\, f \subset \mathrm{int}Q \subset \Omega.$$

Let Q be such a polytope, and let $S_1, S_2, \ldots, S_m$ be the collection of all $(n-1)$-faces of Q. First we will construct a special decomposition of

Q:

$$Q = \cup_{j=1}^{m} Q_j, \qquad \mathrm{int} Q_j \cap Q_k = \emptyset \quad \text{for} \quad j \neq k, \tag{26}$$

where Q_j are convex and compact sets. Namely, for each $x' \in \mathrm{int} S_j$ we define

$$\varphi_j(x') = \max\{t \in \mathbb{R}_+ : B(x' + t\nu(x'), t) \subset Q\},$$

where $\nu(x')$ is the interior normal to S_j at the point x', $B(x,t)$ is the ball $\{y \in \mathbb{R}^n : |y-x| \leq t\}$, $\mathbb{R}_+ = [0, \infty)$. We easily obtain that

$$Q_j := S_j \cup \{x = x' + t\nu(x') : 0 < t \leq \varphi_j(x')\}$$

is a closed, n-dimensional and convex set, and $Q_1, Q_2, \ldots, Q_m$ satisfy (26). Due to convexity of Q_j, $\mathrm{mes}_n \cup_{j=1}^{m} \partial Q_j = \emptyset$. Hence, for any function $g \in L^1(Q)$

$$\int_Q g(x)\, dx = \sum_{j=1}^{m} \int_{Q_j} g(x)\, dx$$

and, by Fubini's theorem

$$\int_{Q_j} g(x)\, dx = \int_{S_j} dx' \int_0^{\varphi_j(x')} g(x' + t\nu(x'))dt. \tag{27}$$

For any $x = x' + t\nu(x') \in Q_j$ one has

$$\delta(x) = t, \qquad \delta(x) \leq \varphi_j(x') \leq \delta_0, \tag{28}$$

where $\delta = \delta(x) = \mathrm{dist}(x, \partial Q)$ and δ_0 is its maximum in Q.

Suppose that $p \geq 1$, $s > 1$ and $f \in C_0^\infty(Q)$. By using (27) and (28) for the function

$$g(x' + t\nu(x')) = |f(x' + t\nu(x'))|^p \left(\frac{1}{t^s} + \frac{1}{(s-1)\delta_0^s} \right),$$

we get

$$\int_{Q_j} |f(x)|^p \left[\frac{1}{\delta^s(x)} + \frac{1}{(s-1)\delta_0^s} \right] dx$$

$$\leq p \int_{S_j} dx' \int_0^{\varphi_j(x')} \left(\frac{1}{t^s} + \frac{1}{(s-1)\delta_0^s} \right) dt \int_0^t |f(y)|^{p-1} \left| \frac{\partial f(y)}{\partial \tau} \right| d\tau$$

$$= \frac{p}{s-1} \int_{S_j} dx' \int_0^{\varphi_j(x')} \frac{|f(y)|^{p-1}}{\tau^{s-1}} \left| \frac{\partial f(y)}{\partial \tau} \right| A(x', \tau) d\tau,$$

where $y = x' + \tau\nu(x')$, and

$$A(x', \tau) = 1 - \frac{\tau^{s-1}}{\varphi_j^{s-1}} + \frac{\tau^{s-1}}{\delta_0^s}[\varphi_j(x') - \tau] \leq 1,$$

By using this and the inequality $\left|\frac{\partial f}{\partial \tau}\right| \leq |\nabla f|$ and by summing over $j = 1, 2, \ldots, m$ we get

$$I_Q := \int_Q |f|^p \left(\frac{1}{\delta^s} + \frac{1}{(s-1)\delta_0^s}\right) dx \leq \frac{p}{s-1} \int_Q \frac{|f|^{p-1}}{\delta^{s-1}} |\nabla f| dx. \qquad (29)$$

If $p = 1$, then (29) is the inequality (25) for $\Omega = Q$. Following Hardy (see [25], Theorem 330), in the case $p > 1$ we apply Hölder's inequality in (29) to get

$$I_Q \leq \frac{p}{s-1} \left(\int_Q \left(\frac{|f|^{p-1}}{\delta^{s-s/p}}\right)^{p'} dx\right)^{1/p'} \left(\int_Q \left(\frac{|\nabla f|}{\delta^{s/p-1}}\right)^p dx\right)^{1/p}$$

$$= \frac{p}{s-1} \left(\int_Q \frac{|f|^p}{\delta^s} dx\right)^{1/p'} \left(\int_Q \frac{|\nabla f|^p}{\delta^{s-p}} dx\right)^{1/p},$$

where $1/p' = 1 - 1/p$. Consequently,

$$I_Q \leq \left(\frac{p}{s-1}\right)^p \int_Q \frac{|\nabla f|^p}{\delta^{s-p}} dx.$$

This completes the proof of Theorems 9 and 11.

Proof of Theorem 10. Suppose that $p \geq 1$, $s > 1$ and $\sigma(\Omega)$ is finite. Let us denote

$$X = \int_\Omega \frac{|u|^p}{\delta^s} dx, \quad Y = \int_\Omega \frac{|\nabla u|^p}{\delta^{s-p}} dx, \quad \delta = \mathrm{dist}(x, \partial\Omega).$$

For $\varepsilon \in (0,1)$ and $u = u_\varepsilon(x) = \delta^{(s-1+\varepsilon)/p}$ we have

$$X = M_{-1+\varepsilon}(\Omega), \qquad Y = \left(\frac{s-1+\varepsilon}{p}\right)^p M_{-1+\varepsilon}(\Omega), \qquad (30)$$

where $M_{-1+\varepsilon}(\Omega)$ is the following moment of Ω about its boundary

$$M_{-1+\varepsilon}(\Omega) = \int_\Omega \delta^{-1+\varepsilon} dx.$$

Using (30), the equation

$$\lim_{\varepsilon \to +0} \sup \varepsilon M_{-1+\varepsilon}(\Omega) \leq \sigma(\Omega) \qquad (31)$$

and the Calderon - Zigmund - Burenkov theorem (see [14], P. 78), we obtain that

$$c_p(s, \Omega) \geq p(s-1)^{-1}.$$

To prove (31) we remark that

$$\gamma_k := \sup\left\{\frac{A(t)}{t} \,:\, 0 < t < \frac{\delta_0}{k}\right\} - \sigma(\Omega) \to 0$$

as $k \to \infty$. Consequently,

$$b := \lim_{\varepsilon\to+0} \sup \varepsilon \int_0^{\delta_0} \frac{A(t)}{t^{2-\varepsilon}} dt = \lim_{k\to\infty} \lim_{\varepsilon\to+0} \sup \varepsilon \int_0^{\delta_0/k} \frac{A(t)}{t^{2-\varepsilon}} dt$$

$$\leq \lim_{k\to\infty} [\sigma(\Omega) + \gamma_k] \lim_{\varepsilon\emptyset 0} \left(\frac{\delta_0}{k}\right)^{\varepsilon} = \sigma(\Omega).$$

Integrating by parts and using the known formulas (see [43], Chapter 1), we have

$$(1-\varepsilon) \int_0^{\delta_0} \frac{A(t)}{t^{2-\varepsilon}} dt = \frac{A(\delta_0)}{\delta_0^{1-\varepsilon}} + \int_0^{\delta_0} \frac{dA(t)}{t^{1-\varepsilon}} = M_{-1+\varepsilon}(\Omega).$$

for any $\varepsilon \in (0,1)$. Consequently,

$$\lim_{\varepsilon\to+0} \sup \varepsilon M_{-1+\varepsilon}(\Omega) = b \leq \sigma(\Omega)$$

which proves (31).

The proof of Theorem 10 is complete.

Example 4. Suppose that $p \geq 1$ and $s > 1$. Our aim is to obtain an upper estimate for $\lambda > 0$ in the inequality

$$\int_\Omega \frac{|u|^p}{\delta^s} dx + \frac{\lambda}{\delta_0^s} \int_\Omega |u|^p dx \leq \left(\frac{p}{s-1}\right)^p \int_\Omega \frac{|\nabla u|^p}{\delta^{s-p}} dx, \quad \forall u \in C_0^\infty(\Omega), \quad (32)$$

for convex domains (compare formula (25)).

We will examine the domains

$$\Omega_\varepsilon = (-\varepsilon, 1+\varepsilon)^{n-1} \times (-\varepsilon, \varepsilon) \subset \mathbf{R}^n$$

and functions u_ε defined by

$$u_\varepsilon(x) = \delta^{(s-1+\varepsilon)/p}, \qquad 0 < \varepsilon < 1,$$

in the case $n = 2$ only. Note that $\delta_0 = \sup \delta = \varepsilon$ for Ω_ε.

For $n = 2$, straightforward computations give

$$X = \frac{2\varepsilon^\varepsilon}{\varepsilon} + \frac{8\varepsilon^\varepsilon}{1+\varepsilon}, \qquad Y = \left(\frac{s-1+\varepsilon}{p}\right)^p X,$$

$$Z := \frac{\lambda}{\delta_0^s} \int_{\Omega_\varepsilon} u_\varepsilon^p dx = \frac{2\lambda\varepsilon^\varepsilon}{s+\varepsilon} \left(1 + \frac{4\varepsilon}{s+1+\varepsilon}\right)$$

and

$$\frac{X + Z - \left(\frac{p}{s-1}\right)^p Y}{2\varepsilon^\varepsilon} = \frac{\lambda}{s} - \frac{p}{s-1} + O(\varepsilon).$$

If (32) is true then

$$\lambda \leq \frac{ps}{s-1}.$$

Hence, the best possible value of λ in (32) satisfies the inequalities

$$\frac{1}{s-1} \leq \lambda \leq \frac{ps}{s-1} \quad (\forall p \geq 1, s > 1).$$

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